Φ geometry in physics: a unified visual language and numerical stability

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We present "φ-geometry," a visual re-parameterization of familiar relations by a single angle on a unit (or scaled) circle: $\beta = \sin\varphi = v/c$, $\gamma^{-1} = \cos\varphi$, and $\gamma\beta = \tan\varphi$. In this language, key results of special relativity—time dilation, length contraction, the energy–momentum invariant—and collinear velocity addition reduce to elementary trigonometric identities. We also give compact sin/cos forms for basic electromagnetic transformations and the de Broglie pair, introduce an atomic scale $\varphi(Z,n)$ from $v/c \approx Z\alpha/n$, and read the string momentum–winding plane as $(\cos\varphi, \sin\varphi)$. The aim is clarity and pedagogy: one diagram organizes diverse formulas and helps trace transitions across domains. A practical benefit is numerical robustness: switching to the complementary branch (sin ↔ cos) mitigates division-by-zero and precision loss near limiting angles. We do not propose new dynamics. Rather, we present a re-parameterization aimed at clarity and pedagogy. The physical content of SR/GR remains unchanged; what we provide is an alternative geometric language. The accuracy for specific processes and regimes requires further study, analysis, and refinement, which we leave to future work.

Keywords: φ-geometry; unit-circle parameterization; sin/cos re-parameterization; visual pedagogy; scientific visualization; special relativity; Lorentz factor (γ); Lorentz transformations; energy–momentum invariant; de Broglie relations; electromagnetism; velocity addition; complementary branches (sin↔cos); numerical stability/robustness

1. Basic Geometry

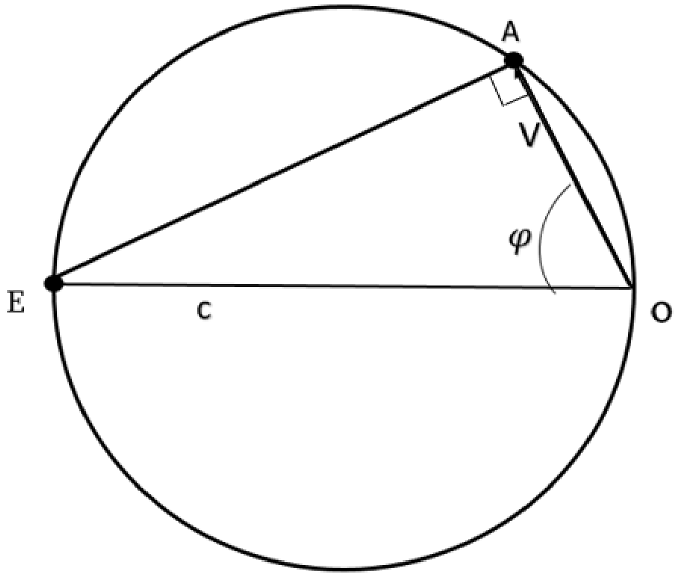

Figure. 1. An inscribed right triangle (angle φ at O). The hypotenuse OE is scaled to c (or to 1).

Axes / normalization.
Hypotenuse: $OE = c$ (or $OE = 1$).
Legs: $OA = OE \cdot \cos\varphi$, $\quad EA = OE \cdot \sin\varphi$.

Kinematics.

$$\beta = \sin\varphi, \quad \gamma^{-1} = \cos\varphi, \quad \gamma\beta = \tan\varphi\,.$$

Hence: $\Delta t = \gamma\Delta\tau = \Delta\tau/\cos\varphi\,;\quad L = L_0/\gamma = L_0\cos\varphi$ [1].

Mnemonic interpretations of the legs.

- **OA** — axial "rest-like" direction: $\gamma^{-1} = \cos\varphi$. Think of this as the fraction of a "rest" component. In gravitational examples there is an analogy $\cos\varphi \leftrightarrow \sqrt{1 - \frac{2GM}{rc^2}}$ (outside the horizon) [2].
- **EA** — transverse "dynamic" direction: $\beta = \sin\varphi$. The normalized momentum is $\frac{p}{mc} = \gamma\beta = \tan\varphi$ (mnemonic) [1,3].

Invariant.
$\cos^2\varphi + \sin^2\varphi = 1$. It follows that $\sec^2\varphi - \tan^2\varphi = 1$.
Equivalently, $E^2 - (pc)^2 = (mc^2)^2$.

Numerical stability with physical meaning.
Standard "stable rewrites" in numerical analysis (see, e.g., Higham [4]) mitigate loss of significance by changing the algebraic form.
In the φ-notation, stability arises naturally from circle geometry.
When $\cos\varphi$ is small, switch to the complementary angle $\Phi = 90° - \varphi$ (i.e., swap sin ↔ cos) and work with $\sin\Phi$ instead — and vice versa.
This avoids division by small quantities and reduces precision loss. The same trick is useful for kinematics, energetics, gravitational estimates, atomic examples, and beyond.

Note. This is a simple re-parameterization of standard formulas. We do not introduce new dynamical postulates.

Complementary symmetry.
In the φ-language there is a natural discrete "mirror" symmetry $\varphi \leftrightarrow 90° - \varphi$ (i.e., sin ↔ cos). In practice, this yields a stable branch switch near limiting regimes. Below we illustrate the same symmetry in kinematic, electromagnetic, and atomic examples.

2. Kinematics and velocity addition (φ-parameterization)

Textbook baseline.
Standard special-relativistic language uses rapidity η. It gives three relations: $\beta = \tanh \eta$, $\gamma = \cosh \eta$, and $\gamma\beta = \sinh \eta$.
They are equivalent to the φ-parameterization: $\beta = \sin \varphi$, $\gamma^{-1} = \cos \varphi$, and $\gamma\beta = \tan \varphi$.
The identity $\cosh^2 \eta - \sinh^2 \eta = 1$ (2.1) also holds.
If additivity of collinear boosts is the priority, rapidity is convenient. If visual clarity and numerical stability are the priority, a single angle φ is convenient ("one angle, one diagram") [1,2].
A key advantage of rapidity is additivity for collinear boosts, $\eta_{12} = \eta_1 + \eta_2$, which immediately yields Einstein's velocity-addition law $\beta_{12} = (\beta_1 + \beta_2)/(1 + \beta_1\beta_2)$ (2.2) (see e.g. [2,3]).

"φ-language" (circular re-parameterization).
The same quantities can be rewritten by an angle on the unit (or c-scaled) circle ("φ-triangle"): $\beta = \sin \varphi$, $\gamma^{-1} = \cos \varphi$, $\gamma\beta = \tan \varphi$.
This is equivalent to rapidity ($\tanh \eta \leftrightarrow \sin \varphi$, $\cosh \eta \leftrightarrow \sec \varphi$, $\sinh \eta \leftrightarrow \tan \varphi$) but more visual: one diagram organizes the formulas.

Velocity addition in φ-notation.

With $\beta_1 = \sin \varphi_1 = v_1/c$ and $\beta_2 = \sin \varphi_2 = v_2/c$, the collinear result is

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{\sin \varphi_1 + \sin \varphi_2}{1 + \sin \varphi_1 \cdot \sin \varphi_2} \quad (2.3)$$

This makes the bound $|\beta_{12}| < 1$ explicit. Geometrically, all motion lives inside the circle, which is helpful for quick estimates; see Fig. 2 [1–3].

