Lorentz-Covariant Hamiltonian Mechanics and its Quantization

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Abstract The Lorentz-covariant upgrade of Newton's Second Law sets a particle's mass times the second derivative with respect to its Lorentz-invariant proper time of its observed space-time location equal to the four-force applied to it. Hamiltonian mechanics yields the Lorentz-covariant upgrade of Newton's Second Law when the Hamiltonian is a Lorentz-invariant function of the particle's observed space-time location and conjugate four-momentum, and the time derivatives of these two Lorentz-covariant four-vectors are taken with respect to the particle's Lorentz-invariant proper time. Very simple Lorentz-invariant Hamiltonians that yield the electromagnetic Lorentz Force Law and the gravitational geodesic equation are pointed out, as is the straightforward quantization of Lorentz-covariant Hamiltonian mechanics. A very weak short-range relativistic complement to the hydrogen atom's Coulomb potential is found, which seems negligible for most purposes, but might affect studies of the proton's charge radius.

1. The Lorentz-covariant upgrades of Newton's Second Law and Hamiltonian mechanics

An effort to acheive a Lorentz-covariant upgrade of Newton's Second Law, i.e.,

$$m\big(d^2\mathbf{x}/dt^2\big) = \mathbf{F}, \tag{1.1}$$

must deal with the two closely related facts that *the particle's observed spatial location* $\mathbf{x}$ *alone isn't a Lorentz-covariant entity,* and that *the universal time* t *of* Eq. (1.1) *doesn't exist in Lorentzian relativity.* The particle's three-vector observed spatial location $\mathbf{x}$ alone *is therefore replaced by its Lorentz-covariant four-vector observed space-time location* $x^\mu = (ct, \mathbf{x})$*, where* t *is the time given by a clock at rest with respect to the observer,* and the Eq. (1.1) second derivative with respect to universal time t *is replaced by the second derivative with respect to the particle's Lorentz-invariant proper time* τ*, which is given by a clock at rest with respect to the particle instead of at rest with respect to the observer.* The three-vector force $\mathbf{F}$ of Eq. (1.1) *is thereby replaced by a Lorentz-covariant four-vector force* f^μ, and the three-vector Newton's Second Law of Eq. (1.1) *is upgraded to the following Lorentz-covariant four-vector form,*[1]

$$m\big(d^2x^\mu/d\tau^2\big) = f^\mu. \tag{1.2}$$

Ordinary nonrelativistic Hamiltonian mechanics has a Hamiltonian $H(\mathbf{x},\mathbf{p})$ which is a scalar function of the particle's observed three-vector spatial location $\mathbf{x}$ and conjugate momentum $\mathbf{p}$. The first-order in universal time t pair of three-vector ordinary nonrelativistic Hamiltonian equations of particle motion,

$$d\mathbf{x}/dt = \nabla_{\mathbf{p}} H(\mathbf{x},\mathbf{p}) \;\text{ and }\; d\mathbf{p}/dt = -\nabla_{\mathbf{x}} H(\mathbf{x},\mathbf{p}), \tag{1.3}$$

yields Newton's Eq. (1.1) Second Law as follows,

$$\begin{gathered} m\big(d^2\mathbf{x}/dt^2\big) = m\big(d(d\mathbf{x}/dt)/dt\big) = m\big(d(\nabla_{\mathbf{p}}H(\mathbf{x},\mathbf{p}))/dt\big) = \\ m\big[\big((d\mathbf{x}/dt)\cdot\nabla_{\mathbf{x}}\big)(\nabla_{\mathbf{p}}H(\mathbf{x},\mathbf{p})) + \big((d\mathbf{p}/dt)\cdot\nabla_{\mathbf{p}}\big)(\nabla_{\mathbf{p}}H(\mathbf{x},\mathbf{p}))\big] = \\ m\big[\big((\nabla_{\mathbf{p}}H(\mathbf{x},\mathbf{p}))\cdot\nabla_{\mathbf{x}}\big) - \big((\nabla_{\mathbf{x}}H(\mathbf{x},\mathbf{p}))\cdot\nabla_{\mathbf{p}}\big)\big](\nabla_{\mathbf{p}}H(\mathbf{x},\mathbf{p})) = \mathbf{F}(\mathbf{x},\mathbf{p}). \end{gathered} \tag{1.4}$$

To yield the Eq. (1.2) Lorentz-covariant upgrade, $m(d^2x^\mu/d\tau^2) = f^\mu$, of Newton's Second Law, the particle's three-vector spatial location $\mathbf{x}$ and conjugate momentum $\mathbf{p}$ of Eq. (1.3) ordinary nonrelativistic Hamiltonian mechanics must respectively become the Lorentz-covariant four-vectors x^μ and p_ν. The Hamiltonian $H(\mathbf{x},\mathbf{p})$, which is a scalar function of $\mathbf{x}$ and $\mathbf{p}$, must become a Lorentz-invariant function $H(x,p)$ of x^μ and p_ν, and the Eq. (1.3) pair of nonrelativistic three-vector equations of particle motion must become the following pair of Lorentz-covariant four-vector equations of particle motion,

$$dx^\mu/d\tau = \partial H(x,p)/\partial p_\mu \;\text{ and }\; dp_\mu/d\tau = -\partial H(x,p)/\partial x^\mu, \tag{1.5}$$

in which time derivatives are with respect the particle's Lorentz-invariant proper time τ. The Eq. (1.5) Lorentz-covariant upgrade of ordinary nonrelativistic Hamiltonian mechanics yields Eq. (1.2) as follows,

$$\begin{gathered} m\big(d^2x^\mu/d\tau^2\big) = m\big(d(dx^\mu/d\tau)/d\tau\big) = m\big[d(\partial H(x,p)/\partial p_\mu)/d\tau\big] = \\ m\big[(\partial^2 H(x,p)/\partial p_\mu\partial x^\nu)(dx^\nu/d\tau) + (\partial^2 H(x,p)/\partial p_\mu\partial p_\nu)(dp_\nu/d\tau)\big] = \\ m\big[(\partial^2 H(x,p)/\partial p_\mu\partial x^\nu)(\partial H(x,p)/\partial p_\nu) - (\partial^2 H(x,p)/\partial p_\mu\partial p_\nu)(\partial H(x,p)/\partial x^\nu)\big] = f^\mu(x,p). \end{gathered} \tag{1.6}$$

