An Alternative Derivation of Length Contraction Using Einstein's Own Thought Experiment

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Abstract

By now, every high school Physics student has been familiar with the famous Special Theory of Relativity proposed by Einstein. The foundation of Special Theory of Relativity rests on two postulates: (1) The laws of Physics are the same in all inertial frames of reference and (2) The speed of light in a vacuum $c = 3 \times 10^8$ m/s is the same in all inertial frames of reference. The consequences of such postulates are the famous Time Dilation and Length Contraction formulas. Historically, in his paper *On the Electrodynamics of Moving Bodies* (1905), he derived Length Contraction through Lorentz Transformation [1]. However, this article explores another way Length Contraction could have been discovered: through changing one bit of Einstein's thought experiment, using the mathematical tool of infinite sum.

1. Introduction

The Time Dilation and Length Contraction are stated in the following form:

$$\Delta t = \Delta t_0 \gamma \qquad (1)$$

$$L = \frac{L_0}{\gamma} \qquad (2)$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Δt_0 is time measured in the inertial frame of reference (FOR) in which the clock used to measure time is at rest relative to the observer.

Δt is time measured in the FOR in which the clock used to measure time is moving at a constant speed v relative to the observer.

L_0 is the length measured in the FOR in which the two endpoints of the measured length are at rest relative to the observer.

L is the length measured in the FOR in which the two endpoints of the measured length are not at rest relative to the observer.

2. Time Dilation derivation

To understand the derivation of Length Contraction, we begin by recalling Einstein's famous thought experiment to derive Time Dilation.

The second postulate states that the speed of light $c = 3 \times 10^8$ m/s is the same in all FORs. To explore what consequence this postulate could pose to the time perceived by observers in different FORs, let us imagine the following set-up:

In observer A's FOR, he is on the ground observing another observer B on the train which moves with the constant speed v. On the train, there is a device used to measure time: A photon clock. The clock contains a photon particle and two opposite mirrors separated by a distance L.
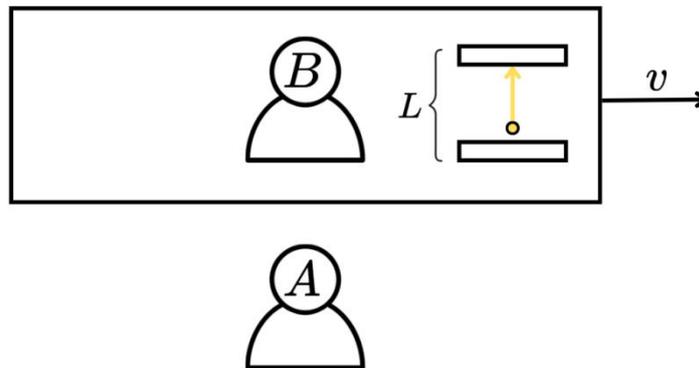

Figure 1. Experimental set-up for Time Dilation derivation

Both agree that a "tick" is the time between the photon moving from mirror 1 to mirror 2 and from mirror 2 back to mirror 1.

From B's FOR, he measures a "tick", denoted Δt_0, as:
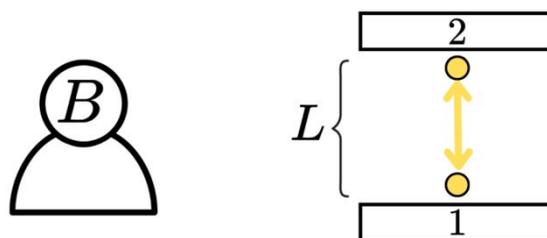
$$\Delta t_0 = \frac{2L}{c} \tag{3}$$

Figure 2. Light's trajectory in B's FOR

From A's FOR, light travels in the following way:
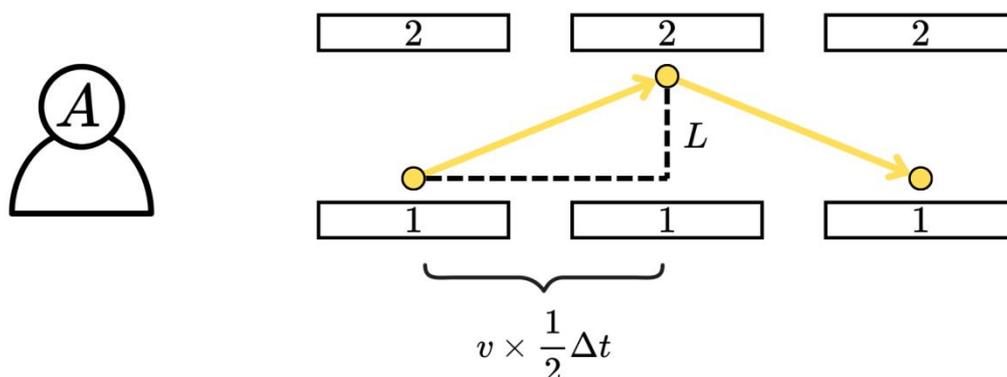

$$v \times \frac{1}{2}\Delta t$$

Figure 3. Light's trajectory in A's FOR

A has also agreed that his full "tick" is the time between the photon moving from mirror 1 to mirror 2 and backwards, denoted Δt. Thus, the time light moves from

mirror 1 to mirror 2, in this set-up, is only half of A's "tick", or $\frac{1}{2}\Delta t$, and the distance mirror 1 covered in that time is $v \times \frac{1}{2}\Delta t$.