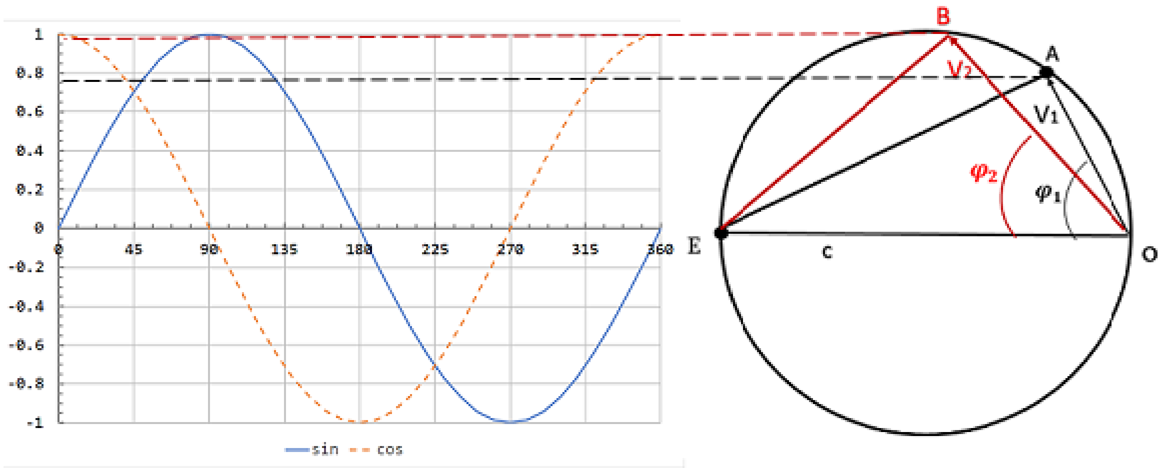

Figure 2 — *φ-triangle with sine/cosine map.* *Point A corresponds to speed v_1 (angle φ_1); point B to v_2 (angle φ_2). The unit (or c-scaled) circle encodes $\beta = \sin\varphi$ and $\gamma^{-1} = \cos\varphi$. The resultant speed O→B follows relativistic velocity addition. Left panel: as the point slides along arc OE, sin φ rises 0→1 while cos φ falls 1→0; the horizontal line highlights $|\beta| < 1$.*

Reading from the diagram.
Basic effects can be read directly: time dilation $\Delta t = \Delta\tau/\cos\varphi$ and length contraction $L = L_0 \cos\varphi$ [1].

Numerical stability (complementary angle).
When cos φ or sin φ is small, switch to the complementary angle Φ = 90° − φ (swap sin ↔ cos). Small denominators disappear, and the representation stays equivalent and numerically stable [4]. The limiting positions φ = 0° and φ = 90° (the triangle degenerates to a segment) are not analyzed here; they are a topic for a separate discussion.

Why two branches (sin and cos)

From Fig. 3: $v_r = v\sqrt{1 - \frac{v^2}{c^2}}$ (2.4)

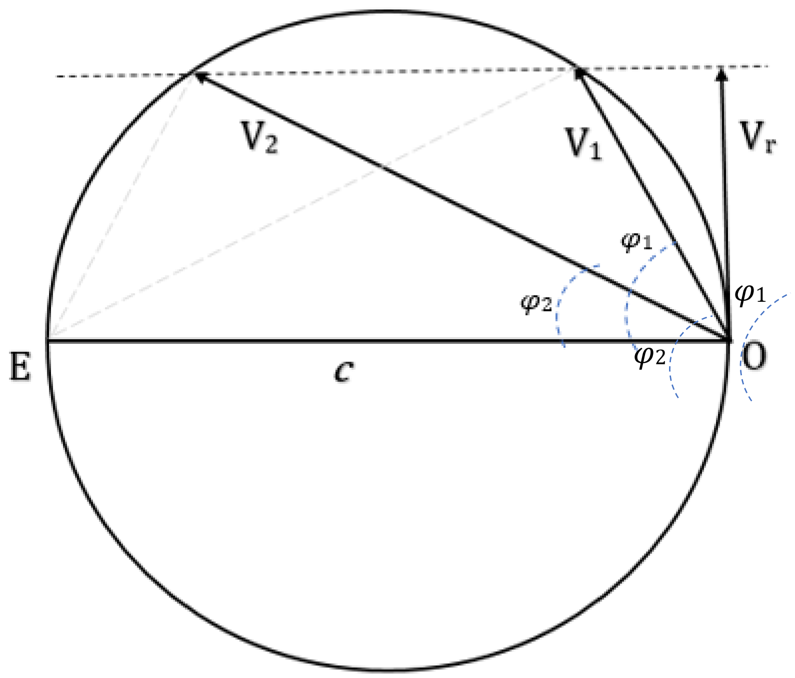

Figure 3 — *Two complementary velocity branches. Angles φ_1, φ_2 are measured at O between OE and OV_1, OV_2, with $0° < \varphi_1 < \varphi_2 < 90°$.*

Let $\beta = v / c$ and $\beta_r = v_r / c$ (2.5)
Then $\beta_r^2 = \beta^2(1 - \beta^2) \Rightarrow \beta^4 - \beta^2 + \beta_r^2 = 0$, $0 \le \beta_r \le \frac{1}{2}$ (2.6).
Define the angle φ by — $\sin(2\varphi) = 2\beta_r$ (so $0 \le \beta_r \le \frac{1}{2}$)).
Then $\sqrt{1 - 4\beta_r^2} = |\cos(2\varphi)|$, and $\beta^2 = \frac{1}{2}(1 \pm |\cos(2\varphi)|)$ (2.7).
Equivalently, $\beta^2 = sin^2\varphi \ or \ \beta^2 = cos^2\varphi$, i.e., two branches.
Hence, $\beta = \sin\varphi$ or $\beta = \cos\varphi$: two complementary branches of the same φ-triangle; the switch is φ ↔ 90° − φ.

3. The φ-triangle: definitions and identities

We consider a right triangle inscribed in the unit (or scaled) circle, with the hypotenuse OE laid along the diameter (typically Fig. 1 uses OE = c; Fig. 4 uses OE = 1).

