

ON THE PRODUCT DISTRIBUTION ON RUNS IN A CHAIN

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ABSTRACT. Let

$$E(n) : 1 = s_0 < s_1 = 2 < \dots < s_h = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. We study the distribution of the logarithmic partial sum $\sum \log(s_i)$ on maximal consecutive steps of a given type.

1. PRELIMINARIES AND SETUP

Let $l(n)$ be the length of an addition chain leading to n , denoted $E(n)$, of the form

$$E(n) : s_0 = 1, s_1 = 2, \dots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$ such that $l(n) := \beta(m)$. By adapting the ideas of the paper [1], we partition the steps in an addition chain into the following classes of steps

$$\mathcal{A} := \{i : s_i = 2s_{i-1}\} \quad (\text{doubling steps})$$

$$\mathcal{B} := \{i : \gamma s_{i-1} \leq s_i < 2s_{i-1}\} \quad (\text{large steps})$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden ratio*

$$\mathcal{C} := \{i : (1 + \delta)s_{i-1} \leq s_i < \gamma s_{i-1}\} \quad (\text{medium - sized steps})$$

where $\delta := \delta(m) \rightarrow 0$ as $m \rightarrow \infty$. In particular

$$\delta := \delta(m) = \frac{1}{\log m}$$

$$\mathcal{D} := \{i : s_i < (1 + \delta)s_{i-1}\} \quad (\text{small steps}).$$

We denote the cardinality of the sets to be

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

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We call steps in $\mathcal{B}, \mathcal{C}, \mathcal{D}$ as *non-doubling steps*. We have therefore the relation

$$A + B + C + D = \beta(m).$$

Because the non-doubling steps in an addition chain cannot grow faster than a Fibonacci sequence, we have (by induction) the inequality

$$2^m \leq n \leq 2^A \gamma^{B+C+D} = 2^{\beta(m)} \left(\frac{\gamma}{2}\right)^{B+C+D}$$

and we deduce from this relation an upper control for the total number of non-doubling steps in an addition chain of length $\beta(m)$ to be

Lemma 1.1. *Put*

$$E(n) : s_0 = 1, s_1 = 2, \dots, s_{l(n)} = n$$

be an addition chain with $l(n) := \beta(m)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain of length $\beta(m)$ with cardinality A, B, C, D , respectively. Then we have

$$B + C + D \leq \frac{\beta(m) - m}{1 - \log_2 \gamma}.$$

It turns out that the non-doubling steps in an addition chain have certain structural pattern.

Lemma 1.2. *If $j \in \mathcal{B}$, then $j - 1 \in \mathcal{C} \cup \mathcal{D}$. In particular, each large step in an addition chain must be preceded by either a small step or a medium-sized step.*

Proof. Let $j \in \mathcal{B}$ (large step) then we have by definition

$$\gamma s_{j-1} \leq s_j < 2s_{j-1}$$

where $\gamma := \frac{1+\sqrt{5}}{2}$ is the *golden ratio*. Write $s_j = s_k + s_l$ with $k \geq l$. The inequality $s_j = s_k + s_l \leq s_{j-1} + s_l$ with $s_j < 2s_{j-1}$ implies that

$$\gamma s_{j-1} \leq s_j \leq s_{j-1} + s_{j-2}$$

which further implies

$$(\gamma - 1)s_{j-1} < s_{j-2} \iff s_{j-1} < \gamma s_{j-2}$$

since $\gamma = \frac{1}{\gamma-1}$. This proves $j - 1 \in \mathcal{C} \cup \mathcal{D}$. □

1.1. The notion of a run and product distribution. We begin this section with the following definition.

Definition 1.3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be steps in an addition chain as in the setup. We call a **maximal** consecutive sequence of steps of a given type

$$j_1 < j_2 < \cdots < j_d$$

such that $j_1 - 1, j_d + 1$ cannot be a step of the given type a **run** of the step type. We call the number of terms in the *run* the **run length**.