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The following definition of *the antisymmetric relativistic Poisson bracket* $\{F(x,p), G(x,p)\}$ *of any pair of functions* $F(x,p)$ *and* $G(x,p)$ *of relativistic phase space* (x,p),

$$\{F(x,p), G(x,p)\} \stackrel{\text{def}}{=} (\partial F(x,p)/\partial x^\nu)(\partial G(x,p)/\partial p_\nu) - (\partial G(x,p)/\partial x^\nu)(\partial F(x,p)/\partial p_\nu) = -\{G(x,p), F(x,p)\}, \tag{1.7}$$

enables Eq. (1.6) to be alternatively obtained and exhibited in the very compact notation,

$$m\big(d^2x^\mu/d\tau^2\big) = m\big(d(dx^\mu/d\tau)/d\tau\big) = m\{\{x^\mu, H(x,p)\}, H(x,p)\} = f^\mu(x,p), \tag{1.8}$$

because the chain rule and Eq. (1.5) together imply that $(dF(x,p)/d\tau) = \{F(x,p), H(x,p)\}$ for any function $F(x,p)$ of relativistic phase space (x,p) since,

$$(dF(x,p)/d\tau) = (\partial F(x,p)/\partial x^\nu)(dx^\nu/d\tau) + (\partial F(x,p)/\partial p_\nu)(dp_\nu/d\tau) = (\partial F(x,p)/\partial x^\nu)(\partial H(x,p)/\partial p_\nu) - (\partial F(x,p)/\partial p_\nu)(\partial H(x,p)/\partial x^\nu) = \{F(x,p), H(x,p)\}. \tag{1.9}$$

Eqs. (1.9) and (1.7) yield in particular that $(dH(x,p)/d\tau)$ vanishes because,

$$(dH(x,p)/d\tau) = \{H(x,p), H(x,p)\} = -\{H(x,p), H(x,p)\} = 0. \tag{1.10}$$

2. Obtaining Lorentz-covariant Lagrangian mechanics from its Hamiltonian counterpart

Obtaining the Eq. (1.3) equations of ordinary nonrelativistic Hamiltonian mechanics from the equation of ordinary nonrelativistic Lagrangian mechanics, namely,

$$d\mathbf{p}/dt = \nabla_{\mathbf{x}} L(d\mathbf{x}/dt, \mathbf{x}), \text{ where } \mathbf{p} = \nabla_{d\mathbf{x}/dt} L(d\mathbf{x}/dt, \mathbf{x}), \tag{2.1}$$

and the relation $H(\mathbf{x},\mathbf{p}) = (d\mathbf{x}/dt)\cdot\mathbf{p} - L(d\mathbf{x}/dt, \mathbf{x})$ of the ordinary nonrelativistic Hamiltonian $H(\mathbf{x},\mathbf{p})$ to the ordinary nonrelativistic Lagrangian $L(d\mathbf{x}/dt, \mathbf{x})$ *is a standard presentation of nonrelativistic mechanics* [2]. We will now, speaking roughly, *invert* that presentation to obtain the *relativistic* equation of Lorentz-covariant Lagrangian mechanics from the second Eq. (1.5) *relativistic* equation of Lorentz-covariant Hamiltonian mechanics and the Lorentz-invariant Lagrangian $L(dx/d\tau,\, x)$ given by,

$$L(dx/d\tau,\, x) = (dx^\nu/d\tau)p_\nu - H(x,p). \tag{2.2}$$

Taking the partial derivative with respect to x^μ of both sides of Eq. (2.2) yields,

$$\partial L(dx/d\tau,\, x)/\partial x^\mu = -\partial H(x,p)/\partial x^\mu. \tag{2.3}$$

Inserting the content of the second Eq. (1.5) relativistic Hamiltonian equation of particle motion, which is $dp_\mu/d\tau = -\partial H(x,p)/\partial x_\mu$, into Eq. (2.3) yields,

$$dp_\mu/d\tau = \partial L(dx/d\tau,\, x)/\partial x^\mu. \tag{2.4}$$

Taking the partial derivative with respect to $(dx^\mu/d\tau)$ of both sides of Eq. (2.2) yields,

$$p_\mu = \partial L(dx/d\tau,\, x)/\partial(dx^\mu/d\tau). \tag{2.5}$$

Eq. (2.4), with p_μ given by Eq. (2.5), is the Lorentz-covariant equation of relativistic Lagrangian mechanics,

$$dp_\mu/d\tau = \partial L(dx/d\tau,\, x)/\partial x^\mu, \text{ where } p_\mu = \partial L(dx/d\tau,\, x)/\partial(dx^\mu/d\tau). \tag{2.6}$$

Conversely, the two Eq. (1.5) equations of relativistic Hamiltonian mechanics similarly follow from the Eq. (2.6) equation of relativistic Lagrangian mechanics and the Lorentz-invariant Hamiltonian given by $H(x,p) = (dx^\nu/d\tau)p_\nu - L(dx/d\tau,\, x)$; the interested reader may wish to fill in the calculational details. Relativistic Lagrangian mechanics is therefore *equivalent* to its relativistic Hamiltonian counterpart.