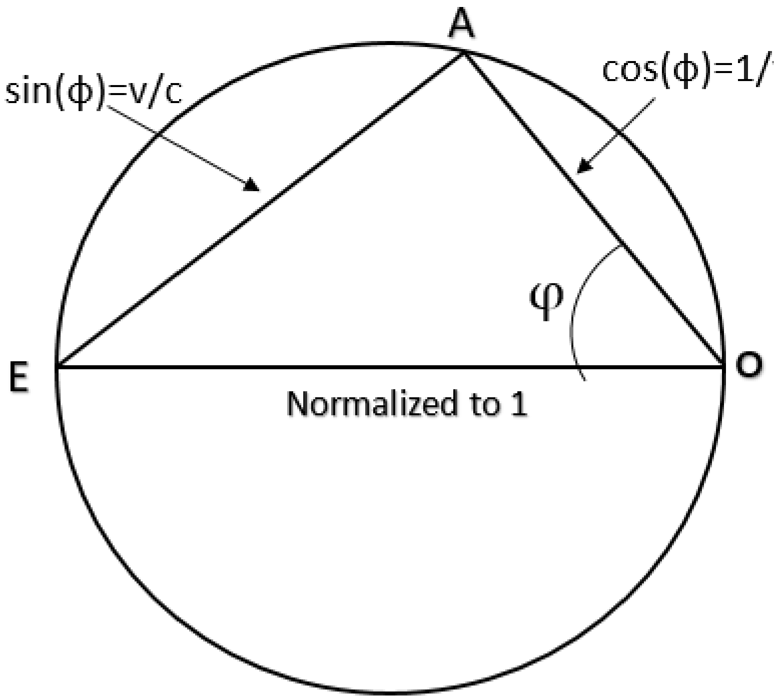

Figure 4 — *φ-triangle with hypotenuse normalized to OE ≡ 1.*

Definitions (circular parametrization).

$$\beta = \frac{v}{c} = sin\varphi, \ \ \frac{1}{\gamma} = cos\varphi \ , \ \ \gamma\beta = tan\varphi \ (3.1).$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = sec\ \varphi \ , \qquad \frac{E}{mc^2} = \gamma = sec\ \varphi \ , \ \ \frac{p}{mc} = \gamma\beta = tan\varphi \ (3.2)$$

Energy–momentum invariant (in a "one-line" form).

$$sec^2\varphi - tan^2\varphi = 1 \ \Leftrightarrow \left(\frac{E}{mc^2}\right)^2 - \left(\frac{p}{mc}\right)^2 = 1 \ \Leftrightarrow \ E^2 - (pc)^2 = (mc^2)^2 \ (3.3).$$

(see, e.g., textbook expositions [1,3])

Energy projections.

$$Ecos\varphi = mc^2, \qquad Esin\varphi = pc \implies E = \frac{mc^2}{cos\varphi}, \qquad pc = \frac{mc^2 sin\varphi}{cos\varphi} = mc^2 tan\varphi \ (3.4).$$

Hence

$$E^2 = (mc^2)^2 + (pc)^2 \ (3.5),$$

which is consistent with the standard relation [1,3].

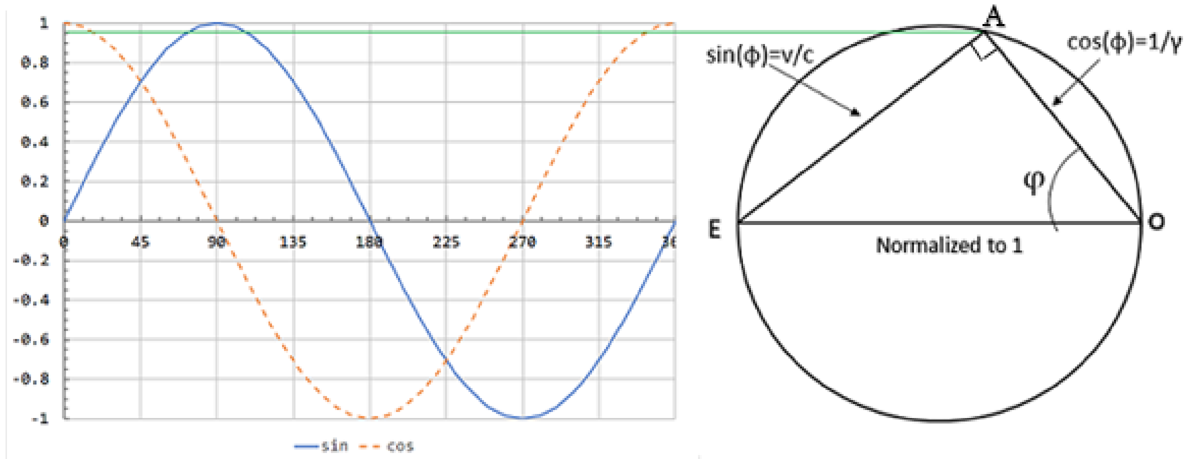

Figure 5 — *Sine/cosine and the φ-triangle: motion of point A along arc OE.*

As A moves, the angle φ runs from 0° to 90°. In the left panel of Fig. 5 this corresponds to sin φ increasing (0→1) while cos φ decreases (1→0), or vice versa. At any position of A one can read off, "at a glance":

$$\beta = \frac{v}{c} = sin\varphi, \quad \frac{1}{\gamma} = cos\varphi, \quad \gamma\beta = tan\varphi, L = L_0 cos\varphi \ (3.6).$$

If cos φ or sin φ becomes small, it is convenient to switch to the complementary angle Φ = 90° − φ (swap sin ↔ cos). This removes small denominators and "infinities"; the representation remains equivalent and numerically stable.

Quick practical cross-check (particles).

Given independent (E, p, m), compute

$$sin\varphi = \beta = \frac{pc}{E}, \qquad cos\varphi = \frac{1}{\gamma} = \frac{mc^2}{E} \ (3.7)$$

Then

$$son^2\varphi + cos^2\varphi = \frac{p^2c^2 + m^2c^4}{E^2} \approx 1 \ (3.8)$$

which is just the invariant (up to rounding). Illustrations (rounding standard; masses from PDG [5]):

- Electron, $K = 5MeV: \beta \approx 0.9957, \frac{1}{\gamma} \approx 0.0927, cos^2\varphi + sin^2\varphi = 1$;

- Muon, $E \approx 3GeV$: $\beta \approx 0.99938$, $\frac{1}{\gamma} \approx 0.0352$, $cos^2\varphi + sin^2\varphi = 1$;
- Au-197 at T ~ 300 K (estimate): $\beta \sim 6.5 \times 10^{-7}$, $1/\gamma \approx 1$, $cos^2\varphi + sin^2\varphi = 1$.

(For atomic estimates one may use v/c ≈ Z α/n; α — CODATA [6].)

Remark (scope). This is a simple re-parametrization of standard formulas; no new dynamical postulates are introduced.

Appendix A. φ-map identities (short proofs)

4. GPS: a short check of φ-geometry (kinematics + gravity)

Idea in a figure.

On the φ-diagram (Fig. 6), point A is a satellite in orbit; point B is a clock on Earth's surface.

From special-relativistic kinematics we read

$$\cos \varphi_{\mathrm{k}}(\mathrm{v}) = \sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}} \quad (4.1)$$

from gravity (spherically symmetric weak field) we read

$$\cos \varphi_{\mathrm{g(r)}} = \sqrt{1 - \frac{2\mathrm{GM}}{\mathrm{r\,c}^2}} \quad (4.2)$$

Here the left side, cos φ_{g}, is our φ-geometry label that links the standard Schwarzschild factor to the diagram in Fig. 1 [8]; the right side is the standard clock-rate factor for a static observer (see [2, 9]).