Example 1.4. Consider the addition chain

$$E(n) : 1 = s_0, s_1 = 2, s_2 = 3, s_4 = 10, s_5 = 20, s_6 = 23.$$

Here, we observe that $2^4 < 23 < 2^5$ so that $m := 4$ and $\delta := \delta(m) = \frac{1}{\log m}$. Furthermore, we observe that $s_1 = 2s_0, s_2 < s_1(1 + \delta), s_2 \cdot \gamma < s_3 < s_2 \cdot 2, s_4 = 2s_3, s_5 = 2s_4, s_6 < 23(1 + \delta)$. Thus, $\{1, 4, 5\}$ are steps of type \mathcal{A} , $\{2, 6\}$ are steps of type \mathcal{D} and $\{3\}$ is the only step of type \mathcal{B} . We observe that $1 < 2$ is a consecutive sequence of steps of type $\mathcal{A} \cup \mathcal{D}$. Similarly, $4 < 5 < 6$ is a consecutive sequence of steps of type $\mathcal{A} \cup \mathcal{D}$. Since these are the only consecutive sequences of steps of type $\mathcal{A} \cup \mathcal{D}$ in the addition chain and the sequence $4 < 5 < 6$ contains more steps of type $\mathcal{A} \cup \mathcal{D}$ than $1 < 2$, it follows that the sequence of consecutive steps

$$4 < 5 < 6$$

is a run of type $\mathcal{A} \cup \mathcal{D}$. Also, $2 < 3$ is the only sequence of consecutive steps of type $\mathcal{B} \cup \mathcal{D}$. Thus, $2 < 3$ is a run of type $\mathcal{B} \cup \mathcal{D}$. Furthermore, we observe that $3 < 4 < 5$ is the only consecutive sequence of steps of type $\mathcal{A} \cup \mathcal{B}$ so that

$$3 < 4 < 5$$

is a run of type $\mathcal{A} \cup \mathcal{B}$.

Lemma 1.2 hints at the core idea that a **run** of step type $\mathcal{C} \cup \mathcal{B}$ or $\mathcal{D} \cup \mathcal{B}$ will always appear among the non-doubling steps in any addition chain, whether or not optimal. More likely it is for chains that are not optimal to have many **run** of types $\mathcal{C} \cup \mathcal{B}$. We begin with the following Lemmas

In the sequel, we develop bounds for the distribution function $\sum \log(s_i)$ on a run of various types in an addition chain leading to n of the form

$$E(n) : s_0 = 1, s_1 = 2, \dots, s_{l(n)} = n$$

with $2^m \leq n < 2^{m+1}$ such that $l(n) := \beta(m)$.

Lemma 1.5. *Let*

$$E(n) : s_0 = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type \mathcal{A} . Then*

$$\frac{d(d+1)}{2} \log 2 \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Let

$$j_1 < \cdots < j_d$$

be **run** of a step of type \mathcal{A} . By induction, we can write

$$2^{d+\beta_o} \geq s_{j_d} = 2s_{j_{d-1}} \cdots = 2^{d-1}s_{j_1} \geq 2^d$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, we can write

$$2^{d-1+\beta_1} \geq s_{j_{d-1}} = 2s_{j_{d-2}} \cdots = 2^{d-2}s_{j_1} \geq 2^{d-1}$$

for some $\beta_1 := \beta_1(d) > 0$. By induction, we write

$$2^{d-r+\beta_r} \geq s_{j_{d-r}} = 2s_{j_{d-r-1}} \cdots = 2^{d-r-1}s_{j_1} \geq 2^{d-r}$$

for any $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. Setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o := \alpha_o(d)$, we get

$$(\log 2) \sum_{s=1}^d s \leq \sum_{k=1}^d \log(s_{j_k}) \leq (\log 2) \sum_{s=1}^d (s + \alpha_o).$$