It is furthermore readily shown that the Eq. (2.6) equation of relativistic Lagrangian mechanics follows from variation of the Lorentz-invariant Lagrangian action,

$$S_L[x(\tau)] = \int L(dx(\tau)/d\tau,\, x(\tau))\, d\tau, \tag{2.7}$$

and that the Eq. (1.5) *equivalent* two equations of relativistic Hamiltonian mechanics follow from variation *of the same action written in terms of the Lorentz-invariant Hamiltonian* $H(x,p)$ *by the use of* Eq. (2.2),

$$S_H[x(\tau),\, p(\tau)] = \int [(dx^\nu(\tau)/d\tau)p_\nu(\tau) - H(x(\tau), p(\tau))]\, d\tau. \tag{2.8}$$

3. The nonrelativistic and relativistic Hamiltonians for a free particle

The Eq. (1.1) *nonrelativistic* case of Newton's Second Law of course has $\mathbf{F} = \mathbf{0}$ for *a free particle,* i.e.,

$$m\big(d^2\mathbf{x}/dt^2\big) = \mathbf{0}. \tag{3.1a}$$

The standard nonrelativistic Hamiltonian which corresponds to Eq. (3.1a) is, of course,

$$H(\mathbf{p}) = \big(|\mathbf{p}|^2/(2m)\big). \tag{3.1b}$$

The two three-vector equations of particle motion which follow from the Eq. (3.1b) Hamiltonian are,

$$d\mathbf{x}/dt = \nabla_{\mathbf{p}} H(\mathbf{p}) = \mathbf{p}/m \;\text{ and }\; d\mathbf{p}/dt = -\nabla_{\mathbf{x}} H(\mathbf{p}) = \mathbf{0}, \tag{3.1c}$$

and together they clearly imply Eq. (3.1a).

We next seek *a Lorentz-invariant relativistic Hamiltonian $H(p)$ whose two four-vector equations of particle motion $dx^\mu/d\tau = \partial H(p)/\partial p_\mu$ and $dp_\mu/d\tau = -\partial H(p)/\partial x^\mu = 0$ together imply the $f^\mu = 0$ free-particle case of the* Eq. (1.2) *Lorentz-covariant upgrade of Newton's Second Law,* i.e.,

$$m\big(d^2x^\mu/d\tau^2\big) = 0, \tag{3.2a}$$

and which reduces to the Eq. (3.1b) *nonrelativistic free-particle Hamiltonian $H(\mathbf{p}) = \big(|\mathbf{p}|^2/(2m)\big)$ in the nonrelativistic limit that time is universal, i.e., in the limit that $(dt/d\tau) \to 1$, where $t = (x^0/c)$.*

We note in this regard that *the class of all Lorentz-invariant relativistic Hamiltonians of the form,*

$$H(p) = H_0 - (\eta^{\mu\nu} p_\mu p_\nu/(2m)), \tag{3.2b}$$

where H_0 is an arbitrary constant with the dimension of energy, and $\eta^{\mu\nu}$ is the Minkowski metric, i.e.,

$$\eta^{00} = 1,\, \eta^{11} = \eta^{22} = \eta^{33} = -1 \text{ and } \eta^{\mu\nu} = 0 \text{ when } \mu \neq \nu, \tag{3.2c}$$

satisfy the two Hamiltonian equations of motion,

$$dx^\mu/d\tau = \partial H(p)/\partial p_\mu = -\eta^{\mu\nu} p_\nu/m \;\; \textit{and} \;\; dp_\mu/d\tau = -\partial H(p)/\partial x^\mu = 0, \tag{3.2d}$$

which together imply that Eq. (3.2a) *is satisfied, namely that* $m\big(d^2x^\mu/d\tau^2\big) = 0$.

Furthermore, the matrix inverse $\eta_{\lambda\mu}$ of the Eq. (3.2c) Minkowski metric $\eta^{\mu\nu}$ exists; in fact $\eta_{\lambda\mu}$ is equal to $\eta^{\lambda\mu}$. Therefore multiplying both sides of the first Eq. (3.2d) result $dx^\mu/d\tau = -\eta^{\mu\nu} p_\nu/m$ by $-\eta_{\lambda\mu} m$ and summing over the index μ yields that $p_\lambda = -\eta_{\lambda\mu} m(dx^\mu/d\tau)$. Since $(dx^\mu/d\tau) = (dx^\mu/dt)(dt/d\tau)$, as $(dt/d\tau) \to 1$, $p_\lambda \to -\eta_{\lambda\mu} m(dx^\mu/dt) = (-m(dx^0/dt),\, m(d\mathbf{x}/dt)) = (-mc,\, m(d\mathbf{x}/dt))$. Therefore as $(dt/d\tau) \to 1$, $p_0 \to -mc$. Eq. (3.2b) and the fact that $p_0 \to -mc$ as $(dt/d\tau) \to 1$ together yield that,

$$\begin{aligned} H(p) = H_0 - (\eta^{\mu\nu} p_\mu p_\nu/(2m)) &= \\ H_0 - \big((p^0)^2/(2m)\big) + \big(|\mathbf{p}|^2/(2m)\big) &\to \\ H_0 - \big(mc^2/2\big) + \big(|\mathbf{p}|^2/(2m)\big) \text{ as } (dt/d\tau) &\to 1. \end{aligned} \tag{3.2e}$$

Setting the constant H_0 to $\big(mc^2/2\big)$ in the Eq. (3.2b) class of relativistic Hamiltonians $H(p) = H_0 - (\eta^{\mu\nu} p_\mu p_\nu/(2m))$ ensures that $H(p)$ indeed reduces to the Eq. (3.1b) nonrelativistic free-particle Hamiltonian $H(\mathbf{x}, \mathbf{p}) = \big(|\mathbf{p}|^2/(2m)\big)$ in the $(dt/d\tau) \to 1$ nonrelativistic limit that time is universal.