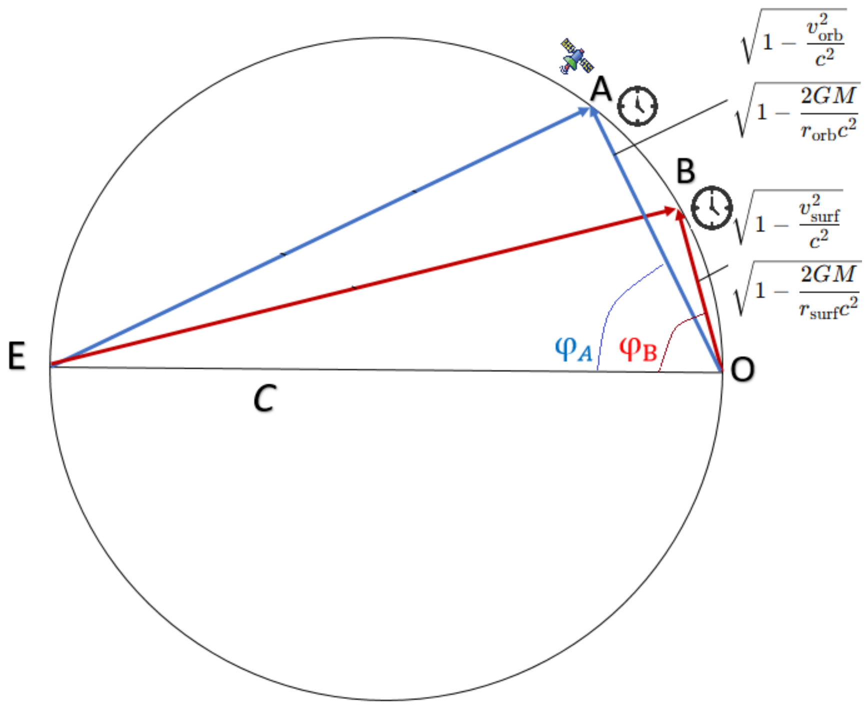

Figure 6 — GPS in φ-geometry: *for satellite A and ground station B, both the kinematic and gravitational geometry are shown.*

Daily clock-rate difference (GPS)

The daily clock-rate offset consists of two contributions:

Kinematic term

$$\Delta t(kin) = \Delta t\big(cos\varphi_k(v_{\text{surf}}) - cos\varphi_k(v_{\text{orb}})\big) = \Delta t\left(\sqrt{1-\frac{\mathrm{v}_{\text{surf}}^2}{\mathrm{c}^2}} - \sqrt{1-\frac{\mathrm{v}_{\text{orb}}^2}{\mathrm{c}^2}}\right) \quad (4.3)$$

Gravitational term

$$\Delta t(grav) = \Delta t\left(cos\varphi_g(r_{orb}) - cos\varphi_g(r_{surf})\right) = \Delta t\left(\sqrt{1-\frac{2\mathrm{GM}}{r_{surf}\mathrm{c}^2}} - \sqrt{1-\frac{2\mathrm{GM}}{r_{orb}\mathrm{c}^2}}\right) \quad (4.4)$$

Total

$$\Delta T = \Delta t(grav) + \Delta t(kin)\ (4.3), \qquad \Delta\mathrm{t} = 86400\ \mathrm{s}(4.5)$$

Substitution (nominal values):

v_{surf}=465.1 m/s, v_{orb}≈3874 m/s;

r_{surf}=6371 km, r_{orb}=26560 km; G, $M_{\otimes}$, c — стандартные константы [5,6].

Then:

$\Delta t(kin) = -7.11\mu\mathrm{s/day}$ (satellite clock runs slower due to motion),

$\Delta t(grav) = 45.72\mu\mathrm{s/day}$ (orbital clock runs faster in the weaker field),

$\Delta T = 45.72 - 7.11 = 38.5$ μs/day (net: satellite clocks run faster) [7].

In the φ-language this is straightforward: for GPS the kinematic point lies near 0°, so sin φ_k = β is very small; hence the kinematic contribution is small and negative relative to the gravitational one.

Gravitational illustration (Schwarzschild, outside the horizon)

This is the same φ-triangle from Fig. 1, but with a gravitational angle φ_g that encodes the clock rate in a spherically symmetric field. Figure 7 shows a "gravitational twin" of the basic scheme in Fig. 1.

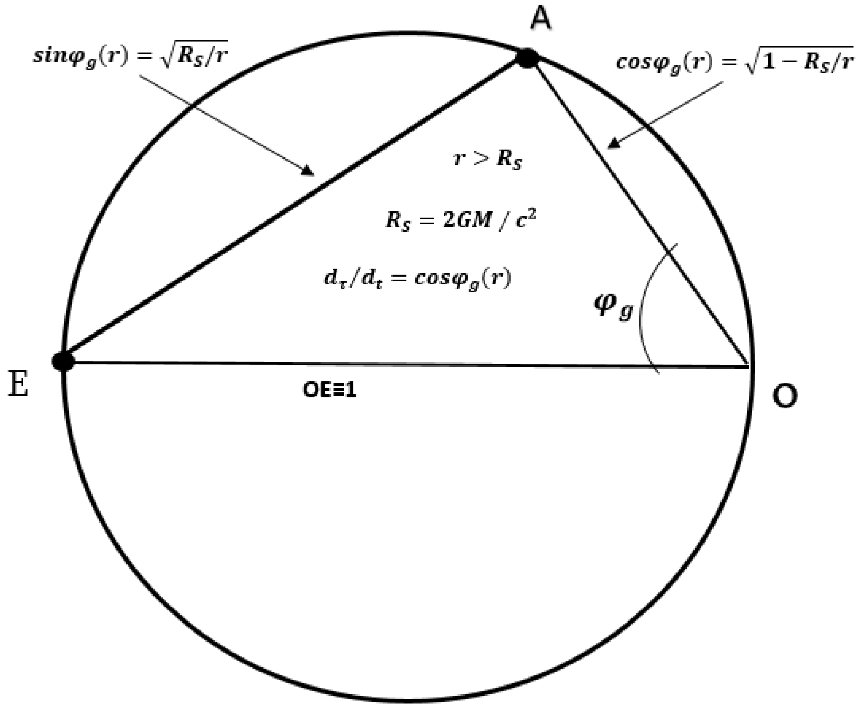

Figure 7 - *Gravitational φ-geometry*

Definitions. Let $R_S = 2GM / c^2$ be the Schwarzschild radius, and $r > R_S$ the radial coordinate of a static observer. Then

$$cos\varphi_g(r) = \sqrt{1 - \frac{R_S}{r}}\ (4.6), \qquad sin\varphi_g(r) = \sqrt{\frac{R_S}{r}}\ (4.7)$$

the clock rate (static observer) is $d_\tau/d_t = cos\varphi_g(r)$ (4.6), $(r > R_S)$

How to read it from the picture. The closer point A is to the body's center (smaller r), the larger sin φ_g and the smaller cos φ_g — so the clock runs more slowly. The difference in clock rates between two radii r_1 and r_2 over an interval Δt (e.g., one day):

$$\Delta t_{grav} = \Delta t\left[cos\varphi_g(r_2) - cos\varphi_g(r_1)\right]\ (4.8)$$

This is exactly formula (4.2), which we used for the Earth's surface ↔ GPS-orbit pair.