□

Lemma 1.6. *Let*

$$E(n) : s_0 = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type \mathcal{C} . Then*

$$\frac{d(d+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log \gamma + d\alpha_o \log \gamma$$

for some $\alpha_o := \alpha_o(d) > 0$, where $\delta := \delta(m) = \frac{1}{\log m}$ and $\gamma := \frac{1+\sqrt{5}}{2}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type \mathcal{C} . We can write

$$(1 + \delta)s_{j_{d-1}} \leq s_{j_d} < \gamma s_{j_{d-1}}$$

so that by induction, we deduce further

$$(1 + \delta)^d \leq s_{j_d} < \gamma^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, we obtained (by induction) the inequality

$$(1 + \delta)^{d-1} \leq s_{j_{d-1}} < \gamma^{d-1+\beta_1}$$

for $\beta_1 := \beta_1(d) > 0$. We get

$$(1 + \delta)^{d-r} \leq s_{j_{d-r}} < \gamma^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We obtain for the lower bound

$$\sum_{k=1}^d \log(s_{j_k}) \geq (\log(1 + \delta)) \sum_{s=1}^d s \geq \delta \frac{d(d+1)}{4}$$

with $\delta := \delta(m) = \frac{1}{\log m}$. By setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$, we get for the upper bound

$$\sum_{k=1}^d \log(s_{j_k}) < \sum_{s=1}^d (s + \alpha_o) \log \gamma = \frac{d(d+1)}{2} \log \gamma + d\alpha_o \log \gamma.$$

□

Lemma 1.7. *Let*

$$E(n) : s_0 = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type $\mathcal{B} \cup \mathcal{C}$. Then

$$\frac{d(d+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2$$

for some $\alpha_o := \alpha_o(d) > 0$ and $\gamma := \frac{1+\sqrt{5}}{2}$ and $\delta := \delta(m) = \frac{1}{\log m}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{B} \cup \mathcal{C}$. Then we can write

$$(1 + \delta)s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1}$$

so that by induction, we get

$$(1 + \delta)^d \leq s_{j_d} < 2^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, by induction, we can write

$$(1 + \delta)^{d-1} \leq s_{j_d-1} < 2^{d-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Thus

$$(1 + \delta)^{d-r} \leq s_{j_d-r} < 2^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. By setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$, we get for the logarithmic sum

$$\sum_{k=1}^d \log(s_{j_k}) \leq \sum_{s=1}^d (s + \alpha_o) \log 2 = \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2.$$

Similarly for the lower bound of the logarithmic sum, we get

$$\sum_{k=1}^d \log(s_{j_k}) > \sum_{s=1}^d s \log(1 + \delta) \geq \frac{d(d+1)}{4} \delta$$

where $\delta := \delta(m) = \frac{1}{\log m}$. □

Lemma 1.8. *Let*

$$E(n) : s_0 = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type $\mathcal{D} \cup \mathcal{C}$. Then

$$\frac{d'(d'+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log \gamma + d\alpha_o \log \gamma$$

for some $\alpha_o := \alpha_o(d) > 0$ where d' is the number of steps of type \mathcal{C} in the run and $\gamma := \frac{1+\sqrt{5}}{2}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{D} \cup \mathcal{C}$. Then we can write

$$(1 + \delta)s_{j_{d-1}} \leq s_{j_d} < \gamma s_{j_{d-1}}$$

if $j_d \in \mathcal{C}$. On the other hand, we get

$$s_{j_d} < (1 + \delta)s_{j_{d-1}}$$

if $j_d \in \mathcal{D}$. By induction, we can write

$$(1 + \delta)^{d'} \leq s_{j_d} < \gamma^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$ and where d' is the number of steps of type \mathcal{C} in the run. Similarly, we obtain (by induction) the inequality

$$(1 + \delta)^{d'-1} \leq s_{j_{d-1}} < \gamma^{d-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Thus

$$(1 + \delta)^{d'-r} \leq s_{j_{d-r}}$$

for $0 \leq r \leq d' - 1$ and

$$s_{j_{d-r}} < \gamma^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ with $\beta_r := \beta_r(d) > 0$. By setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$, we get for the logarithmic sum

$$\sum_{k=1}^d \log(s_{j_k}) \leq \sum_{s=1}^d (s + \alpha_o) \log \gamma = \frac{d(d+1)}{2} \log \gamma + d\alpha_o \log \gamma.$$