Using Pythagorean theorem, the total distance light travels is:

$$c \times \Delta t = 2 \times \sqrt{L^2 + (v \times \frac{1}{2}\Delta t)^2}$$

$$\Rightarrow c^2 \times \Delta t^2 = 4 \times \left[L^2 + (v \times \frac{1}{2}\Delta t)^2\right]$$

$$\Rightarrow c^2 \times \Delta t^2 = 4L^2 + v^2\Delta t^2$$

$$\Rightarrow \Delta t^2 = \left(\frac{2L}{c}\right)^2 + \frac{v^2}{c^2}\Delta t^2 \tag{4}$$

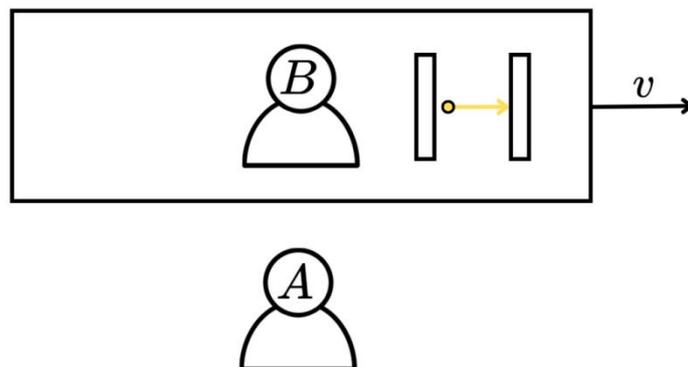Substitute (3) in (4) we have:

$$\Delta t^2 = \Delta t_0{}^2 + \frac{v^2}{c^2}\Delta t^2$$

$$\Rightarrow \Delta t^2 (1 - \frac{v^2}{c^2}) = \Delta t_0{}^2$$

$$\Rightarrow \Delta t^2 = \frac{\Delta t_0{}^2}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \Delta t_0 \gamma$$

This completes the derivation for (1).

3. Length Contraction derivation

To derive Length Contraction formula, we now orient the photon clock longitudinally with respect to the train's motion in the same set-up in *Figure 1*.

Figure 4. Experimental set-up for Length Contraction derivation

Let the length measured between two mirrors in B's FOR and A's FOR be L_0 and L, respectively.

From B's FOR, he measures a "tick" as:

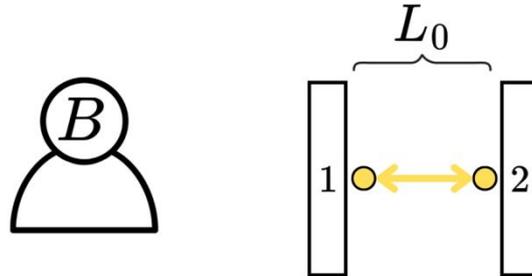$$\Delta t_0 = \frac{2L_0}{c} \tag{5}$$

Figure 5. Light's trajectory in B's FOR

From A's FOR, a "tick" is measured separately: The time elapsed when the photon moves from mirror 1 to mirror 2 and vice versa.

In the first period Δt_1 when the light moves from mirror 1 to mirror 2, the light travels in the following way:
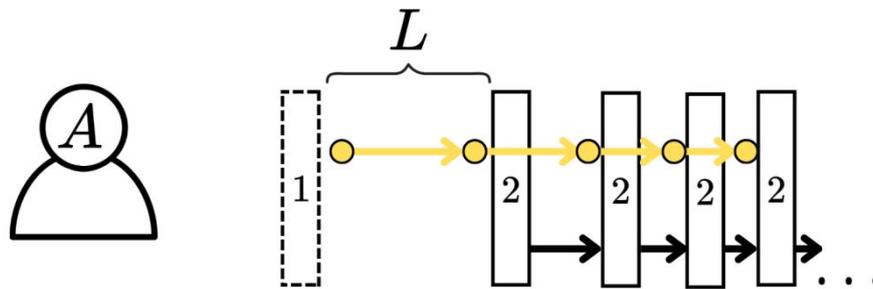