For brief numerical illustrations, see [7]; for the link to the φ-diagram, see [8].

Numerical stability.

cos φ_g or sin φ_g becomes very small (near the boundary of the domain), it is convenient to switch to the complementary angle $\Phi g = 90° - \varphi_g$ (swap sin↔cos). The representation remains equivalent and numerically stable. In this note we deliberately work only with $r > R_S$.

For orbital dynamics and complicated trajectories, the φ-notation serves as a clear overlay on top of the standard geodesic calculation.

A simple heuristic for practical computations.

When choosing between equivalent algebraic forms, prefer the one without a small denominator. In the φ-parameterization this means: if |cos φ| ≥ |sin φ|, it is convenient to rewrite the expression in terms of cos φ; otherwise, use sin φ. This rule is fully equivalent to the original formulas yet avoids division by small numbers and thus improves numerical stability. Such techniques are standard in numerical analysis; here they are stated compactly for the φ-language, as motivated by the φ-geometry.

5. Quantum Mechanics (De Broglie: Phase/Group)

Standard relations.

$$E = \hbar\omega \text{ (5.1)}; \quad p = \hbar k \text{ (5.2)}.$$

For a free relativistic particle: $E^2 = (pc)^2 + (mc^2)^2$ (5.3).

Phase and group velocities: $v_{ph} = \frac{\omega}{k}, v_g = \frac{d\omega}{dk}$ (5.4).

for a free particle: $v_g = v$ and $v_{ph} v_g = c^2$ (5.5).

φ-representation (see Fig. 8).

Let $v \equiv v_g = \frac{d\omega}{dk} = c \cdot \sin\varphi$ (5.6)

Then $v_{ph} = \frac{\omega}{k} = \frac{c}{\sin\varphi} = \frac{c^2}{v}$, $\gamma = \sec\varphi$ (5.7)

$$\frac{pc}{E} = sin\ \varphi , \qquad \frac{mc^2}{E} = cos\ \varphi \text{ (5.8)}$$

Equivalent to: $k = \frac{p}{\hbar} = \frac{E}{\hbar c}\sin\varphi , \omega = \frac{E}{\hbar}$ (5.9)

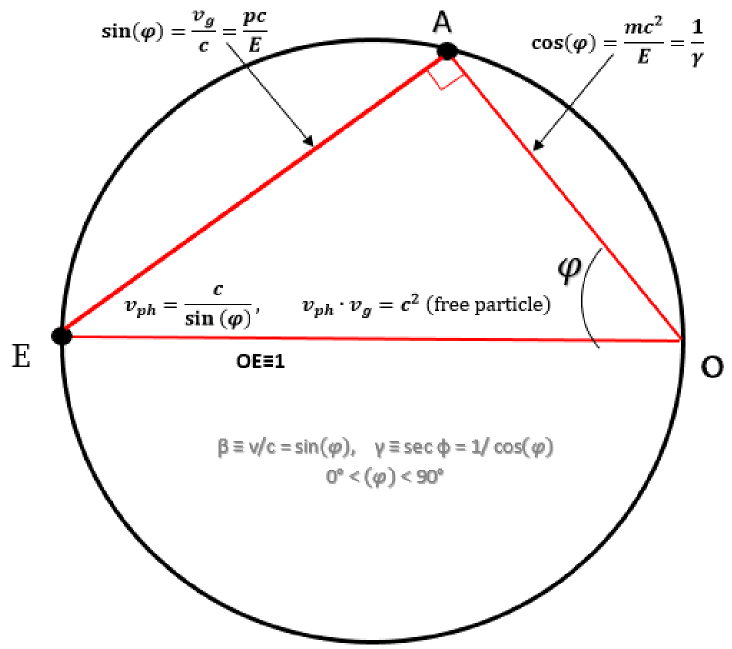

Figure 8— *De Broglie–φ (base). sin φ* $= \frac{v_g}{c}$ *, cos φ* $= \frac{mc^2}{E}$*,* $v_{ph} = \frac{c}{\sin\varphi}$*, so that* $v_{ph} \cdot v_g = c^2$ *(free particle)*

Visual reading.

On the unit circle in Fig. 9, point A(φ1):

the leg AE represents the "dynamic" projection (momentum / group velocity):

$$\sin\varphi = \frac{v_g}{c} = \frac{pc}{E} \quad (5.10),$$

While the leg OA represents the "rest" projection:

$$\cos\varphi = \frac{mc^2}{E} = \frac{1}{\gamma} \quad (5.11).$$

The equality $v_{ph}v_g = c^2$ is read off instantly: $\left(\frac{c}{\sin\varphi}\right) \cdot (c\sin\varphi) = c^2$ (5.12)

Monotonicity of velocities (free particle, vacuum).

As the angle φ∈(0°,90°) increases:

$$v_g = c \cdot sin\varphi$$

grows monotonically and approaches c from below, while $v_{ph} = \frac{c}{\sin\varphi}$ decreases monotonically and approaches c from above; the product (5.5) $v_{ph}v_g = c^2$ remains preserved.

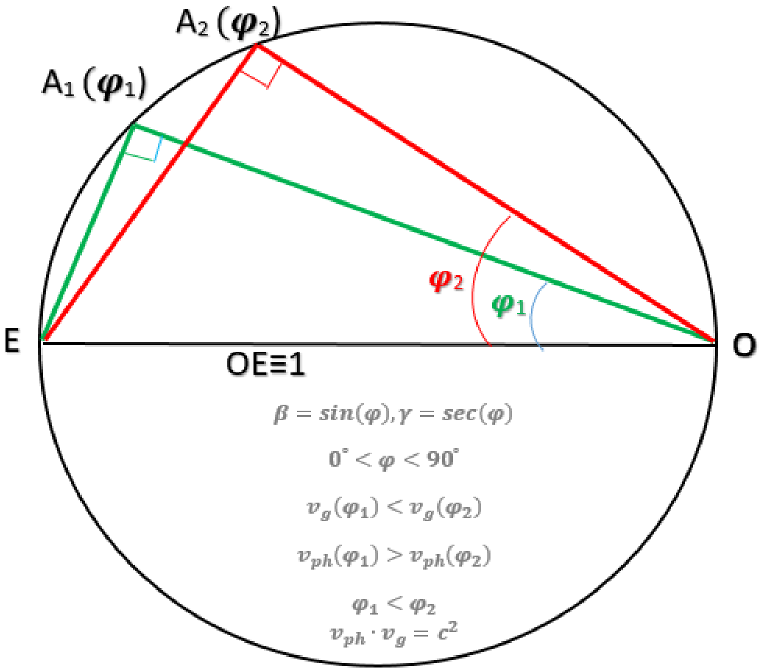

Figure 9 — *De Broglie–φ (base). For* $\varphi_1 < \varphi_2$*:* $v_g(\varphi_1) < v_g(\varphi_2)$, $v_{ph}(\varphi_1) > v_{ph}(\varphi_2)$*, and* $v_{ph} \cdot v_g = c^2$*(free particle).*

Connection with $cos^2\varphi + sin^2\varphi = 1$.