We the lower bound

$$\sum_{k=1}^d \log(s_{j_k}) > \sum_{s=1}^{d'} s \log(1 + \delta) \geq \frac{d'(d'+1)}{4} \delta$$

where $\delta := \delta(m) = \frac{1}{\log m}$. □

Lemma 1.9. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type \mathcal{D} . Then*

$$\sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2 \log m} + \frac{d\alpha_o}{\log m}$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type \mathcal{D} . Then we write

$$s_{j_d} < (1 + \delta)s_{j_{d-1}}$$

so that (by induction), we can write

$$s_{j_d} < (1 + \delta)^{d+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$. Similarly, we obtain (by induction) the inequality

$$s_{j_{d-1}} < (1 + \delta)^{d-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Thus

$$s_{j_{d-r}} < (1 + \delta)^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We deduce for the logarithmic sum

$$\sum_{k=1}^d \log(s_{j_k}) < \sum_{s=1}^d (s + \alpha_o) \log(1 + \delta) \leq \frac{d(d+1)}{2} \delta + d\alpha_o \delta$$

where $\alpha_o = \max\{\beta_r\}_{r=0}^{d-1}$, where $\delta := \delta(m) = \frac{1}{\log m}$. \square

Lemma 1.10. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type $\mathcal{A} \cup \mathcal{B}$. Then

$$\frac{d(d+1)}{2} \log \gamma \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2$$

for some $\alpha_o := \alpha_o(d) > 0$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{A} \cup \mathcal{B}$. If $j_d \in \mathcal{A}$, then we can write

$$s_{j_d} = 2s_{j_{d-1}}$$

so that (by induction) we obtain the inequality

$$\gamma^{d''} 2^{d'} \leq s_{j_d} \leq 2^{d''+d'+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$, where d' is the number of steps of type \mathcal{A} in the run. Again, if $j_d - 1 \in \mathcal{A}$, then we get (by induction) the inequality

$$2^{d'-1} \gamma^{d''} \leq s_{j_d-1} \leq 2^{d''+d'-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. On the other hand, if $j_d \in \mathcal{B}$, then we write

$$\gamma s_{j_d-1} \leq s_{j_d} < 2s_{j_d-1}$$

so that by induction, we get

$$\gamma^{d'+d''} \leq s_{j_d} < 2^{d'+d''+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$, and where d'' is the number of steps of type \mathcal{B} in the run. Similarly, if $j_d - 1 \in \mathcal{B}$, then we write (by induction)

$$\gamma^{d'+d''-1} \leq s_{j_d-1} < 2^{d'+d''-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Combining both cases, we can write

$$\gamma^{d-r} \leq s_{j_d-r} < 2^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We deduce

$$\sum_{k=1}^d \log(s_{j_k}) \geq \sum_{s=1}^d s \log \gamma = \frac{d(d+1)}{2} \log \gamma$$

For the upper bound, we set $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$ and obtain

$$\sum_{k=1}^d \log(s_{j_k}) \leq \sum_{s=1}^d (s + \alpha_o) \log 2 = \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2.$$

□

Lemma 1.11. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)}$$

be an addition chain leading to n with $\beta(m) \geq m$ and $2^m \leq n < 2^{m+1}$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the set-up, with $\#\mathcal{A} = A, \#\mathcal{B} = B, \#\mathcal{C} = C, \#\mathcal{D} = D$. Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{A} \cup \mathcal{C}$. Then*

$$\frac{d(d+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2$$

for some $\alpha_o := \alpha_o(d) > 0$, where $\gamma := \frac{1+\sqrt{5}}{2}$.

