Photon as a Soliton

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Abstract

We have used the logical arguments for the proof that photon is a soliton. Neutrino is also soliton due to the the experimental argument following from the neutron decay. However, with regard to the fact that every elementary particle has also so-called soliton parity, it is not excluded that the soliton parity of K^0-mesons and π^0-ions are also broken. This experimental possibility was not still confirmed. Nevertheless, it can be confirmed by the brilliant interferometric experiments because the broken soliton parity leads to the broken interferometry maxima and minima.

1 Introduction

The history of solitons began in the science of hydrodynamics. In August 1834 the Victorian Scottish civil engineer and naval architect John Scott Russell observed "a most beautiful and extraordinary phenomenon". This was the launching and propagation of a solitary wave (Russell, 1834) .

Theoretical investigations of the solitary wave were carried out by de Boussinesq (1871), Lord Rayleigh (1876), St. Venant (1885) and others. A most important seminal paper, however, was that due to Korteweg and de Vries (1895), working in Amsterdam, the city of canals. Up to that time there appeared to be some doubt as to whether a solitary wave could propagate without change of form when viscosity was negligible.

If we write the Korteweg-deVries equation in the general form (Vadati, 2001)

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{1}$$

then the solution of the last equation is in the form (Vadati, 2001):

$$u(x,t) = \frac{3v}{\alpha}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{v}{\beta}}(x - vt)\right). \tag{2}$$

Zabusky and Kruskal (1965) numerically analyzed the KdV equation and observed a particle-like behavior. In particular, they observed that solitary waves retained their shape and speed after collision. As a result, they coined the word "solitons" to describe solitary waves.

Then, the concept of soliton has been applied in several areas in the applied sciences. The soliton theory was developed by Gardner, et al. (1967) and is arguably considered the most important discovery in the field of mathematical physics in the 20th century (Vadati, 2001).

2 The mechanical bootstrap soliton

Everybody knows that the term "bootstrap model" is used for a class of theories that use very general consistency criteria to determine the form of a quantum theory from some assumptions on the spectrum of particles. It is a form of S-matrix theory.

Geoffrey Chew and others formulated the question of the distinction between composite and elementary particles, advocating a "nuclear democracy" where the idea that some particles were more elementary than others was discarded. Instead, they elaborated a theory of the strong interaction from plausible assumptions about the S-matrix, which describes what happens when particles of any sort collide, an approach advocated by Werner Heisenberg two decades earlier.

Here we consider, with the Galileo and Faraday simplicity, the table-top experiment with the classical bootstrap. So, we consider the classical bootstrap in the plane of the desk, or, table, where the left end of the bootstrap is fixed and the right end of it is in motion.

If the right end of the bootstrap is quickly moving down at the adequate trajectory, then, the energy of the right segment of the bootstrap is transmitted to the left part of the bootstrap forming the bootstrap soliton. So, quick motion is the origin of the soliton and the form of the soliton is determined by trajectory of the motion of the right end of the bootstrap. At present time there is no mathematical differential equation for the determination of the soliton form on the dependence on the right motion of the end of the bootstrap. Obviously, this soliton is not the Korteweg-de Wries soliton but it is so called the bootstrap soliton. At the same time we know that this is not the solution of the propagation of some pulse in the real strings and rods (Pardy, 2005).

Using the analogy with this experimental soliton we can say that photon is created as soliton by the quick electron transition in the Bohr and Sommerfeld model of the hydrogen atom as the analogy with the bootstrap situation.

3 Photon as a soliton from the Bohr model

Bohr model of atom was based on two postulates: 1. every atom can exist in the discrete series of states in which electrons do not radiate even if they are moving at acceleration (the postulate of the stationary states), 2. transiting electron from the stationary state to other, emits the energy according to the law $\hbar\omega = E_i - E_f$, called the Bohr formula, where E_i is the energy of an electron in the initial state, and E_f is the energy of the final state of an electron to which the transition is made and $E_i > E_f$.

Let us remark still that the Bohr theory does not involve the physical mechanism of creation of photons and the adequate model of photon. However, it follows from quantum theory of fields, that photon is excited state of vacuum and at the same time also an electron is the excited state of vacuum, which follows from the elementary experimental equation $\gamma + \gamma \rightleftharpoons e^+ + e^-$ (Berestetzkii et al., 1982).

At present time we know from the most general quantum field theory that all matter and antimatter in universe are excited states of vacuum.