Therefore the Lorentz-invariant relativistic free-particle Hamiltonian $H(p)$ we desire that implies the Eq. (3.2a) relativistic free-particle equation of motion $m\big(d^2x^\mu/d\tau^2\big) = 0$ via the two relativistic Hamiltonian four-vector equations of particle motion $dx^\mu/d\tau = \partial H(p)/\partial p_\mu$ and $dp_\mu/d\tau = -\partial H(p)/\partial x^\mu = 0$, and which reduces to the Eq. (3.1b) nonrelativistic free-particle Hamiltonian $H(\mathbf{p}) = \big(|\mathbf{p}|^2/(2m)\big)$ in the $(dt/d\tau) \to 1$ nonrelativistic limit that time is universal, where $t = (x^0/c)$, is given by,

$$H(p) = \big(mc^2/2\big) - (\eta^{\mu\nu} p_\mu p_\nu/(2m)). \tag{3.2f}$$

4. The relativistic Hamiltonian for a charged particle in an electromagnetic four-potential

Free-particle Hamiltonians such as the $H(p)$ of Eq. (3.2f) are typically modified to incorporate the effect of the interaction of the particle's charge e with an electromagnetic four-potential $A_\mu(x)$ by implementing a "minimal coupling" prescription [3] which adds $(e/c)A_\mu(x)$ to the free particle's four-momentum p_μ. We next work out the equation of charged particle motion which follows from the resulting Hamiltonian, namely,

$$H(A(x), p) = \big(mc^2/2\big) - \big(\eta^{\mu\nu}(p_\mu + (e/c)A_\mu(x))(p_\nu + (e/c)A_\nu(x))/(2m)\big). \tag{4.1}$$

We will obtain that the equation of charged particle motion in an electromagnetic four-potential is the relativistic Lorentz Force Law [4], as of course is expected. The first Hamiltonian equation of motion which the Hamiltonian $H(A(x), p)$ of Eq. (4.1) yields is,

$$dx^\mu/d\tau = \partial H(A(x),\, p)/\partial p_\mu = -\eta^{\mu\nu}(p_\nu + (e/c)A_\nu(x))/m. \tag{4.2a}$$

The second Hamiltonian equation of motion which the Hamiltonian $H(A(x), p)$ of Eq. (4.1) yields is,

$$dp_\lambda/d\tau = -\partial H(A(x),\, p)/\partial x^\lambda =$$
$$\eta^{\mu\nu}(e/c)(\partial A_\mu(x)/\partial x^\lambda)(p_\nu + (e/c)A_\nu(x))/m = -(e/c)(\partial A_\mu(x)/\partial x^\lambda)(dx^\mu/d\tau), \tag{4.2b}$$

where the final equality in Eq. (4.2b) follows from Eq. (4.2a).

We next differentiate both sides of the Eq. (4.2a) result with respect to τ, followed by inserting into the right side of that result the consequence of the Eq. (4.2b) result for $dp_\nu/d\tau$,

$$d^2x^\mu/d\tau^2 = -\eta^{\mu\nu}\big(dp_\nu/d\tau + (e/c)(\partial A_\nu(x)/\partial x^\kappa)(dx^\kappa/d\tau)\big)/m =$$
$$-\eta^{\mu\nu}\big(-(e/c)(\partial A_\kappa(x)/\partial x^\nu)(dx^\kappa/d\tau) + (e/c)(\partial A_\nu(x)/\partial x^\kappa)(dx^\kappa/d\tau)\big)/m. \tag{4.2c}$$

Tidying up the Eq. (4.2c) result produces a Lorentz-covariant equation of charged particle motion of the Eq. (1.2) category, $m\big(d^2x^\mu/d\tau^2\big) = f^\mu$, that indeed is the relativistic Lorentz Force Law [4],

$$m\big(d^2x^\mu/d\tau^2\big) = f^\mu = (e/c)\,(-\eta^{\mu\nu})\,F_{\nu\kappa}(x)\,(dx^\kappa/d\tau) \ \text{ where}$$
$$F_{\nu\kappa}(x) = \big((\partial A_\nu(x)/\partial x^\kappa) - (\partial A_\kappa(x)/\partial x^\nu)\big). \tag{4.2d}$$

5. The relativistic Hamiltonian for a particle in a Lorentz-covariant gravitational metric

The modification of the free-particle Hamiltonian $H(p)$ of Eq. (3.2f) to incorporate the particle's interaction with a Lorentz-covariant gravitational metric $g^{\mu\nu}(x)$ replaces the Minkowski metric $\eta^{\mu\nu}$ by $g^{\mu\nu}(x)$, i.e.,

$$H(g(x),\, p) = \big(mc^2/2\big) - (g^{\mu\nu}(x)p_\mu p_\nu/(2m)). \tag{5.1}$$

Eq. (5.1) *imposes* $g^{\mu\nu}(x) = g^{\nu\mu}(x)$. *It is in addition assumed that* $g^{\mu\nu}(x)$ *has a matrix inverse* $g_{\lambda\mu}(x)$ *at all* x *such that* $g_{\lambda\mu}(x)g^{\mu\nu}(x) = \delta^\nu_\lambda$. The first Hamiltonian equation of motion that Eq. (5.1) yields is,

$$dx^\mu/d\tau = \partial H(g(x),\, p)/\partial p_\mu = -g^{\mu\nu}(x)p_\nu/m. \tag{5.2a}$$

