Characterization of Perspective Affinities that Preserve Arc Length and Curvature

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Abstract

In this work, a precise characterization is given of those perspective affine reflections that preserve both the arc length and the magnitude of curvature for any smooth planar curve. The result shows that the only non-trivial transformation with this property is the orthogonal reflection with respect to the given axis, i.e., the perpendicular affine symmetry with characteristic constant $k = -1$.

1 Introduction

Perspective affine reflections play an important role in descriptive and affine geometry. In this work, we prove that preserving metric quantities (arc length and curvature) uniquely selects the orthogonal symmetry(and k=-1) as the only non-trivial reflection. This is a novel result, to the author's knowledge not previously published.

2 Theoretical Preparations

Let us work in the Euclidean plane $\mathbb{R}^2$. A regular twice-differentiable curve is given by $\gamma : I \to \mathbb{R}^2$, $t \mapsto \gamma(t) = (x(t), y(t))$, with $\gamma'(t) \neq 0$.

Arc length element.

$$ds = \|\gamma'(t)\| \, dt, \qquad \|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

Curvature.

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{\big((x'(t))^2 + (y'(t))^2\big)^{3/2}} = \frac{\|\gamma'(t) \wedge \gamma''(t)\|}{\|\gamma'(t)\|^3}. \tag{1}$$

If we consider the signed curvature:

$$\kappa_{\text{sgn}}(t) = \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3},$$

the sign depends on the local orientation of the parametrization.

Perspective Affinity. A perspective affinity is defined by a fixed axis s and a direction of rays S_∞ (with characteristic constant $k \neq 0$). In affine coordinates adapted to $s = \{y = 0\}$, a typical example of an affinity with parallel rays has the linear form:

$$A_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Orthogonal reflection with respect to s has the matrix:

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R^\top R = I, \quad \det R = -1.$$

The conditions $S_\infty \perp s$ and $k = -1$ identify this second case.

3 Main Lemma

Lemma 1 (Transformation of ds and κ)**.** *Let $F(x) = Ax + b$ be an affine map with linear part $A \in GL(2, \mathbb{R})$. For $\tilde{\gamma} = F \circ \gamma$, we have:*

$$d\tilde{s} = \|A\gamma'(t)\|\, dt, \qquad \kappa_{\tilde{\gamma}}(t) = \frac{|\det A|}{\|A\gamma'(t)\|^3}\, \kappa_\gamma(t)\, \|\gamma'(t)\|^3.$$

Proof. Since $\tilde{\gamma}' = A\gamma'$ and $\tilde{\gamma}'' = A\gamma''$, we get $d\tilde{s} = \|A\gamma'\|\, dt$. For the curvature, the pseudoproduct transforms as $\|A\gamma' \wedge A\gamma''\| = |\det A| \cdot \|\gamma' \wedge \gamma''\|$. □

Lemma 2 (Condition for metric preservation)**.** *If F preserves arc length for every curve, then $A^\top A = I$, i.e., A is an orthogonal matrix and $|\det A| = 1$. Consequently, from Lemma 1 it follows that $|\kappa|$ is preserved only in this case.*

4 Main Theorem

Theorem 1 (Florian's Characterization)**.** *In a perspective affine reflection with axis s, ray direction S_∞, and characteristic constant k, the arc length and* magnitude *of curvature $|\kappa|$ are preserved* if and only if*:*

$$S_\infty \perp s, \qquad k = -1,$$

i.e., the transformation is the orthogonal reflection with respect to s.

Proof. ($\Rightarrow$) From Lemma 2, the linear part A must be orthogonal with $|\det A| = 1$. Orthogonal matrices that leave s invariant are the identity I (trivial case) and the reflection R (non-trivial case). The latter requires $S_\infty \perp s$ and $k = -1$.

($\Leftarrow$) If $S_\infty \perp s$ and $k = -1$, the affinity is an orthogonal reflection, with $R^\top R = I$ and $|\det R| = 1$. By Lemma 1, ds and $|\kappa|$ are preserved. The signed curvature changes sign, as expected from the orientation reversal. □

5 Conclusions and Remarks

The result uniquely selects the orthogonal reflection as the perspective affine map preserving metric magnitudes. Any deviation from $S_\infty \perp s$ or $k = -1$ causes shear or anisotropic stretching, systematically changing ds and κ.

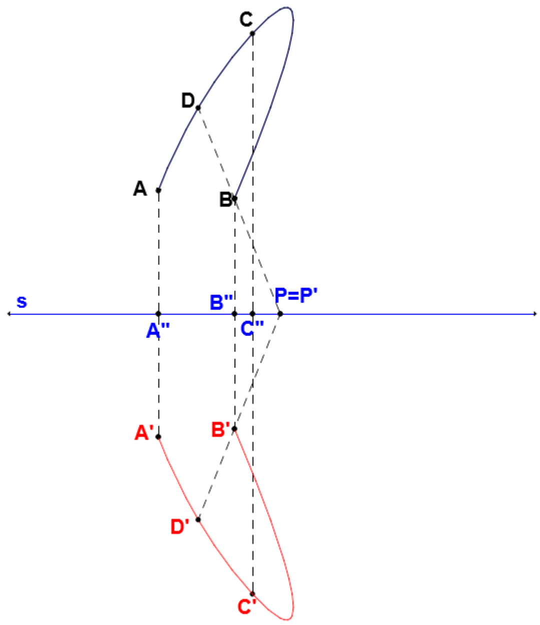
Remark 1. *In descriptive geometry terminology, the conditions $S_\infty \perp s$ and $k = -1$ describe the perpendicular affine symmetry – the only one preserving both arc length and curvature.*

6 Examples and Exercises

Exercise 99

Present the perspective affine reflection of a given curve, assuming:

- the axis of affinity s,
- curvature between two points of the curve is equal.

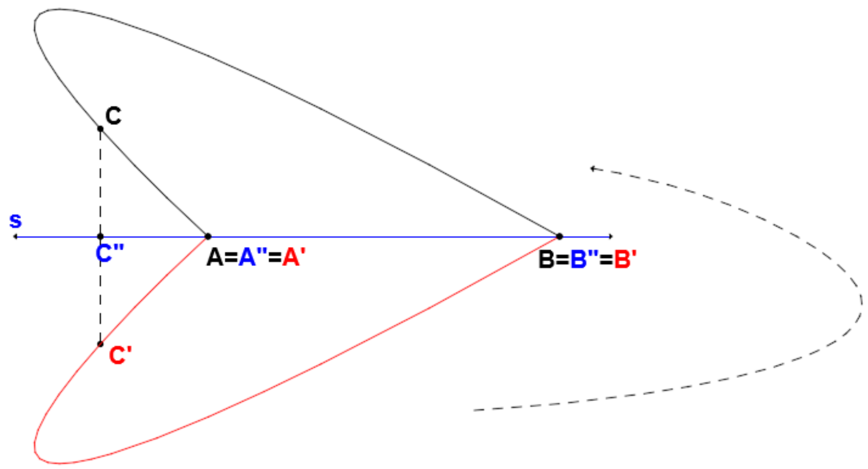


Solution: According to Theorem 1, the required transformation is the orthogonal reflection along the axis s. The resulting reflected image is exact and symmetric.

Exercise 100

Determine the reflected figure of a semicircular curve, assuming:

- the axis of affinity s contains two points of the curve,
- curvature between two points of the curve is equal.



Solution: Again, the orthogonal reflection with respect to s gives the second half of the figure, completing the symmetric shape (full heart figure).

Acknowledgments

This result is original and is the author's work.