4 Photon as a soliton from the Sommerfeld model

Let us find the classical equation of motion and the trajectory of an electron according to relativistic theory. In this case, the r and φ polar coordinates change with different frequencies. Or, the motion is quasi-periodic. We determine the angle through which the perihelion of the electron is shifted during "one" revolution. Then we obtain the Sommerfeld formula for the energy levels and find their splitting. We follow he the monograph by Sokolov et al. (1966)

Using the relativistic Lagrangian function

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2} + \frac{Ze^2}{r}, \tag{3}$$

where

$$\beta^2 = \frac{v^2}{c^2} = \frac{1}{c^2}(\dot{r}^2 + r^2\dot{\varphi}^2), \tag{4}$$

we obtain the equation of motion in the form:

$$\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1-\beta^2}} = -\frac{Ze^2}{r^3}\mathbf{r}. \tag{5}$$

For the generalized momenta

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, \quad p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{r}}, \tag{6}$$

we get from eq. (6):

$$p_r = \frac{p_\varphi}{r^2}r', \quad r' = \frac{dr}{d\varphi} \tag{7}$$

Now, it follows, in accordance with the law of conservation of energy, that

$$E = c\sqrt{mc^2 + p_r^2 + \frac{p_\varphi^2}{r^2}} - \frac{Ze^2}{r} = const, \tag{8}$$

which implies that

$$r' = \frac{r^2}{cp_\varphi}\sqrt{\left(E + \frac{Ze^2}{r}\right)^2 - m^2c^4 - \frac{c^2p_\varphi^2}{r^2}}. \tag{9}$$

The solution of the last equation can be realized by the substitution $r = 1/u$. Then, after some mathematical operations we get the differential equation for u and therefore for r. The solution of the differential equation for u is then the equation of the trajectory r:

$$r = \frac{q}{1 + \varepsilon \cos \gamma\varphi}, \tag{10}$$

where

$$\gamma = \sqrt{1 - \frac{Z^2 e^4}{c^2 p_\varphi^2}} \tag{11}$$

$$q = \frac{\gamma^2 c^2 p_\varphi^2}{Z e^2 E}. \tag{12}$$

$$\varepsilon = \sqrt{1 + \frac{\gamma^2 \left(1 - \frac{m^2 c^4}{E^2}\right)}{(1 - \gamma^2)}}. \tag{13}$$

It is apparent from Eq. (10) that the motion is quasi-periodic. For the shift $\Delta\varphi$ of the perihelion, we have from (10)

$$\Delta\varphi = \frac{2\pi(1-\gamma)}{\gamma} \approx \frac{\pi Z^2 e^4}{c^2 p_\varphi^2}. \tag{14}$$

With the help of

$$\oint p_i dx_i = I_i, \tag{15}$$

where I_i are so called the Ehrenfest adiabatic invariants, with p_i, x_i being generalized coordinates, we get

$$I_\varphi = 2\pi p_\varphi. \tag{16}$$

$$I_r = 2\pi \left(\frac{B}{\sqrt{A}} - \sqrt{C} \right), \tag{17}$$

where

$$A = m^2 c^2 \left(1 - \frac{E^2}{m^2 c^4}\right). \tag{18}$$

$$B = \frac{Z e^2 E}{c^2}, \quad C = p_\varphi^2 - \frac{Z^2 e^4}{c^2} \tag{19}$$

Then, for the energy E, we obtain the expression

$$E = m^2 c^2 \left\{ 1 + \frac{Z^2 e^4}{c^2 \left[\frac{I_r}{2\pi} + \sqrt{\frac{I_\varphi^2}{4\pi^2} - \frac{Z^2 e^4}{c^2}} \right]^2} \right\}^{-1/2}. \tag{20}$$

We immediately see that the frequencies

$$\omega_\varphi = \frac{\partial E}{\partial I_\varphi}, \quad \omega_r = \frac{\partial E}{\partial r} \tag{21}$$

are different.

Using the Bohr-Sommerfeld quantization conditions

$$\oint p_\varphi d\varphi = n_\varphi \hbar, \quad \oint p_r dr = n_r \hbar, \tag{22}$$

where n_φ, n_r are azimutal and radial quantum numbers, we can transform the energetic formula (20) in the new form:

$$E_{n_r,n_\varphi} = E - mc^2 = mc^2 \left\{ 1 + \frac{Z^2\alpha^4}{\left[n_r + \sqrt{n_\varphi{}^2 - Z^2\alpha^2}\right]^2} \right\}^{-1/2} - mc^2, \tag{23}$$

where $\alpha = e^2/c\hbar \approx 1/137$ is so called the fine structure constant. Expanding the formula (23) into a series in α^2 and restricting ourselves to quantities of the order of α^2, we have:

$$E_{n,n_\varphi} = -\frac{R\hbar Z^2}{n^2}\left[1 + \frac{\alpha^2 Z^2}{n^2}\left(\frac{n}{n_\varphi} - \frac{3}{4}\right)\right] \tag{24}$$

Since n_φ varies from 1 to n_φ it follows from Eq. (24) that the energy levels, which is determined by the principal quantum number $n = n_r + n_\varphi$ are split into n closely sub-levels (this close spacing is a consequence of the smallness of α^2).