An important consequence of Eq. (5.2a) and the fact that $g_{\gamma\delta}(x)g^{\delta\nu}(x) = \delta^\nu_\gamma$ is that,

$$p_\gamma = -mg_{\gamma\delta}(x)(dx^\delta/d\tau). \tag{5.2b}$$

The second Hamiltonian equation of motion which the Hamiltonian $H(g(x),\, p)$ of Eq. (5.1) yields is,

$$dp_\nu/d\tau = -\partial H(g(x),\, p)/\partial x^\nu = (\partial g^{\alpha\beta}(x)/\partial x^\nu)p_\alpha p_\beta/(2m). \tag{5.2c}$$

We next differentiate with respect to τ the first and the third of the equal entities of Eq. (5.2a) to obtain,

$$d^2x^\mu/d\tau^2 = -((\partial g^{\mu\nu}(x)/\partial x^\kappa)(dx^\kappa/d\tau)p_\nu/m) - (g^{\mu\nu}(x)(dp_\nu/d\tau)/m). \tag{5.2d}$$

We now insert the result of Eq. (5.2c) into the second term of Eq. (5.2d) to obtain,

$$d^2x^\mu/d\tau^2 = -((\partial g^{\mu\nu}(x)/\partial x^\kappa)(dx^\kappa/d\tau)p_\nu/m) - (g^{\mu\nu}(x)(\partial g^{\alpha\beta}(x)/\partial x^\nu)p_\alpha p_\beta/(2m^2)). \tag{5.2e}$$

We next eliminate all occurrences of p_γ from Eq. (5.2e) by systematic application of Eq. (5.2b) to obtain,

$$d^2x^\mu/d\tau^2 = [((\partial g^{\mu\nu}(x)/\partial x^\kappa)g_{\nu\lambda}(x)) - \tfrac{1}{2}(g^{\mu\nu}(x)(\partial g^{\alpha\beta}(x)/\partial x^\nu)g_{\alpha\kappa}(x)g_{\beta\lambda}(x))](dx^\kappa/d\tau)(dx^\lambda/d\tau). \tag{5.2f}$$

Eq. (5.2f) is simplified by applying to it the consequence of the partial differentiation of,

$$g^{\gamma\delta}(x)\,g_{\delta\rho}(x) = \delta^\gamma_\rho\,, \tag{5.3a}$$

with respect to x^σ to obtain,

$$(\partial g^{\gamma\delta}(x)/\partial x^\sigma)g_{\delta\rho}(x) = -g^{\gamma\delta}(x)(\partial g_{\delta\rho}(x)/\partial x^\sigma). \tag{5.3b}$$

Two applications of Eq. (5.3b) followed by one of, $g^{\alpha\beta}(x)g_{\alpha\kappa}(x) = \delta^\beta_\kappa$, to Eq. (5.2f) change it to,

$$d^2x^\mu/d\tau^2 = -g^{\mu\nu}(x)[(\partial g_{\nu\lambda}(x)/\partial x^\kappa) - \tfrac{1}{2}(\partial g_{\kappa\lambda}(x)/\partial x^\nu)](dx^\kappa/d\tau)(dx^\lambda/d\tau). \tag{5.4}$$

The Eq. (5.4) result is conventionally presented in the following gravitational geodesic equation form [5],

$$d^2x^\mu/d\tau^2 + \tfrac{1}{2}g^{\mu\nu}(x)[(\partial g_{\nu\lambda}(x)/\partial x^\kappa) + (\partial g_{\nu\kappa}(x)/\partial x^\lambda) - (\partial g_{\kappa\lambda}(x)/\partial x^\nu)](dx^\kappa/d\tau)(dx^\lambda/d\tau) = 0. \tag{5.5}$$

6. The quantization of classical Lorentz-covariant Hamiltonian mechanics

The transition from classical Lorentz-covariant Hamiltonian mechanics to its quantized counterpart converts real-valued functions $F(x,p)$ of the phase space (x,p), such as its components x^μ and p_ν, and also Lorentz-invariant Hamiltonians $H(q,p)$, into Hermitian linear operators $\widehat{F}(\widehat{x},\widehat{p})$ that act on a complex-valued wave function $\psi(x)$ whose argument $x=(x^0,\mathbf{x})$ is the particle's observed four-vector space-time location—we recall that the component x^0 of $x=(x^0,\mathbf{x})$ is equal to ct, where t is the time given by a clock which is at rest with respect to the observer. When the particle's wave function $\psi(x)$ has been normalized such that,

$$\int \psi^*(x)\psi(x)\, d^4x = 1, \tag{6.1a}$$

$\psi^*(x)\psi(x)$ is interpreted as the probability density of the particle's observed space-time location.

