Riemann Rearrangement Theorem -
A Counterexample

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Abstract It has been recently explained that Riemann rearrangement theorem is wrong [1], and that it has never been correctly proved. In [1] was demonstrated, on a famous example, that it is not the rearrangement of the elements, but rather the omission of elements of the conditionally convergent series, that would lead to a different summation result. The example that was used in [1] does not strictly follow the Riemann rearrangement method that is proposed in his theorem. It was correctly detected by Google's AI module. In this paper an example that follows the Riemann rearrangement method is going to be presented and again, it is going to be explained that the reason the “rearranged” series has a different sum is the omission of the infinite number of elements of the original series. Generally speaking, the rearrangement method has no critical impact on the summation result – the summation result depends on the sum of elements that are not included in the sum, and that is very simple to understand.

1 Introduction

Here, a part of the paper [1] is going to be repeated for the reason of the completeness of this paper. First, Riemann series (rearrangement) theorem [2] is going to be very briefly introduced.

Riemann series theorem: Suppose that we have a sequence of real numbers m_k, and that

$$R=\lim_{n\to\infty}\sum_{k=1}^{n} m_k ,$$

is conditionally convergent. Let M be the real number (it also can take the values of ∞ or $-\infty$) .

Then, there exists a permutation σ such that the following holds

$$R_\sigma=\sum_{k=1}^{\infty} m_{\sigma(k)}=M.$$

In the following sections, it is going to be shown that this is not true for a specific series and its "rearrangement". Different value of the sum for "rearranged" series is direct consequence of the omission of infinite number of elements of the original series – although you cannot specify a single element that is missing [1], without using a limes.

2 A Counterexample

Here, as it was done in [1], one example of the series is going to be presented. Let us start with the following series z_k (zero sum harmonic series – or ZSH series [1]):

$$z_1=b_1=1,\ z_2=a_1=-1,\ z_3=b_2=\frac{1}{2},\ z_4=a_2=-\frac{1}{2},\ z_5=b_3=\frac{1}{3},\ z_6=a_3=-\frac{1}{3},\ldots$$

The odd index elements of the series that are positive are marked as b, while the even index elements of the series that are negative are marked as a. That will simplify some notation later in the text. It is easy to understand that the following holds:

$$B=\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k}=0.$$

It is trivial to understand that ZSH series is conditionally convergent.

Now, it is going to be shown, step by step, how we can "rearrange" the ZSH series to obtain the value ln(2) using Riemann method.

STEP 1. We take the first positive element of the series $b_1 = 1$ and create a sum $S_{1up}= b_1$. It is easy to understand that $S_{1up} > \ln(2)$. So, we need to add the first negative element of series a_1, so we have sum $S_1 = S_{1up} + a_1 = 0$. Since $0 < \ln(2)$, we need to make the next step.

STEP 2. Since $0 < \ln(2)$, we need to add next positive elements of the initial series until their sum is bigger than ln(2). We create $S_{2up}= b_2 + b_3 = 1/2 +1/3 > \ln(2)$. Now, we need to add next negative elements until sum becomes smaller than ln(2). That leads to $S_2 = S_{2up}$ + a2 = 1/3, that is smaller than ln(2). Now, we should continue with the next steps using the same strategy.

In the Step K we will have the following situation

STEP K.

$$S_{Kup} = \sum_{k=K}^{2K-1} \frac{1}{k}, \tag{1}$$

while

$$S_K = \sum_{k=K+1}^{2K-1} \frac{1}{k} = S_{Kup} - \frac{1}{K}. \tag{2}$$

By using graphic method proposed in the Mathologer's video: **The best A-A≠0 paradox**, it is easy to understand that $S_{Kup} > \ln(2)$ and $S_K < \ln(2)$ (the second inequality is not graphically presented in the video – however, it is easy to understand it - when the first rectangle is removed and all the rectangles that are left and are oriented to the right, are flipped and oriented to the left, inequality follows from the fact that function $1/x$ is monotonically decreasing). From the same video (and many others, as well as the numerous literature) we know that the following holds

$$S = \lim_{K \to \infty} S_{Kup} = \lim_{K \to \infty} S_K = \ln(2).$$

However, from equations (1) and (2) we can easily understand that second half of negative series a is not used in the sum S. So, we have a clear counterexample that the rearrangement method proposed by Riemann does not use all the elements of the original series, and that is the reason that the "rearranged" series has a different sum.

3. Conclusion

In this paper a correct counterexample for the Riemann series (rearrangement) theorem has been presented and it has been shown that the reason that the "rearranged" series has a different sum is the omission of the infinite number of elements of the original series.

References

[1] M. V. Jankovic (2025). A Flow in the Proof of Riemann Rearrangement Series – An Incomplete Mathe-magic Trick. hal-05173608, version 1, or viXra:2507.0153.

[2] B. Riemann (2004). On the representation of a function by a trigonometric series. *Collected papers*. Translated by Baker, Rodger; Christenson, Charles; Orde, Henry. Translation of 1892 German edition, Heber City, UT, Kendrick Press.