The fine-structure splitting is

$$\Delta E_{n,n_\varphi} = E_{n,n_\varphi} - E_{n,n_\varphi - 1} = \frac{R\hbar Z^4\alpha^2}{n^2 n_\varphi(n_\varphi - 1)}, \tag{25}$$

where

$$R = \frac{me^4}{2\hbar^3} \tag{26}$$

being the Rydberg constant.

The splitting, or fine structure, of the levels, is a characteristic result of relativistic effects and it is essentially different from the predictions of the non-relativistic theory.

Let us remark that Dirac formula for H-atom is related to the Sommerfeld formula Or, if we replace the numbers n_φ by $(j + 1/2)$ and n_r by $(j - 1/2)$ in the formula (23), we get:

$$E_{n_r,n_\varphi} = E - mc^2 = mc^2 \left\{ 1 + \frac{Z^2\alpha^4}{\left[n + (j-1/2) + \sqrt{(j+1/2)^2 - Z^2\alpha^2}\right]^2} \right\}^{-1/2} - mc^2, \tag{27}$$

which is original the Dirac formula for the electron with spin 1/2. The coincidence is miraculous and till this time the explanation of the physical origin of this coincidence was not given. More information of this problem is in the Petrov article (Petrov, 2020).

Let us remark still that the Sommerfeld theory does not involve the physical mechanism of creation of photons and the adequate model of photon.

5 The synchrotron photon as a soliton from the bremsstrahlung of an electron in a magnetic field

Synchrotron radiation, or, magnetobremsstrahlung is the electromagnetic radiation emitted when relativistic charged particles are forced to an acceleration perpendicular to their velocity.

Because in most accelerators the particle trajectories are bent by magnetic fields, synchrotron radiation is also called magneto-bremsstrahlung. The emitted spectrum is broadband from the microwave (harmonics of the driving RF field) to x-ray spectral regions. The radiation is vertically collimated and polarized.

It is the direct consequence of Maxwell's equations for accelerated charged particles which always emit electromagnetic radiation. Synchrotron radiation is the special case of charged particles moving at relativistic speed under going acceleration perpendicular to their direction of motion, typically in a magnetic field. In such a field, the force due to the field is always perpendicular to both the direction of motion and to the direction of field, as shown by the Lorentz force law. The motion is circular.

The circular classical trajectory of the electron is created by the Lorentz force $F = (e/c)(\mathbf{v} \times \mathbf{H})$. The trajectory is stationary when the radiative reaction is not considered. The radiative reaction causes the transformation of the circular trajectory to the spiral trajectory. In quantum mechanics, the trajectory is stationary when neglecting the interaction of an electron with the vacuum field. The interaction of an electron with the vacuum field, causes the electron jumps from the higher energetic level to the lower ones. In quantum electrodynamics description of the motion of electron in a homogeneous magnetic field, the stationarity of the trajectories is broken by including the mass operator into the wave equation. Then, it is possible from the mass operator to derive the power spectral formula (Schwinger, 1973*; Tsai, 1974).

So, The quantum interpretation of the synchrotron radiation is the production of photons by charged particle when moving in magnetic field along trajectories when electron perform transition from one trajectory to other. The duration of transition is zeptoseconds and it means that the emitted photons are solitons.

6 The soliton-wave solution of the nonlinear Schrödinger equation

The soliton solution is also involved in the nonlinear Schrödinger equation of the special form. However, this is the analogue of the Korteweg-de Vries equation. So, the proposed nonlinear Schrödinger equation which has possible physical meaning is of the following form (Pardy, 1993; 2001):

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi - b(\ln|\Psi|^2)\Psi \tag{28}$$

with the possible test, described by author in the simmilar situation (Pardy, 1994).

The solution of eq. (28) is as it follows. Let be $c, (\mathrm{Im}\ c = 0), v, k, \omega$ some parameters and let us insert function

$$\Psi(x,t) = cG(x - vt)e^{ikx - i\omega t} \tag{29}$$

into the one-dimensional equation (28) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m} \tag{30}$$

and for function G we get the following nonlinear equation (the symbol $'$ denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0, \tag{31}$$

where

$$A = \frac{2m}{\hbar}\omega - k^2 + \frac{2m}{\hbar^2} b \ln c^2 \tag{32}$$

$$B = \frac{4mb}{\hbar^2}. \tag{33}$$

After multiplication of eq. (31) by G' we get:

$$\frac{1}{2}\Big[G'^2\Big]' + \frac{A}{2}\Big[G^2\Big]' + B\left[\frac{G^2}{2}\ln G - \frac{G^2}{4}\right]' = 0, \tag{34}$$

or, after integration

$$G'^2 = -AG^2 - BG^2 \ln G + \frac{B}{2}G^2 + const. \tag{35}$$

If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $const. = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B\ln G}} = d\xi, \tag{36}$$

where

$$a = \frac{B}{2} - A. \tag{37}$$

Eq. (36) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{B}} e^{-\frac{B}{4}(\xi + d)^2}, \tag{38}$$

where d is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x,t) = ce^{\frac{a}{B}} e^{-\frac{B}{4}(x - vt + d)^2} e^{ikx - i\omega t}. \tag{39}$$