The action of a configuration-space operator component $\widehat{x^\mu}$ on a wave function $\psi(x)$ is to simply multiply $\psi(x)$ by the value of the corresponding component x^μ, i.e.,

$$\widehat{x^\mu}\,\psi(x) \stackrel{\text{def}}{=} x^\mu\psi(x). \tag{6.1b}$$

For a normalized wave function $\psi(x)$, the mean values $\langle\widehat{x^\mu}\rangle$ of the four operators $\widehat{x^\mu}$ are of course,

$$\langle\widehat{x^\mu}\rangle = \int \psi^*(x)x^\mu\psi(x)\, d^4x. \tag{6.1c}$$

We next want to define the actions on a wave function $\psi(x)$ of the quantizations $\widehat{p_\nu}$ of the four components p_ν of the particle's conjugate four-momentum p in a way which ensures that *the following complete set of Poisson bracket relations of the components of a classical particle's phase space* (x,p),

$$\{x^\mu,x^\nu\}=0,\ \ \{p_\mu,p_\nu\}=0 \ \text{ and } \ \{x^\mu,p_\nu\}=\delta^\mu_\nu\,, \tag{6.2a}$$

are precisely mirrored by the quantized particle's Hermitian linear-operator phase space $(\widehat{x},\widehat{p})$, i.e.,

$$\{\widehat{x^\mu},\widehat{x^\nu}\}\psi(x)=0,\ \ \{\widehat{p_\mu},\widehat{p_\nu}\}\psi(x)=0 \ \text{ and } \ \{\widehat{x^\mu},\widehat{p_\nu}\}\psi(x)=\delta^\mu_\nu\,\psi(x). \tag{6.2b}$$

Given that $\widehat{x^\mu}\,\psi(x)=x^\mu\psi(x)$, as stated in Eq. (6.1b), and making the further assumption that the Poisson bracket $\{\widehat{o_a},\widehat{o_b}\}$ of any two linear operators $\widehat{o_a}$ and $\widehat{o_b}$ is equal to $k_1[\widehat{o_a}\widehat{o_b}-\widehat{o_b}\widehat{o_a}]$, where k_1 is a constant, implies that $\{\widehat{x^\mu},\widehat{x^\nu}\}\psi(x)=k_1[x^\mu x^\nu - x^\nu x^\mu]\psi(x)=0$, as required by the *first* equality of Eq. (6.2b).

We next apply *to the last two required equalities of* Eq. (6.2b) *the same assumption that* $\{\widehat{o_a},\widehat{o_b}\} = k_1[\widehat{o_a}\widehat{o_b}-\widehat{o_b}\widehat{o_a}]$ *for any two linear operators* $\widehat{o_a}$ *and* $\widehat{o_b}$. Consequently those last two equalities read,

$$\widehat{p_\mu}(\widehat{p_\nu}\psi(x))=\widehat{p_\nu}(\widehat{p_\mu}\psi(x)) \quad\text{and}\quad k_1[x^\mu(\widehat{p_\nu}\psi(x))-\widehat{p_\nu}(x^\mu\psi(x))]=\delta^\mu_\nu\,\psi(x). \tag{6.2c}$$

It is easily verified that *both* Eq. (6.2c) required equalities are satisfied if $\widehat{p_\nu}\psi(x) = -(\partial\psi(x)/\partial x^\nu)/k_1$. In order for the linear operator $\widehat{p_\nu}$ to have the dimension of momentum, the constant k_1 must have the dimension of inverse action, and in order for $\widehat{p_\nu}$ to be a *Hermitian* linear operator, k_1 must be imaginary. Empirical physics additionally requires k_1 to have the particular value $(1/(i\hbar))$, so,

$$\widehat{p_\nu}\psi(x) = -i\hbar(\partial\psi(x)/\partial x^\nu), \tag{6.2d}$$

and the quantum Poisson bracket $\{\widehat{o_a},\widehat{o_b}\}$ of any two linear quantum operators $\widehat{o_a}$ and $\widehat{o_b}$ is given by,

$$\{\widehat{o_a},\widehat{o_b}\}=k_1(\widehat{o_a}\widehat{o_b}-\widehat{o_b}\widehat{o_a})=(\widehat{o_a}\widehat{o_b}-\widehat{o_b}\widehat{o_a})/(i\hbar)=[\widehat{o_a},\,\widehat{o_b}]/(i\hbar), \tag{6.2e}$$

where $[\widehat{o_a},\,\widehat{o_b}] \stackrel{\text{def}}{=} (\widehat{o_a}\widehat{o_b}-\widehat{o_b}\widehat{o_a})$ *is the commutator bracket of the linear quantum operators* $\widehat{o_a}$ *and* $\widehat{o_b}$.

With the quantum Poisson bracket concept well-defined via Eq. (6.2e), we can incorporate into Lorentz-covariant quantum mechanics the concept of differentiating a linear quantum operator $\widehat{o}(\widehat{x},\widehat{p})$ with respect to Lorentz-invariant proper particle time τ, which is given by a clock at rest with respect to the particle instead of at rest with respect to the observer, by applying Eq. (1.9) in the following quantum Poisson bracket form,

$$\begin{aligned} d\widehat{o}(\widehat{x},\widehat{p})(\tau)/d\tau = \{\widehat{o}(\widehat{x},\widehat{p})(\tau),\,\widehat{H}(\widehat{x},\widehat{p})\} = [\widehat{o}(\widehat{x},\widehat{p})(\tau),\,\widehat{H}(\widehat{x},\widehat{p})]/(i\hbar) = \\ (-i/\hbar)(\widehat{o}(\widehat{x},\widehat{p})(\tau)\,\widehat{H}(\widehat{x},\widehat{p}) - \widehat{H}(\widehat{x},\widehat{p})\,\widehat{o}(\widehat{x},\widehat{p})(\tau)), \end{aligned} \tag{6.3a}$$

a first-order linear differential operator equation, which is readily verified to have the solution,

$$\widehat{o}(\widehat{x},\widehat{p})(\tau) = \exp(+i\widehat{H}(\widehat{x},\widehat{p})(\tau-\tau_0)/\hbar)\ \ \widehat{o}(\widehat{x},\widehat{p})(\tau_0)\ \ \exp(-i\widehat{H}(\widehat{x},\widehat{p})(\tau-\tau_0)/\hbar). \tag{6.3b}$$

In the above *quantum Heisenberg picture,* the τ-dependent Hermitian-operator result $\widehat{o}(\widehat{x},\widehat{p})(\tau)$ *parallels* the τ-dependent real-number result $o(x(\tau),p(\tau))$ of *classical* Lorentz-covariant Hamiltonian mechanics.

