Few Commentary on $a^3+b^3 = 2(2^5 - c^3)$

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with the Expounding of PayDie(2024)

Abstract
Diophantine equation of the form $a^3+b^3 = 2(2^5 – c^3)$ relates to the quadratic equation $1 + 4n + 4n^2$ which further relates to $1x^3 + 4y^3 + 4z^3 = 512$ whose parametric solution for x is exactly is the twice of the square of odd number > 1, i.e. $2(2n+1)^2$. Considering φ(sum of the odd divisors of $2(2n+1)^2$) is exactly equal to φ(sum of the even divisors of $2(2n+1)^2$), some interesting properties involving Piltz functions and Jordan totients were conjectured.

1. Divisor Summation and Totients

Define order-k Piltz function $Ɂ_k(n)$ as the summation that is being carried on the k-th powered factors of multiplicity 3 of n. Consider $\sigma_k(n)$ as order-k divisor summation function, it sums along k-th powered factors of multiplicity 2 of n, then $Ɂ_k$ is obtained after performing Dirichlet multiplication with $\mathbb{1}$ against σ_k.

Define order-k Jordan totient $J_k(n)$ as Dirichlet multiplication between Möbius μ and order-k divisor summation function σ_k. Consequently $J_1(n) = \varphi(n)$, with φ being the Euler totient function, since φ is the result of the Dirichlet multiplication of μ with σ_1. Applying yet another Möbius μ as Dirichlet multiplier against J_k, one obtains $\mu_k(n)$, i.e. the order-k Möbius function.

Proposition:
n $Ɂ_2(n) \equiv 1 \pmod 3$ for all cube n with $27\nmid n$ and $8\nmid n$.
n $Ɂ_2(n) \equiv 2 \pmod 3$ for all cube n with $27\nmid n$ and $8\mid n$.
n $Ɂ_2(n) \equiv 0 \pmod 3$ otherwise.
Proof, since both functions $\sigma_2(n)$ and $Ɂ_2(n)$ are multiplicative (whenever coprime), thus for each factor p with exponent a, $Ɂ_2(p^a)$ is

$$\sum_{j=0}^{a} \sigma_2(p^j) = \sum_{j=0}^{a} \frac{p^{2(j+1)} - 1}{p^2 - 1}$$

which leads to $Ɂ_2(p^a)$ being equivalent to

$$\frac{1}{p^2-1}\left(\sum_{j=0}^{a} p^{2(j+1)} - (a+1)\right) = \frac{p^2\left(p^{2a+2}-1\right)}{(p^2-1)^2} - \frac{a+1}{p^2-1}$$

That is the residue of n $Ɂ_2(n)$ modulo 3 being the product of the residues for each prime factor. If $3\mid n$ then automatically n $Ɂ_2(n) \equiv 0 \pmod 3$. Now consider $3\nmid n$, with n being composed of prime factors with their own exponent, n $Ɂ_2(n) \not\equiv 0 \pmod 3$ iff every exponent of such constituent primes $\not\equiv 2 \pmod 3$, however by requiring all exponent to be congruent to 0 or 1 (mod 3) this is equivalently stating that n must be a cube due to each prime exponent being multiple of 3. This automatically covers the criterion to be not divisible by $27 = 3^3$, since if such cube factor does, then n must contain the factor 3, which can't be possible considering this is the $3\nmid n$ case. Consequently, when n is a cube and $3\nmid n$, every exponent from the factors of n must be congruent to 0 (mod 3), leading each exponent (say a) when added by 1 then divided by 2 (due to the a+1 over 2 term, in said $Ƥ_2(p^a)$ equation above) to be indivisible by 3. So far, this shows that $n\,Ƥ_2(n) \not\equiv 0 \pmod 3$ iff n being a cube number and $3 \nmid n$. The remaining works are only matter of parity determination, due to the presence of $(-1)^a$ times 2 for prime factor $p \equiv 2 \pmod 3$, yet for $p \equiv 1 \pmod 3$ the exponent a always contribute as the residue 1 (mod 3). The only thing that can change the parity of the exponent of 2 is whether an odd number of the primes $p \equiv 2 \pmod 3$ able to appear with an odd exponent, but because every prime factors being a cube here (considering the case of $n\,Ƥ_2(n) \not\equiv 0 \pmod 3$), **the only prime that can give "odd parity exponent a" while still being cube number, is the prime 2 itself**. That is, if the entire factors lacked the prime 2, such parity will never flip, and the residue stays at 1 (mod 3), the only case that it able to flip, thereby leading to 2 (mod 3), is when the entire factors contain the constituent 2^{3a} alias 8^a. This concludes the "switch" of divisibility by 8: if the cube number, n, which is not being congruent to 0 (mod 27) is divisible by 8, then $n\,Ƥ_2(n) \equiv 2 \pmod 3$, else such n will yield $n\,Ƥ_2(n) \equiv 1 \pmod 3$.

