ON THE LOW ENERGY ADDITION CHAIN EQUATION

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Abstract. We develop a criterion for an addition chain to have low energy and pose a related classification problem.

1. Introduction

Let $\mathcal{C}_n : s_0 = 1 < s_1 < \cdots < s_h = n$ be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(\sigma(i) \geq \tau(i), 1 \leq i \leq h)$. In [1], we define the energy of the addition chain $\mathcal{C}_n$ as

$$\mathbb{E}(\mathcal{C}_n) := \sum_{i=1}^{h} s_{\tau(i)}.$$

We say that the energy of the addition chain is *low* if $\mathbb{E}_n = n - 1$. On the other hand, we say that the addition chain has *excess* energy if $\mathbb{E}_n \geq n$. In this paper, we introduce a criterion for the energy of an addition to be low. We pose the following classification problem related to low-energy addition chains:

Problem 1.1. Let $n \geq 3$. Classify all possible addition chains

$$\mathcal{C}_n : s_0 = 1 < s_1 < \cdots < s_h = n$$

with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(\sigma(i) \geq \tau(i),\ 1 \leq i \leq h)$ and

$$m_t := \#\{i \in \{1, \ldots, h\}\ :\ \sigma(i) = t, 0 \leq t \leq h-1\}$$

satisfying

$$\sum_{t=1}^{h-1} (1 - m_t) s_t = m_0 - 1.$$

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2. The low energy chain equation

In this section, we develop an identity for the energy of a chain in terms of the target and introduce an equation whose solubility is connected with chains with low energy.

Proposition 2.1 (The low energy chain equation)**.** *Let* $\mathcal{C}_n : s_0 = 1 < s_1 < \cdots < s_h = n$ *be an addition chain leading to* n *with* $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(\sigma(i) \geq \tau(i), 1 \leq i \leq h)$*. Define the multiplicity*

$$m_t := \#\{i \in \{1, \ldots, h\} \ : \ \sigma(i) = t, 0 \leq t \leq h-1\}$$

which may be zero for some t*. Then*

$$\mathbb{E}(\mathcal{C}_n) = n - m_0 + \sum_{t=1}^{h-1}(1-m_t)s_t.$$

Furthermore, the energy $\mathbb{E}(\mathcal{C}_n) = n-1$ *if and only if*

$$\sum_{t=1}^{h-1}(1-m_t)s_t = m_0 - 1.$$

Proof. Set

$$m_t := \#\{i \in \{1, \ldots, h\} \ : \ \sigma(i) = t, 0 \leq t \leq h-1\}$$

which may be zero for some t. This is the number of times the index $\sigma(i)$ appears as indices for each term s_t, where $m_t \geq 0$. We write

$$\begin{aligned}\mathbb{E}(\mathcal{C}_n) :&= \sum_{i=1}^{h} s_{\tau(i)}\\ &= \sum_{i=1}^{h} s_i - \sum_{i=1}^{h} s_{\sigma(i)}\\ &= \sum_{i=1}^{h} s_t - \sum_{i=0}^{h-1} m_t s_t, \quad (\sigma(i) \in \{1, \ldots, i-1\})\\ &= s_h - m_0 + \sum_{t=1}^{h-1}(1-m_t)s_t\\ &= n - m_0 + \sum_{t=1}^{h-1}(1-m_t)s_t.\end{aligned}$$

Using this identity, we deduce $\mathbb{E}(\mathcal{C}_n) = n-1$ if and only if

$$\sum_{t=1}^{h-1}(1-m_t)s_t = m_0 - 1.$$

□

3. Consequences and related problems

The following observations can be made for addition chain satisfying the low energy chain equation.

- We observe that if each multiplicity $m_t = 1$ for all $t = 0, \dots, h-1$ which means that the largest summand used to generate each subsequent term must be used exactly once, then the energy $\mathbb{E}(n) = n - 1$. In this case, the addition chain is necessarily Brauer.

- If the multiplicity $m_t = 0$ for some indices (never used as a larger summand) then the sum
$$\sum_{t=1}^{h-1}(1 - m_t)s_t$$
contains some positive s_t terms. Therefore, for the chain energy to be $\mathbb{E}(\mathcal{C}_n) = n-1$ (low energy), those positive s_t must be offset by some negative contributions of $(1 - m_t)s_t$ with $m_t \geq 2$ or with $m > 1$.

Hence, we pose the following related problem.

Problem 3.1 (Low energy chain classification problem). Let $n \geq 3$. Classify all possible addition chains
$$\mathcal{C}_n : s_0 = 1 < s_1 < \cdots < s_h = n$$
with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(\sigma(i) \geq \tau(i),\ 1 \leq i \leq h)$ and
$$m_t := \#\{i \in \{1, \dots, h\} \ : \ \sigma(i) = t, 0 \leq t \leq h-1\}$$
satisfying
$$\sum_{t=1}^{h-1}(1 - m_t)s_t = m_0 - 1.$$

References

1. T.Agama, *The energy method in addition chains*, Researchgate, 2025.