Proof. Consider a run

$$j_1 < \cdots < j_d$$

of a step of type $\mathcal{A} \cup \mathcal{C}$. If $j_d \in \mathcal{A}$, then we can write

$$s_{j_d} = 2s_{j_d-1}$$

so that (by induction) we obtain the inequality

$$(1 + \delta)^{d''} 2^{d'} \leq s_{j_d} \leq \gamma^{d''} 2^{d'+\beta_o}$$

for some $\beta_o := \beta_o(d) > 0$, where d' is the number of steps of type \mathcal{A} in the run. Again, if $j_d - 1 \in \mathcal{A}$, then we get (by induction) the inequality

$$(1 + \delta)^{d''} 2^{d'-1} \leq s_{j_d-1} \leq \gamma^{d''} 2^{d'-1+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. On the other hand, if $j_d \in \mathcal{C}$ then

$$(1 + \delta)s_{j_d-1} \leq s_{j_d} < \gamma s_{j_d-1}$$

so that (by induction)

$$(1 + \delta)^{d''} 2^{d'} \leq s_{j_d} < 2^{d'+\beta_o} \gamma^{d''}$$

for some $\beta_o := \beta_o(d) > 0$ and where d'' is the number of steps of type \mathcal{C} in the run. Similarly, if $j_d - 1 \in \mathcal{C}$ then

$$2^{d'} (1 + \delta)^{d''-1} \leq s_{j_d-1} < \gamma^{d''-1} 2^{d'+\beta_1}$$

for some $\beta_1 := \beta_1(d) > 0$. Combining both cases, we can write

$$(1 + \delta)^{d-r} \leq s_{j_d-r} < 2^{d-r+\beta_r}$$

for $0 \leq r \leq d-1$ and $\beta_r := \beta_r(d) > 0$. We deduce for the lower bound

$$\sum_{k=1}^d \log(s_{j_k}) \geq \sum_{s=1}^d s \log(1 + \delta) \geq \frac{d(d+1)}{4} \delta$$

where $\delta := \delta(m) = \frac{1}{\log m}$. Setting $\max\{\beta_r\}_{r=0}^{d-1} = \alpha_o$, we deduce for the upper bound

$$\sum_{k=1}^d \log(s_{j_k}) < \sum_{s=1}^d (s + \alpha_o) \log 2 = \frac{d(d+1)}{2} \log 2 + d\alpha_o \log 2.$$

□

Remark 1.12. From now on, we fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D$$

for an addition chain

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

leading to $n \in [2^m, 2^{m+1})$. Consequently, the numbers $c, c' > 0$ appearing in the theorems are absolute constants, because once the proportion of each step type is fixed, these do not vary with m . We establish explicit bounds for logarithmic partial sums of terms in an addition chain in a **run** of any type. In contrast, each appearance of α_o in the lemmas depended only on d , while each α_o in the theorem depends only on m .

Theorem 1.13. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the setup and fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{C} \cup \mathcal{D}$. Then*

$$\sum_{k=1}^d \log(s_{j_k}) \geq \frac{(c')^2}{4 \log m} (\beta(m))^2 + \frac{c'}{4 \log m} \beta(m)$$

and

$$\sum_{k=1}^d \log(s_{j_k}) \leq \frac{c^2 \log \gamma}{2} (\beta(m))^2 + \left(\frac{c \log \gamma}{2} + c \alpha_o \log \gamma \right) \beta(m)$$

for some fixed $c', c > 0$ and $\alpha_o := \alpha_o(m) > 0$.

Proof. Consider a **run** of type $\mathcal{C} \cup \mathcal{D}$ of the form

$$j_1 < \cdots < j_d.$$

Lemma 1.8 gives

$$\frac{d'(d'+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log \gamma + d \alpha_o \log \gamma$$

for some $\alpha_o := \alpha_o(d) > 0$ where d' is the number of steps of type \mathcal{C} in the run and $\gamma := \frac{1+\sqrt{5}}{2}$. Since $\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type $\mathcal{C} \cup \mathcal{D}$, there exist fixed constants $c', c > 0$ such that $d' = c' \beta(m)$ and $d = c \beta(m)$. \square

Theorem 1.14. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the setup and fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

*be a **run** of a step of type $\mathcal{B} \cup \mathcal{C}$. Then*

$$\sum_{k=1}^d \log(s_{j_k}) \geq \frac{c^2}{4 \log m} (\beta(m))^2 + \frac{c}{4 \log m} \beta(m)$$

and

$$\sum_{k=1}^d \log(s_{j_k}) \leq \frac{c^2 \log 2}{2} (\beta(m))^2 + \left(\frac{c \log 2}{2} + c \alpha_o \log 2 \right) \beta(m)$$

for some $c > 0$.