Proposition:
$p^2\,Ƥ_1(p) = \mu_2(p) + J_2(p) + \sigma_2(p) + \sigma_1(p) + \varphi(p)\,J_2(p)$, for p primes.
Proof, given $Ƥ_1(p) = p+2$, $\mu_2(p) = p^2-2$, $\sigma_1(p) = p+1$, $\sigma_2(p) = p^2+1$, $\varphi(p) = p-1$, $J_2(p) = p^2-1$, it leads to
$p^2\,(p+2) = p^2-2 + p^2-1 + p^2+1 + p+1 + (p-1)(p^2-1)$
expanding both sides and collecting terms, the equations indeed hold.
Let d(n) being the difference when general n is used instead of p
$d(n) = n^2\,Ƥ_1(n) - \mu_2(n) - J_2(n) - \sigma_2(n) - \sigma_1(n) - \varphi(n)\,J_2(n)$

Conjecture:
d(n) is always even unless only at the cases below
For odd n, d(n) is odd iff every factor of n has exponent congruent to 0 (mod 4) or 1 (mod 4).
For even n, d(n) is odd iff n is multiple of 4 and $\omega(n/4) = \Omega(n) - 2$,
with Ω being the sum of exponents from the factors of n and ω being the count of unique prime factors of n. In other words, for even n, the factor 2 must have exponent exactly 2, followed by all other prime factor with each exponent does not exceed 1.

From aforesaid propositions with regard to $n\,Ƥ_2(n)$, the presence of a cube as some factor is required, that is, such residue criterion were determined using $\Omega(n)/\omega(n) \geq 3n$, with n itself being positive integer. However, the case of d(n) is interesting, since for odd n, in order to produce odd d(n), there is no restriction on how large can the prime powers be as long as each of them being exactly 0 (mod 4) or 1 (mod 4), i.e. $\Omega(n)$ does not have to mirror $\omega(n)$ exactly, unlike the propositions of $n\,Ƥ_2(n)$ under the congruence class (mod 3). Yet for even n, in order to produce odd d(n), one being required to remove the factor 4 from n, rendering $n/4 \equiv 1 \pmod 2$, and then $\Omega(n)$ must mirror $\omega(n)$ exactly, just like the propositions of $n\,Ƥ_2(n)$ under the congruence class (mod 3). The situation in here is, d(n) —which constitutes $n^2\,Ƥ_1(n)$ when n is multiple of 4— being able to mimic certain aspects of $n\,Ƥ_2(n)$ that is being constrained to certain finite squared (in $Ƥ_1$) or cubed (in $Ƥ_2$) factors, yet such factors of the very same d(n) be "unbounded" in terms of exponent —when n is simply odd— possibly bears a faint reminiscent of the case of Markov-Hurwitz equation.

Proposition:
$p^3\,\sigma_0(p) = J_2(p) + p\,\sigma_2(p) + \varphi(p)\,J_2(p)$, for p primes.
Proof, given $\sigma_0(p) = 2$, $\sigma_2(p) = p^2+1$, $\varphi(p) = p-1$, $J_2(p) = p^2-1$, it follows that
$p^3\,2 = p^2-1 + p\,(p^2+1) + (p-1)(p^2-1)$
$p^3\,2 = p^2-1 + p^3 + p + 1 - p - p^2 + p^3 = 2p^3$
thus $\mathbf{p^3\,\sigma_0(p) = J_2(p) + p\,\sigma_2(p) + \varphi(p)\,J_2(p)}$

one can also obtain that
$\sigma_1(p) = 1 + (\sigma_2(p) + J_2(p))/2$, for p primes.
Let D(n) being the difference when general n is used instead of p
$D(n) = n^3 \sigma_0(n) - J_2(n) - n \sigma_2(n) - \varphi(n) J_2(n)$.

Proposition:
$6 \mid D(n)$ with n being integer greater than 1. Also for prime p, $D(p) = 0$.
Proof, given $\sigma_0(p) = 2$, $\sigma_2(p) = p^2+1$, $\varphi(p) = p-1$, $J_2(p) = p^2-1$, it follows that
$- J_2(n) - \varphi(n) J_2(n) = (1-p) p (1+p)$, being consecutive.
$p^3 \sigma_0(n) - p \sigma_2(n) = (p-1) p (p+1)$, being consecutive.
Since they are being 3 consecutive integers, thus either one of them must be divisible by 3. Moreover, either 2 even numbers or 1 even number must be present, so they also must be divisible by 2. Since each term of the expression being multiplicative, any prime powers also worked. With careful adjustment, similar way can be applied for squarefree composites, and other composites involving prime powers as well. The caveat is, 0 is not allowed as the power, since the smallest p is 2, yet indeed 1 does not work out, therefore n > 1 being required. Thus **6 | D(n)** with n being integer greater than 1.
Also $D(p) = 0$ since $D(p) = (p-1) p (p+1) + (1-p) p (1+p) = 0$.