On the unit circle take $\sin\varphi = \frac{pc}{E}$, $\cos\varphi = \frac{mc^2}{E}$

Then

$$cos^2\varphi + sin^2\varphi = 1 \Leftrightarrow \left(\frac{mc^2}{E}\right)^2 + \left(\frac{pc}{E}\right)^2 = 1 \Leftrightarrow E^2 = (pc)^2 + (mc^2)^2 (5.13)$$

Together with (5.6)–(5.7) this yields $v_{ph}v_g = c^2$

Quick numerical check (optional).

Take any angle φ (for example, 10°, 30°, 60°, 85°) and substitute:

$$E = \frac{mc^2}{\cos\varphi}, pc = E\sin\varphi, v_{ph} = \frac{c}{\sin\varphi}, v_g = c \cdot \sin\varphi$$

Verify:

$$E^2 - (pc)^2 - (mc^2)^2 \approx 0 \; and \; (v_{ph} / c)(v_g / c) \approx 1$$

Mass examples: m_ec^2=0.51099895MeV (electron), m_pc^2≈938.272 (proton) [5].

6. Atomic illustration (order-of-magnitude estimate)

Standard relations.

For hydrogen-like levels (nonrelativistic limit)

$$\frac{v}{c} \approx \frac{Z_\alpha}{n} \quad (6.1)$$

(see quantum mechanics textbooks [10]; α — according to CODATA [6])

φ-representation.

Let

$$\sin(\varphi_{atom}) \approx \frac{Z_\alpha}{n} \, (6.2).$$

Then

$$cos(\varphi_{atom}) \approx \sqrt{1-\left(\frac{Z_\alpha}{n}\right)^2}\,, \qquad \gamma_{atom} \approx \frac{1}{\sqrt{1-\left(\frac{Z_\alpha}{n}\right)^2}} = sec(\varphi_{atom}). \, (6.3)$$

Visual reading.

A point on the circle with angle φ_{atom} shows the "degree of relativity" of the electron:

In the hydrogen-like estimate, v/c≈Zα/n.

Introducing the angle φ_{atom} from $\sin(\varphi_{atom}) \approx \frac{Z_\alpha}{n}$, we conclude: at fixed α, increasing Z or decreasing n leads to a growth of φ_{atom}; consequently, $v/c = \sin(\varphi_{atom})$ increases, and the Lorentz factor $\gamma = se\,c(\varphi_{atom}) = 1\,/\,cos(\varphi_{atom})$ also grows monotonically (see Fig. 10).

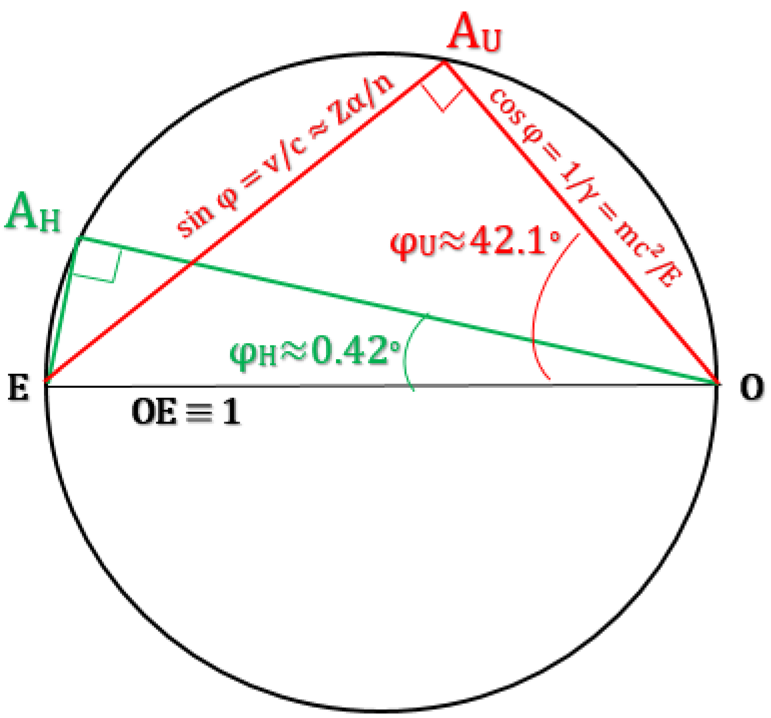

Figure 10 — *Atom–φ (оценка). sin φ ≈ Zα/n (H: φ≈0.42°, U^91+: φ≈42.1°). β=sin φ, γ=sec φ. Diagram not to scale: φ_H is enlarged for clarity.*

Remark (multi-electron case).

The same φ-geometry can be applied to multi-electron atoms by assigning each electron/subshell its own angle φ_i from the order-of-magnitude estimate

$$\beta_i \equiv \frac{v_i}{c} \approx \frac{Z_{eff,i}\alpha}{n_i}, \sin(\varphi_i) = \beta_i \ (6.4)$$

Then, on a single circle, one can construct a "multi-electron profile" as a set of triangles O−Ai−E with different φ_i, where larger φ_i visually mark the more "relativistic" shells. We do not fix a specific screening scheme Z_{eff} and do not draw such a profile here; if needed, it can be chosen according to the problem (for example, using Slater's rules) and constructed independently.

Scope.

This is an order-of-magnitude estimate (Bohr model / nonrelativistic limit). For large Z and real multi-electron atoms, relativistic corrections are required (Dirac, fine structure, finite nuclear size, screening).

7. String sketch (T-duality — visual hint)

Standard relations.

For a closed string on a circle of radius R (dimension: α′ — length2):

$$p_n = \frac{n}{R}, \qquad w_m = \frac{mR}{\alpha'}, \qquad n, m \in Z \ (7.1)$$

left/right momenta:

$$P_L = \frac{n}{R} + \frac{mR}{\alpha'}\ , p_R = \frac{n}{R} - \frac{mR}{\alpha'}\ (7.2).$$

T-duality:
$R \leftrightarrow \alpha'/R\ , n \leftrightarrow m$ (accordingly $P_L \leftrightarrow P_L\ , p_R \leftrightarrow -p_R$) [11–13]. (7.3)

φ-hint (visual).

Normalize the pair of modes to the unit circle and set the dimensionless vector

$$\vec{v}(n,m) = \frac{(n,m)}{\sqrt{n^2+m^2}} = (\cos\tilde{\varphi}\ , \sin\tilde{\varphi})\ , \cos\tilde{\varphi} = \frac{n}{\sqrt{n^2+m^2}}\ , \sin\tilde{\varphi} = \frac{m}{\sqrt{n^2+m^2}}\ (7.4).$$

Then the exchange n↔m (together with $R \leftrightarrow \alpha'/R$) is read as φ↔90∘−φ — that is, a complementary rotation on the circle (interchanging cos↔sin).

Scope.

This is an illustrative sketch of mode symmetry/exchange; we do not draw any conclusions about string dynamics here.