Figure 6. Light's trajectory from mirror 1 to mirror 2 in A's FOR

While light travels a distance of L within time $T_0 = \frac{L}{c}$, mirror 2 moves a distance of $v \times \frac{L}{c}$. Consequently, the photon must cover this additional displacement, which requires an extra interval of time $T_1 = v \times \frac{L}{c^2}$, during which mirror 2 shifts by $v^2 \times \frac{L}{c^2}$. Light now must cover $v^2 \times \frac{L}{c^2}$, which requires a time $T_2 = v^2 \times \frac{L}{c^3}$, during which mirror traverses again by $v^3 \times \frac{L}{c^3}$, and so on ad infinitum. Thus, the total time light travels Δt_1 is the sum of an infinite geometric series of discrete time steps:

$$\Delta t_1 = T_0 + T_1 + T_2 + T_3 + \ldots = \frac{L}{c} + v \times \frac{L}{c^2} + v^2 \times \frac{L}{c^3} + v^3 \times \frac{L}{c^4} + \ldots$$

$$\Rightarrow \Delta t_1 = L \times (\frac{1}{c} + \frac{v}{c^2} + \frac{v^2}{c^3} + \frac{v^3}{c^4} + \ldots) = L \times \lim_{N \to \infty} \sum_{i=0}^{N} \frac{v^i}{c^{i+1}} \tag{6}$$

In contrast to the first period Δt_1, the second period Δt_2 involves the light and mirror 2 moving towards each other, which allows for a simple kinematic equation without using infinite series.
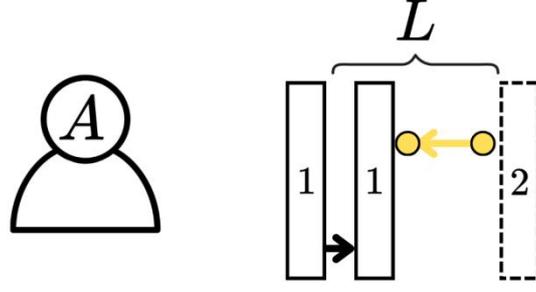
Figure 7. Light's trajectory from mirror 2 to mirror 1 in A's FOR

The light meets mirror 1 when the sum of the distances both travel equals the initial separation L between two mirror 1 and 2. Since they are moving towards each other, we have:

$$v\Delta t_2 + c\Delta t_2 = L$$

$$\Rightarrow \Delta t_2(v + c) = L$$

$$\Rightarrow \Delta t_2 = \frac{L}{v + c} \tag{7}$$

Combine (6) and (7), we have the full "tick" in A's FOR:

$$\Delta t = \Delta t_1 + \Delta t_2 = L \times \lim_{N \to \infty} \sum_{i=0}^{N} \frac{v^i}{c^{i+1}} + \frac{L}{v + c} \tag{8}$$

For ease of calculation, we can express the speed v as some proportion to the speed of light:

$$v = xc \ (0 \le x < 1)$$

Then (8) can be rewritten as:

$$\Delta t = L \times \lim_{N \to \infty} \sum_{i=0}^{N} \frac{(xc)^i}{c^{i+1}} + \frac{L}{xc + c}$$

$$\Rightarrow \Delta t = \frac{L}{c} \times \lim_{N \to \infty} \sum_{i=0}^{N} x^i + \frac{L}{c(1 + x)}$$

$$\Rightarrow \Delta t = \frac{L}{c} \times \left(\lim_{N \to \infty} \sum_{i=0}^{N} x^i + \frac{1}{1 + x} \right) \tag{9}$$

For $0 \le x < 1$, the sum $\sum_{i=0}^{N} x^i$ would converge to $\frac{1}{1-x}$ as $N \to \infty$. Thus, we have:

$$\lim_{N \to \infty} \sum_{i=0}^{N} x^i = \frac{1}{1 - x} \ (0 \le x < 1) \tag{10}$$

Substitute (10) in (9), we have:

$$\Delta t = \frac{L}{c} \times \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right)$$

$$\Rightarrow \Delta t = \frac{2L}{c} \times \left(\frac{1}{1 - x^2}\right) \tag{11}$$

Substitute (1) for Δt in (11):

$$\Delta t_0 \gamma = \frac{2L}{c} \times \left(\frac{1}{1 - x^2}\right) \tag{12}$$

Note that we have set speed v as some proportion to the speed of light c, we can rewrite γ as follows:

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \dfrac{(xc)^2}{c^2}}} = \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \gamma^2 = \frac{1}{1 - x^2} \tag{13}$$

Substitute (13) for $\frac{1}{1-x^2}$ and (5) for Δt_0 in (12), we have:

$$\frac{2L_0}{c}\gamma = \frac{2L}{c}\gamma^2 \Rightarrow \frac{L_0}{\gamma} = L$$

This completes the derivation for (2).

References

[1] Einstein, A. (2005). Zur Elektrodynamik bewegter Körper [AdP 17, 891 (1905)]. Annalen Der Physik, 14(S1), 194–224. https://doi.org/10.1002/andp.200590006