Conjecture:
$D(2(2n+1)^2) \equiv 6 \pmod{12}$, for n positive integer.
Otherwise $12 \mid D(m)$ with all m not being twice of odd squares, exclusively. That is the only exceptions are conjectured to be the only $m = 2(2n+1)^2$ with n being positive integer. For instances
D(18) = 25290; D(50) = 549450; D(98) = 4142754
D(162) = 35574390; D(242) = 62336538; D(338) = 169758810
all of which are the form of $D(2(2n+1)^2)$ in which all of above $\equiv 6 \pmod{12}$.
Further, one can sketch few examples to illustrate that when φ(sum of the odd divisors of m) = φ(sum of the even divisors of m) , i.e. iff $m = 2(2n+1)^2$, the quotient D(m)/6 must be odd, implying D(m) must be always congruent to 6 (mod 12). However, this is not yet ruling out the possibility of squarefree m in which its quotient D(m)/6 is odd.

2. Relation with $a^3+b^3 = 2(2^5 - c^3)$

Using the identity $(m+n)^3 = m^3 + n^3 + 3 m n (m+n)$, we can express $2(2^5 – c^3) = (4-c)^3 + (c-4)^3$ that leads to the expression $a^3+b^3 = 2(2^5 – c^3)$ which reveals the particular solution a = 4 – c, and a = c – 4. Departing from $a^3+b^3 = 2(2^5 – c^3)$, now consider $x + 4y^3 + 4z^3 = 512$, that is the leading coefficient of x,y,z is 1,4,4. Using quadratic change of variable, one can write the (x,y,z) triplet using c, as (2c, 4-c, c-4) with c being the square of odd number > 1, i.e. $(2n+1)^2$. Since x = 2c, y = 4-c, and z = c-4, the parametric solution for $x + 4y^3 + 4z^3 = 512$ can be derived after shifting some parameter $5 - 4n - 4n^2$ into $5 - 4n^2$ and then substituting $c = (2n+1)^2$. Such parametric solution is $(x,y,z) = (2(2n + 1)^2, 5 - 4n^2, 5 – 4(n+1)^2)$. That is $x(n) = 2(2n + 1)^2$, $y(n) = 5 - 4n^2$, $z(n) = 5 – 4(n+1)^2$). The quadratic $1 + 4n + 4n^2$ is exactly $(2n+1)^2$, i.e. odd square number >1.

Considering $x(n) = 2(2n + 1)^2$, with n = 1,2,3,4,… this yielded x(n) = 18, 50, 98, 162,… which are the numbers such that φ(sum of the odd divisors) = φ(sum of the even divisors). This x(n) coincides as a subset of the recurrence series that represents maximum number of regions into which the plane can be divided using n+1 quadrilaterals (allowing concave), which again, partially motivated from the older problem where one tries to find the maximum number of regions defined by n lines in the plane (R. Graham, D. Knuth, O. Patashnik, 1994).

Addendum, Ramblings, The Expounding

This section briefly expounds the former text regarding yesteryear PayDie (Qnoz, 2024).

Revisiting the intro which featured the constant π, such section purported the defense against the rivaling constant (say $k = 2\pi$, here k is used rather than tau since some people remarked that tau appears like half π in spite of being twice). The constant π is indeed able to represent semicircle (as of radian angle) if not whole loads of important paths in contour integral (if somebody claimed the semicircle actually hid the ½ under the rug, consider such contours, in which each of such contour truly is regarded as full path, instead of being half of the procedure) as well as the full circle when one considers the area of a full circle with radius 1. Keep in mind the spirit in here is to avoid using the fractional multiplier, in which the other one, viz. twice of π needs so.
The series expansion of sin(k sin(x)) which yield "nice series" of denominator terms, indeed occurs when $k = 2\pi$, **albeit** such series of denominators in OEIS A085990 only appear if every term in such denominator uses π itself, rather than being expressed in k. Pardon for the poor wording, the former was never meant to say $k = \pi$, yet the main point is with $k = 2\pi$, and then expressing every term of such denominator **using** k wouldn't yield the asserted sequence, however π did.