The normalization and the classical limit of the last equation is with no change the standard probability interpretation of the wave function is as follows.

$$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1. \tag{40}$$

Using the Gauss integral

$$\int_0^\infty e^{-\lambda^2 x^2}\, dx = \frac{\sqrt{\pi}}{2\lambda}, \tag{41}$$

we get with $\lambda = \left(\frac{B}{2}\right)^{\frac{1}{2}}$

$$c^2 e^{\frac{2a}{B}} = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} \tag{42}$$

and the density probability $\Psi^*\Psi = \delta_m(\xi)$ is of the form (with $d = 0$):

$$\delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m \xi^2} \quad ; \quad \alpha = \frac{2b}{\hbar^2}. \tag{43}$$

It may be easy to see that $\delta_m(\xi)$ is the delta-generating function and for $m \to \infty$ is just the Dirac δ-function.

It means that the motion of a particle with sufficiently big mass m is strongly localized and in other words it means that the motion of this particle is the classical one. Such behavior of a particle cannot be obtained in the standard quantum mechanics because the plane wave

$$e^{ikx - i\omega t} \tag{44}$$

corresponds to the free particle with no possibility of localization for $m \to \infty$.

Let us still remark that coefficient c^2 is real and positive number because it is a result of the solution of eq. (42) which can be transformed into equation ($x = c^2$)

$$x^{1-r} = const. \tag{45}$$

Let us remark that the principle of superposition is in our theory broken. If φ_1 and φ_2 are two different solution of the nonlinear Schrödinger equation then the linear combination $\varphi = a\varphi_1 + b\varphi_2$ where a and b are the arbitrary constants is not the solution of the same equation because of its non-linearity. In other words the original principle of superposition of the standard quantum mechanics is broken. The consequence of the breaking of the principle of superposition is the crucial resolution of the Schrödinger cat paradox. It is not excluded that the non-validity of the principle of superposition can play substantial role in the future particle physics (Laurent et al., 1965; Shapiro, 1972).

7 Discussion

We have used the specific arguments for the proof that photon is a soliton. Neutrino can be considered as soliton due to the the experimental argument following from the neutron decay (Greiner et al., 2009):

$$n \to p + e^- + \bar{\nu}_e. \tag{46}$$

We have also equation

$$e^- + e^+ \to \gamma + \gamma, \tag{47}$$

where the duration of this electron-positron annihilation is practically infinitesimal.

With regard to the quark model of neutron and proton, neutron is not composed from proton, electron and anti-neutrino. It means that these three particles are created during process with attosecond, or, zeptosecond duration, which is still necessary to be experimentally confirmed. This is analogous of the creation of photon in the H-atom. So, this analogy enable us to say that proton, electron and neutrino are solitons and not Einstein singularity of vacuum field. This statement is in harmony with the Shapiro soliton model of all elementary particles (Shapiro, 1976).

Now, let as consider the of parity with K-mesons. We have the decay (Greiner et al., 2009)

$$K^0 \rightarrow \pi^0 + \pi^0 \tag{48}$$

and

$$K^0 \rightarrow \pi^0 + \pi^0 + \pi^0, \tag{49}$$

which is the consequence of the experimental fact that parity is not conserved. It means that K^0-meson is the same in both equations.

However, with regard to the fact that every elementary particle has also so-called soliton parity, it is not excluded that the soliton parity of K^0-meson and π^0-ions is also broken. This experimental possibility was not confirmed. Nevertheless, it forms new deal of particle physics (Shapiro, 1976; Kondratyuk, et all., 1977).

It is well-known that graviton being a particle with spin 2 is the analog of photon with spin 1. The theory which describes the ingredients of particles with all spins is involved for instance in the Schwinger theory of sources (Schwinger, 1969; 1970; 1973; 1989). So, graviton as analog of photon is logically particle with soliton parity and sooner or later it will be confirmed by experiments.

Production of gravitons by binary system is involved in the author articles (Pardy, 1983; 1984; 1994a; 1994b; 1994c; 1994d; 2011; 2018a; 2018b; 2019). So, we can honestly make prediction that the future of the particle physics and gravity will be the soliton particle physics and the soliton gravity. And this will be the new deal of CERN and JWST.

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