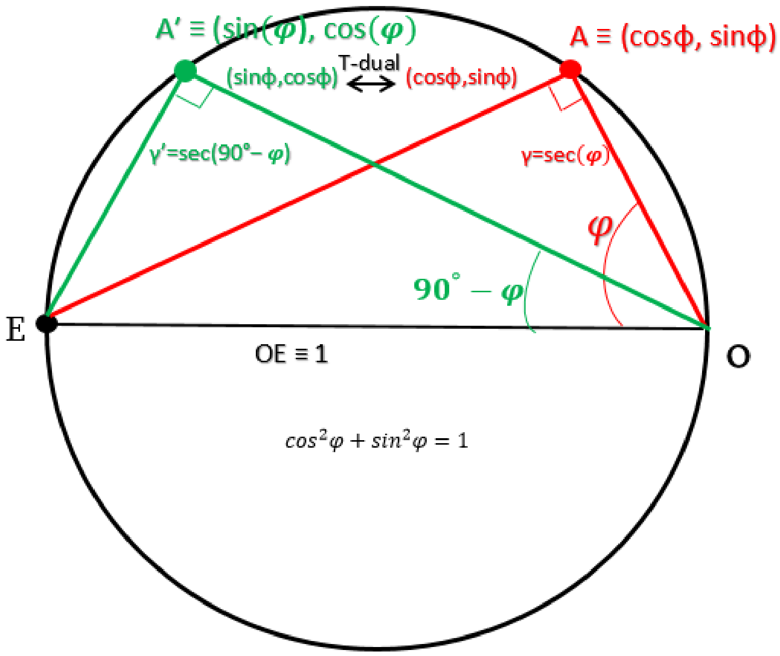

Figure 11 — *T-duality on the φ-circle. Point A=(cos φ, sinφ)=(1/γ,β) and its complementary point A′=(sin φ, cos φ=(β,1/γ) are related by the substitution φ→90°−φ. This visually encodes T-duality R↔α′/R, n↔m (momentum ↔ winding modes). The scaling factor γ=sec φ in this diagram is the geometric scale of the φ-picture, not the Lorentz factor of special relativity.*

Note to Fig. 11

Normalization ($p_n R$, $w_m R_T$) ∝ (n,m) (where R_T=α′/R) makes the diagram dimensionless; under T-duality R↔R_T, the components simply swap n↔m, which corresponds to the rotation φ ↔ 90∘−φ.

Conclusion.

The φ-circle provides a compact picture where the exchange cos↔sin clearly reflects the T-duality operation (momentum ↔ winding). We use this only as a pedagogical visualization layer on top of the standard formulation: the physical content of string theory, its spectra and dynamics, remain unchanged.

8. Electromagnetism

8.1. Standard textbook relations.

Lorentz boost along $\hat{v}$ (for transverse components) [14,15]:

$$\boldsymbol{E}'_{\perp} = \gamma(\boldsymbol{E}_{\perp} + \boldsymbol{v} \times \boldsymbol{B}) \ (8.1)$$

$$\boldsymbol{B}'_{\perp} = \gamma\left(\boldsymbol{B}_{\perp} - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}\right) \ (8.2)$$

Parallel components remain unchanged: $E'_{\parallel} = E_{\parallel}, B'_{\parallel} = B_{\parallel}$.

The invariants are preserved: $I_1 = E^2 - c^2B^2, \quad I_2 = E \cdot B$ (8.3).

8.2. φ-representation (circular reparameterization).

Let β=sin φ, γ=sec φ , $v = csin\,\varphi \cdot \hat{v} \ \Rightarrow$

$$\boldsymbol{E}'_{\perp} = \sec\varphi\,(\boldsymbol{E}_{\perp} + csin\,\varphi \cdot \hat{\boldsymbol{v}} \times \boldsymbol{B}) \ (8.4)$$

$$\boldsymbol{B}'_{\perp} = \sec\varphi\left(\boldsymbol{B}_{\perp} - \frac{sin\,\varphi}{c} \cdot \hat{\boldsymbol{v}} \times \boldsymbol{E}\right) (8.5)$$

Visual reading.

The transverse components scales as sec φ=γ; as φ→90°, notation highlights the growth of γ.

8.3. Connection between relativistic dynamics and electrodynamics.

The same angular parameter φ (with β=sin φ and γ=sec φ) links relativistic dynamics and electrodynamics. As φ increases (that is, as v→c), the transverse components of the fields in the boosted frame scale with γ. The cross terms **v** × **B** and **v** × **E** / c^2 describe the Lorentz mixing of the electric and magnetic components.

Scope.

This is a reparameterization of the standard formulas; Maxwell's equations and the invariants I_1, I_2 remain unchanged.

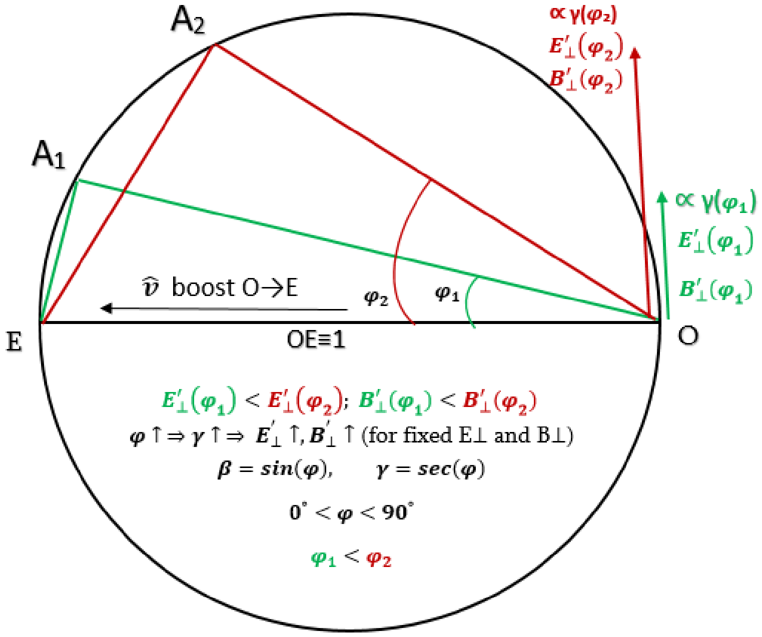

Figure 12 — *EM–φ.* *Boost O→E. With $E_\perp$, $B_\perp$ held fixed, γ=sec φ increases with φ ⇒ $E'_\perp$, $B'_\perp$ increase (shown for $\varphi_1<\varphi_2$); β=sin φ*

8.4. Figure description.

EM–φ (extended) Fig. 12. The circle diagram parameterizes a Lorentz boost by the angle φ at O, with β=sin φ and γ=sec φ (the chord OE is normalized to 1). The boost is along O→E. Two arc positions are shown: A_1 (φ_1) in green and A_2 (φ_2) in red, with $0°<\varphi_1<\varphi_2<90°$. At O, the vertical scale bars illustrate the monotonic scaling: $E'_\perp(\varphi) \propto \gamma(\varphi)$; the out-of-plane dots indicate the same for $B'_\perp(\varphi)$. With $E_\perp$ and $B_\perp$ held fixed, the Lorentz transforms

$$\boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v} \times \boldsymbol{B}) \ (8.1)$$

$$\boldsymbol{B}'_\perp = \gamma(\boldsymbol{B}_\perp - \boldsymbol{v} \times \boldsymbol{E} \,/\, c^2) \ (8.2)$$

Imply $E'_\perp(\varphi_1) < E'_\perp(\varphi_2)$ and $B'_\perp(\varphi_1) < B'_\perp(\varphi_2)$. Parallel components are unchanged: $E'_\parallel = E_\parallel, B'_\parallel = B_\parallel$.