Proof. Consider a **run** of type $\mathcal{B} \cup \mathcal{C}$ of the form

$$j_1 < \cdots < j_d.$$

Lemma 1.7 gives

$$\frac{d(d+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log 2 + d \alpha_o \log 2$$

for some $\alpha_o := \alpha_o(d) > 0$ and $\gamma := \frac{1+\sqrt{5}}{2}$ and $\delta := \delta(m) = \frac{1}{\log m} \dots$. Since $\#\mathcal{A} := A$, $\#\mathcal{B} = B$, $\#\mathcal{C} = C$, $\#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type $\mathcal{B} \cup \mathcal{D}$, there exists a fixed constant $c > 0$ such that $d = c\beta(m)$. \square

Theorem 1.15. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the setup and fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type \mathcal{C} . Then

$$\sum_{k=1}^d \log(s_{j_k}) \geq \frac{c^2(\beta(m))^2}{4 \log m} + \frac{c\beta(m)}{4 \log m}$$

and

$$\sum_{k=1}^d \log(s_{j_k}) \leq \frac{c^2}{2}(\beta(m))^2 \log \gamma + \beta(m) \left(\frac{c}{2} \log \gamma + c\alpha_o \log \gamma \right)$$

for some fixed constant $c > 0$.

Proof. Consider a **run** of type \mathcal{C} of the form

$$j_1 < \cdots < j_d.$$

Lemma 1.6 gives

$$\frac{d(d+1)}{4 \log m} \leq \sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2} \log \gamma + d\alpha_o \log \gamma$$

for some $\alpha_o := \alpha_o(d) > 0$, where $\delta := \delta(m) = \frac{1}{\log m}$ and $\gamma := \frac{1+\sqrt{5}}{2}$. Since $\#\mathcal{A} := A$, $\#\mathcal{B} = B$, $\#\mathcal{C} = C$, $\#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type \mathcal{C} , there exists a fixed constant $c > 0$ such that $d = c\beta(m)$. \square

Theorem 1.16. *Let*

$$E(n) : s_o = 1 < s_1 = 2 < \cdots < s_{\beta(m)} = n$$

be an addition chain leading to $n \in [2^m, 2^{m+1})$. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four types of steps as in the setup and fix

$$\#\mathcal{A} := A, \quad \#\mathcal{B} = B, \quad \#\mathcal{C} = C, \quad \#\mathcal{D} = D.$$

Furthermore, let

$$j_1 < \cdots < j_d$$

be a **run** of a step of type \mathcal{D} . Then

$$\sum_{k=1}^d \log(s_{j_k}) \leq \frac{c^2}{2 \log m} (\beta(m))^2 + \beta(m) \left(\frac{c}{2 \log m} + \frac{c\alpha_o}{\log m} \right)$$

Proof. Consider a **run** of type \mathcal{D} of the form

$$j_1 < \cdots < j_d.$$

Lemma 1.9 gives

$$\sum_{k=1}^d \log(s_{j_k}) \leq \frac{d(d+1)}{2 \log m} + \frac{d\alpha_o}{\log m}$$

for some $\alpha_o := \alpha_o(d) > 0$. Since $\#\mathcal{A} := A$, $\#\mathcal{B} = B$, $\#\mathcal{C} = C$, $\#\mathcal{D} = D$ are each fixed and each term in this run is distributed in the step type \mathcal{D} , there exists a fixed constant $c > 0$ such that $d = c\beta(m)$. \square

REFERENCES

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