In the *quantum Schrödinger picture,* the τ-dependence *instead is in wave functions* $\psi(x,\tau)$, where,

$$\psi(x,\tau) \stackrel{\text{def}}{=} \exp(-i\widehat{H}(\widehat{x},\widehat{p})(\tau-\tau_0)/\hbar)\,\psi(x,\tau_0). \tag{6.4a}$$

Differentiating the above Schrödinger-picture wave function $\psi(x,\tau)$ with respect to τ *shows that it satisfies the Schrödinger equation in Lorentz-invariant particle proper time* τ*, i.e.,*

$$i\hbar(d\psi(x,\tau)/d\tau) = \widehat{H}(\widehat{x},\widehat{p})\,\psi(x,\tau). \tag{6.4b}$$

The τ*-dependence,* subsequent to initial τ_0, *of the mean value of a quantum operator* $\widehat{o}(\widehat{x},\widehat{p})$ *is the same in the Schrödinger picture as it is in the Heisenberg picture* since, from Eqs. (6.4a) and (6.3b),

$$\int \psi^*(x,\tau)\ \widehat{o}(\widehat{x},\widehat{p})(\tau_0)\ \psi(x,\tau)\,d^4x = \int \psi^*(x,\tau_0)\ \widehat{o}(\widehat{x},\widehat{p})(\tau)\ \psi(x,\tau_0)\,d^4x. \tag{6.4c}$$

Working out exact or approximate wave-function solutions of the Eq. (6.4b) Schrödinger equation is usually the most advantageous way to tackle problems in quantum mechanics.

In the next section we undertake the quantization of the Eq. (4.1) Lorentz-invariant Hamiltonian for a charged particle in an electromagnetic four-potential in the special case of the hydrogen atom whose electron has no spin. The static character of the relativistic Hamiltonian in that case precipitates a puzzling ambiguity which is resolved by noting that *the* $c\to\infty$ *limit* of that static relativistic Hamiltonian is required to be the unambiguous *nonrelativistic* Hamiltonian for such a hydrogen atom without spin, i.e.,

$$\widehat{H}_{\text{NR}}(\widehat{\mathbf{x}},\widehat{\mathbf{p}},m,e^2) = (|\widehat{\mathbf{p}}|^2/(2m)) - (e^2/|\widehat{\mathbf{x}}|). \tag{6.5}$$

7. The relativistic Hamiltonian for the hydrogen atom without spin

The Eq. (4.1) Lorentz-invariant Hamiltonian for the motion of a charged particle in an electromagnetic four-potential has terms which are *quadratic* in that potential, but *no* quadratic potential effects occur in the charged particle's Eq. (4.2d) Lorentz-covariant classical equation of motion, nor are there quadratic potential terms in the charged particle's Lorentz-invariant Lagrangian. Quantum mechanics, however, *is based on the Hamiltonian, not* on the equation of motion or the Lagrangian, *and in the special case of the hydrogen atom without spin, where* $(eA_0(x)) = -(e^2/|\mathbf{x}|)$ *and* $A_i(x) = 0$ *for* $i = 1,2,3$, the Lorentz-invariant Eq. (4.1) Hamiltonian $\widehat{H}(A(\widehat{x}),\widehat{p})$ *has a quadratic term in* $(eA_0(x)) = -(e^2/|\mathbf{x}|)$. That Hamiltonian in detail is,

$$\begin{aligned}\widehat{H}(\widehat{\mathbf{x}},(\widehat{p_0},\widehat{\mathbf{p}}),m,e^2,c) = \big(mc^2/2\big) - \big(\big(\widehat{p_0} - \big(e^2/(c|\widehat{\mathbf{x}}|)\big)\big)^2/(2m)\big) + \big(|\widehat{\mathbf{p}}|^2/(2m)\big) = \\ \big(|\widehat{\mathbf{p}}|^2/(2m)\big) + \big((\widehat{p_0}/(mc))(e^2/|\widehat{\mathbf{x}}|)\big) - \big((e^2/|\widehat{\mathbf{x}}|)^2/(2mc^2)\big) + \big((mc^2/2)\big(1-(\widehat{p_0}/(mc))^2\big)\big) = \\ \big(|\widehat{\mathbf{p}}|^2/(2m)\big) + \big(\widehat{\varphi}(e^2/|\widehat{\mathbf{x}}|)\big) - \big((e^2/|\widehat{\mathbf{x}}|)^2/(2mc^2)\big) + \big((mc^2/2)\big(1-(\widehat{\varphi})^2\big)\big),\end{aligned} \tag{7.1}$$

where $\widehat{\varphi} \stackrel{\text{def}}{=} (\widehat{p_0}/(mc))$ is a dimensionless Hermitian operator that *commutes* with $\widehat{\mathbf{x}}$, $\widehat{\mathbf{p}}$, $\widehat{p_0}$ and the full Eq. (7.1) Hamiltonian $\widehat{H}(\widehat{\mathbf{x}},(\widehat{p_0},\widehat{\mathbf{p}}),m,e^2,c)$. Therefore the value of $\widehat{\varphi}$ in the last Eq. (7.1) representation of the full Hamiltonian $\widehat{H}(\widehat{\mathbf{x}},(\widehat{p_0},\widehat{\mathbf{p}}),m,e^2,c)$ could be any dimensionless real number. However that full Hamiltonian *is constrained* by the fact that its $c\to\infty$ nonrelativistic limit is the Eq. (6.5) *nonrelativistic Hamiltonian for the hydrogen atom without spin, i.e.,*

$$\widehat{H}_{\text{NR}}(\widehat{\mathbf{x}},\widehat{\mathbf{p}},m,e^2) = (|\widehat{\mathbf{p}}|^2/(2m)) - (e^2/|\widehat{\mathbf{x}}|). \tag{7.2}$$