9. Full φ-map and two branches

Idea.

We consider the entire unit circle as a parametric space and explicitly mark two branches (sheets). This makes the symmetries and limiting regimes visually transparent. In the full picture, points O and E are equivalent rest states (β=0) on different branches; physical quantities depend on |cos φ|, so the observable physics is the same (see Fig. 13).

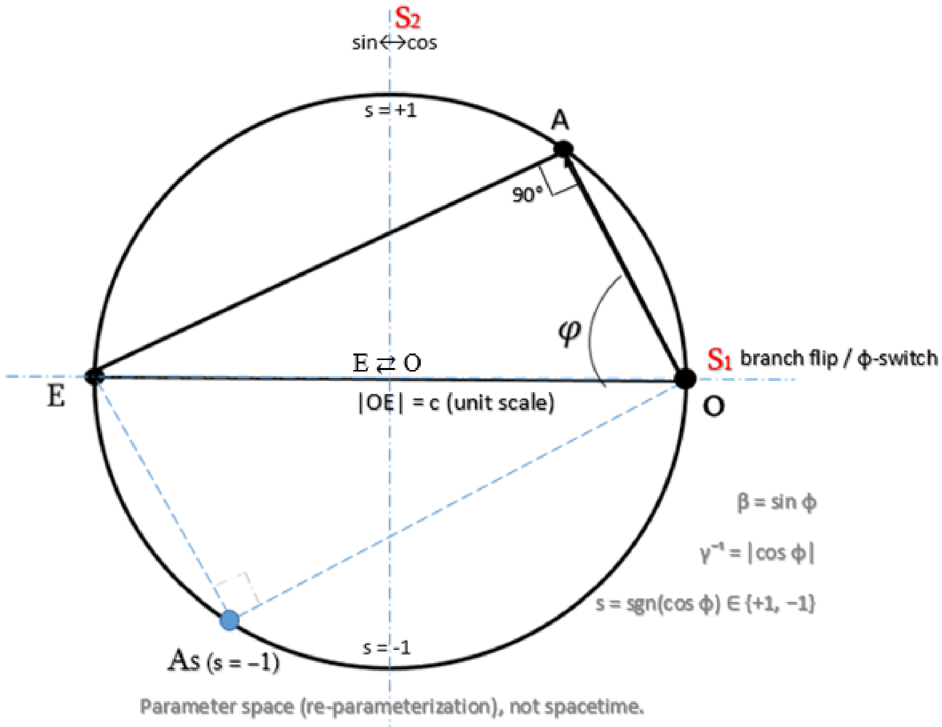

Figure 13 — Full φ-map. *Parameterization:* $\beta = sin\,\varphi\,, \gamma^{-1} = |cos\,\varphi|, s = sgn(cos\,\varphi)$. *Diameter OE: S1 (φ-switch). Perpendicular through the center* — axis *S2. Point A lies on the branch s=+1, mirror As* on *s=−1. O≡E at rest (β=0).*

9.1. Parameterization (without changing the physics)

In this analysis, the circle is used only as a reparameterization, not as a spacetime metric:

$$\beta = \sin\varphi\,, \gamma^{-1} = |\cos\varphi|, s = sgn(\cos\varphi) \in \{+1.-1\}(9.1).$$

The upper semicircle s=+1 coincides with the "working" map of the previous sections; the lower one s=−1 is its conjugate sheet. Points O and E represent the same rest state (β=0) on different branches.

9.2. Two basic symmetries

- S_1 (branch flip / φ-switch) — reflection about the diameter OE: (sin φ,cos φ)⟶(sin φ,−cos φ). |cos φ | is unchanged; only the branch label changes s ⟶ −s.
- S_2 (exchange of contributions) — reflection across the perpendicular to OE (the line sin φ =cos φ): (sin φ, cos φ)⟶(cos φ, sin φ).
  In terms of energies, this corresponds to the exchange pc ↔ mc^2 (see §3).

Example.

For instance, φ=30°: β=sinφ=0.5, 1/γ=cosφ≈0.866.

After S1: β=0.5, 1/γ=0.866, s⟶−1 (only the branch changes).

After S2: β ⟶ cos φ≈0.866, 1/γ ⟶ sin φ=0.5 — the exchange pc↔mc^2.

9.3. How to read the figure

Point A on the branch $s=+1$ defines the angle φ and the velocity $\beta=\sin\varphi$ (the right angle ∠EAO=90° — Thales' theorem). The dashed As is the mirror position on $s=-1$. The diameter OE implements S_1; the perpendicular through the center implements S_2.

9.4. Scope and cautions

The full φ-map is a descriptive language (re-parameterization). We do not alter the equations of SR/GR and do not introduce a new metric; standard results outside limiting regimes are reproduced. The strict dynamics of the "two-sheet" scheme (including possible transitions between branches as $\varphi\to90^\circ$) lies beyond the scope of this work and is noted as a separate direction for future analysis.

10. Universality of φ-geometry

The same circular parameterization remains a neutral container. With an explicit legend (what is placed into sin φ and cos φ), it can be tailored to the problem at hand. Examples: gravity, relativistic kinematics, quantum and atomic physics; energy balances (virial estimate, Eddington parameter Γ); thermal regimes (kT vs. mc^2); cosmological fractions (Ω_m,Ω_Λ).

Conclusion

We have deliberately focused on pedagogical presentation and computational practice. Further implications are left for independent evaluation and future work.

Summary. φ-parameterization is a visual re-expression of standard relativistic formulas through a single angle on the circle. It links kinematics, the energy–momentum invariant, and simple gravitational/atomic estimates into one picture while remaining equivalent to the classical rapidity formalism. The sin↔cos switch provides numerical stability in limiting regimes. Short numerical checks (particles, GPS) confirm consistency.

Advantages

- Unified language. A single angle provides a common scale of effects; everything is read from one diagram.
- Boundaries are visible. Rest and "near-light" states are just different points on the circle.
- Stability. The complementary angle removes small denominators.
- Quick cross-checks. Convenient for verifying independent data (particles, GPS).
- Compatibility. Fully equivalent to SR; the physics remains unchanged.
- Flexibility. Hypotenuse can be normalized to c or 1; easy to "attach" EM, gravity, and atomic estimates.
- Teaching. Less algebra, more intuition; suitable for lectures and quick estimates.

Limitations

- Not a metric. The circle is only a reparameterization; it does not replace GR.
- Two sheets. On the full map one must account for S1 (φ-switch) and the label $s=\mathrm{sgn}(\cos\varphi)$; in the introductory sections we worked on a single sheet.

- Complex regimes. Strong fields/rotation, non-collinear boosts, and subtle EM effects require full calculation.
- Legend required. One must always state explicitly what is placed into sin φ and cos φ to avoid confusion.

Looking ahead. The approach is open to further refinements and extensions (alternative visualizations, robust reformulations for other regimes); constructive feedback is welcome.

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