Conclusion: while it is true that the appearance of "the accompaniment" integer 2 indeed usually pair with π in many cases, the constant π itself cannot be treated as "not fundamental" and therefore need not to be removed entirely (otherwise the non-integer ½ would be the accompaniment instead). However, if π can already express what k is (that is 2π), then the definition of $k = 2\pi$ deemed to be superfluous and consume notational character needlessly (rather than the better use of the that symbol say, Ramanujan τ function, the divisor counter τ, the τ to express nome, and other suitable use of the τ notation, despite that π also been used to denote prime counting function, which still, has true relation with the actual π constant via PNT, also the capital Π meant to denote product, or even the more historic tidbits of π being used as arbitrary placeholder like x these days), hence π constant is not only defended, but is favoured for brevity.

Moving to another constant of interest.
Let $\psi_q(z)$ be q-digamma function
$\psi_q(z) = (1/\Gamma_q(z))\ \partial\Gamma_q(z)/\partial z$
where $\Gamma_q(z)$ being the q-gamma function defined by
$\Gamma_q(z) = ((q;q)_\infty / (q^z;q)_\infty)\ (1-q)^{1-z}$ for $|q| < 1$ using the q-Pochhammer symbol.
and
$\Gamma_q(z) = \Gamma_{1/q}(z)\ q^{(z-2)(z-1)/2}$ for $|q| > 1$.

Let $f(q) = -1 + (1-q)(([(\log(q-1) + \psi_q(2)) / \log(q)] -3/2) – 1/q)$
then calculating $\sum f(q)$ running from $q = 2$ to ∞, with
$\sum f(q) = 0.17278705503139434705946417072339198701844887 3\ldots$ = PayDie(2024)
Assuming the value of f(q) at $q \to 1$ under neutrix "regularisation" is -2
the regularised sum becomes $\sum f(q)$ running from $q \to 1$ to ∞, which is
-2 + PayDie(2024) = -1.827212944…

There is a typesetting mishap at the listed $\sum_p$^∞ f(p) for the 11-adic:
...211397405A91406128600103A2757138A5A9279882829165469A2A611A13837742405474791 yet somehow LaTeX displayed ...405474791 (whitespace) 91 instead, the additional 91 after whitespace should not be there. This might be parts of why this time, word processor being used instead :p

Speaking of which, the mentioned p-adic investigation of that constant of interest, began with the construction of a(n) which is dubbed as the *aiding* function, or nicked aiding-points for local p. The name a(n) itself need not to be arithmetic (i.e. multiplicative like σ or additive like Ω when coprime), the construction is meant to generate terms, i.e. coefficients, further to be multiplied with prime zeta function. Displayed "aiding point" in such text is the series ω(n) – 1 in respective p-base to represent the locus p. Example like 1 0 0 0 0 1 0 0 1 … yet written from right to left in respective p-adic, meant to represent number of unique prime minus 1, so that a(6) gives 1, a(10) gives 1, a(30) gives 2, yet for primes and prime powers these are 0. The decision of using ω(n) – 1 "like" series came from the generating function of the Lambert series except the summation index n i.e. which appears in q^n, only takes n = p = prime number, yielding the unique prime counter function ω(n). For some reasons, at n = 1, a(1) was formerly set as 2, and then everything were subtracted with 1 since there is a -1/p factor in the asymptotic term in a (desperate) attempt to regularise the formed q-analogue. Thus explaining why a(1) = 1, instead of 0. Having a(1) also allows the inverse convolution, which fitting its "aiding" purpose. The modification did not end there, the value a(n) at ω(n) ≥ 2 somehow get subtracted by 1, along with the value a(n-2) whenever ω(n) ≥ 2, gets added by 2, for various reasons, say asymptotic convergence or merely from regularisation point of view. All of these manipulation has nothing to do with the *global* ∑f(q) which value is on the main headlight for hunting. These "valuation toys" are like DIY playground in an attempt to coax integer relation or SVD (singular value decomposition) or inverse-problem-like possibility to discover the link with other established constants, in addition of the kindly provided methods by the authors of various readily-available programs these days.

It is clear that many constants say Euler-Mascheroni γ *et cetera et cetera* are each by itself being "elemental" i.e. cannot be represented in terms of other "elementary" constants without introducing infinite series or sums. Regardless of the ∑f(q) situation, one wants to know if it *might* have something to do with the other constant(s), for example, since AskConstants v5.0 does support zeta of odd integers, Pochhammer maxima abscissa-ordinate, and various q-series constants, one might query the the possibility of proper relation, i.e. whether it is even remotely doable to express ∑f(q) in terms of such special constants, whose known and are well-documented already. So far, program returned terrible margin.

Citations

R. Graham, D. Knuth, O. Patashnik, Concrete Mathematics, 2nd edition, Addison-Wesley, 1994

Fian Qnoz, Preliminary Values for Wild Constant Hunt, Possibly "Nice" Dirichlet Series Originates, accessible at https://vixra.org/pdf/2411.0006v1.pdf viXra e-print, 2024