Examination of the last Eq. (7.1) representation of the full Hamiltonian $\widehat{H}(\widehat{\mathbf{x}},(\widehat{p_0},\widehat{\mathbf{p}}),m,e^2,c)$ *shows that the only value of* $\widehat{\varphi}$ *which is compatible with this* $c\to\infty$ *nonrelativistic-limit requirement is* $\widehat{\varphi} = -1$*, or equivalently,* $\widehat{p_0} = -(mc)$, so the full Hamiltonian of Eq. (7.1) for the hydrogen atom without spin *must be,*

$$\widehat{H}(\widehat{\mathbf{x}},(-(mc),\widehat{\mathbf{p}}),m,e^2,c) = (|\widehat{\mathbf{p}}|^2/(2m)) - (e^2/|\widehat{\mathbf{x}}|) - ((e^2/|\widehat{\mathbf{x}}|)^2/(2mc^2)). \tag{7.3}$$

A very weak short-range relativistic complement $-((e^2/|\widehat{\mathbf{x}}|)^2/(2mc^2))$ to the hydrogen atom's Coulomb potential $-(e^2/|\widehat{\mathbf{x}}|)$ has emerged in Eq. (7.3). Although its effects seem negligible for most purposes, the growth of its effects at short range might affect studies of the proton's charge radius [6].

The 1926-1928 efforts of Schrödinger, Gordon, Klein and Dirac to formulate a relativistic version of quantum mechanics took two debatable ideas for granted [7], (1) time is universal, and (2) the time component of the four-momentum mutiplied by c is to be engineered into becoming the Hamiltonian, e.g.,

$$H = \sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2} - (e^2/|\mathbf{x}|).$$

The issue of the Lorentz covariance of the Hamiltonian equations of particle motion wasn't addressed, and it seems clear that Hamiltonians along the lines of that above, and offshoots of such, prevent Lorentz covariance of the equations of particle motion (e.g., the Dirac particle's *Zitterbewegung* with speed $\sqrt{3}\,c$). It seems arguable that Schrödinger, Gordon, Klein and Dirac ought to have taken explicit Lorentz covariance of the equations of particle motion, and also the nonexistence of universal time, as key theoretical guidelines.

8. Hamiltonians with both the relativistic complement to the Coulomb potential and spin

To account for the electromagnetic effect of the electron's spin, an adequate relativistic upgrade of the Eq. (7.3) Hamiltonian for the hydrogen atom without spin would be to add to it a term of the form,

$$\widehat{H}_{\mathrm{spin}}(\widehat{\mathbf{x}},\widehat{\mathbf{p}},\vec{\sigma}) = -(e\hbar/(8mc))\big[F_{\mu\nu}(\mathbf{E}(\widehat{\mathbf{x}}))\Lambda^{\mu}_{\alpha}(\widehat{\mathbf{p}}/m)\Lambda^{\nu}_{\beta}(\widehat{\mathbf{p}}/m) + \Lambda^{\mu}_{\alpha}(\widehat{\mathbf{p}}/m)\Lambda^{\nu}_{\beta}(\widehat{\mathbf{p}}/m)F_{\mu\nu}(\mathbf{E}(\widehat{\mathbf{x}}))\big]\sigma^{\alpha\beta}(\vec{\sigma}), \quad (8.1)$$

where $F_{\mu\nu}(\mathbf{E}(\widehat{\mathbf{x}}))$ is the covariant antisymmetric electromagnetic tensor corresponding to the proton's electric field $\mathbf{E}(\widehat{\mathbf{x}}) = e(\widehat{\mathbf{x}}/|\widehat{\mathbf{x}}|^3)$, $\sigma^{\alpha\beta}(\vec{\sigma})$ is the contravariant antisymmetric spin-matrix tensor corresponding to the Pauli spin-matrix three-vector $\vec{\sigma}$, and $\Lambda^{\mu}_{\alpha}(\widehat{\mathbf{p}}/m)$ is the Lorentz-transformation tensor of velocity $(\widehat{\mathbf{p}}/m)$.

Alternatively, the relativistic complement $-((e^2/|\widehat{\mathbf{x}}|)^2/(2mc^2))$ to the Coulomb potential can simply be added to the Dirac Hamiltonian for the hydrogen atom, i.e.,

$$\widehat{H}_{\mathrm{D}}(\widehat{\mathbf{x}},\widehat{\mathbf{p}}) = c\,\vec{\alpha}\cdot\widehat{\mathbf{p}} + \beta mc^2 - (e^2/|\widehat{\mathbf{x}}|), \quad (8.2)$$

to produce,

$$\widehat{H}_{\mathrm{D+RC}}(\widehat{\mathbf{x}},\widehat{\mathbf{p}}) = c\,\vec{\alpha}\cdot\widehat{\mathbf{p}} + \beta mc^2 - (e^2/|\widehat{\mathbf{x}}|) - ((e^2/|\widehat{\mathbf{x}}|)^2/(2mc^2)), \quad (8.3)$$

since the Dirac Hamiltonian accounts adequately for the electromagnetic effect of the electron's spin.

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