On Babylonian Mathematics and Real Numbers

They new much more then I thought!

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PAMS Sept. 18, 2025

September 19, 2025

Abstract

Babylonian Algorithm for Square Roots is a precursor of Newton-Cotes numerical method for finding roots, as *numerical approximations* (perturbative method).

"What is a number?" has two complementary aspects: axiomatic viewpoint and as a tool in applications, via mathematical models.

Modern 21-st century Mathematics regards Real Numbers as part of *Complex Dynamics*, in terms of Newton flow, Fatou and Julia sets and fractals, pointing towards their role as trajectories in Dynamics. A brief account is provided.

Their relation to Number Fields, theory resulting from the work of Galois and Artin, starts to be apparent via *Dirichlet Unit Theorem* and the concept of *uniformizer* (PS/Diff. Eq.), extending the correspondence between Number Fields and *Function Fields* of *Algebraic Curves*.

Contents

Motivation for Studying Numbers and L.I. R&D Program

Understanding why *Feynman amplitudes*, which are algebraic integrals called *periods* are *all we need to model reality*, is a long-term project (more then 20 years ... understanding Gravity: more than 50: done!).

It led to my recent claim: *Physics Laws are Period Laws* (see https://vixra.org/abs/2309.0086 [3]), i.e. *Reality is Mathematical* (recall E. Wigner [1]) ... but only after "casting-out the Reals": "Reality", *our "Mathematics Teacher"*, is Algebraic-Geometric, finite, yet unbounded in its mandatory growth ... and AG-Tools [2] are enough to model "Real" Dynamics!

Now, a little change in our mindset: rethink "Real Numbers" (Cauchy equivalence classes of Cauchy sequences) as Dynamics; relate with $SL(2, Z)$ Continuous Fractions representation: *Quantum Dynamics*!? (see my other ISU PAMS talks [4]).

... a lot of work to do ... "in progress" :)

Part I: Newton's Method

Babylonian Mathematics (1800-1600 BCE)

From [1]: *Babylonian algorithm* for extracting square roots yields the remarkable approximation (cuneiforms on *clay tablets*!):

$$a_{n+1} := \frac{1}{2}(a_n + D/a_n) \quad \Longrightarrow \sqrt{2} \approx 1.41421297 \ (a_1 = 1, n = 4)$$

The Babylonian Algorithm was later refined and generalized by Heron, Brahmagupta, Al-Kwarezmi ("algorithms"!), *Newton and Raphson* (17C).

So, "What is a Number?"

There are several stages of development and understanding:

- Ancient, practical use: "one potato / two potato" etc.
- 1850s Cantor: *cardinals* / equivalence classes of sets, "a la" *Grothendieck group ring of a category* / isomorphism moduli space idea;
- Dedekind, Cauchy - Analysis ...
- Galois: "roots" of generic polynomials and Galois Theory (groups of permutations of generic roots);
- Abstract Algebra - Artin: field extensions;

It is of interest to have an understanding of how all these concepts relate: design a *Conceptual Network* (like a city with streets connecting various "businesses").

Recall on Galois to Felix Klein and Sophus Lie

General principals have been essential in providing a deep conceptual advancement of Theory with successful applications (see "On some programs in Math-Physics" [2]).
We will just mention *Galois Principle*:

To study on Object, *associate and study its* group of symmetries.

Later, Felix Klein unified geometries:

A Geometry is a group acting on a set.

The central object defining a "root", a *polynomial*, belongs to Algebra, Geometry and Analysis ...

What kind of “Roots”?

The corresponding concept of “roots” will be called: Galois roots (Klein Geometry), Artin roots (Abstract Algebra) and Newton roots (Analysis).

They are related in a non-trivial way: Number Fields, Algebraic Curves and Function Fields; the Analysis viewpoint leads to *Complex Dynamics* (to be explained).

... to be explained briefly.

Now Babylonian Algorithm is a precursor of *Newton-Cotes-Raphson Numeric Method*, leading to *Newton's Flow* of a polynomial ... it is Dynamics in an Algebraic-Geometry context!

Part II: Real Numbers as Dynamics ("Newtonian")

Role of Numbers: from Geometry to Dynamics

Briefly, "*numbers*" label *axiomatic Abstract Algebra* concepts, e.g. *fields* etc., or, from the application viewpoint, basic tools to build Mathematical Models, e.g. *physics quantities*, *geometric measures* etc.
Less known, or thought this way, *approximations of numbers*, e.g. Cauchy sequences as representatives of "Real Numbers".

• The Pythagorean "conundrum": is $\sqrt{2}$ a "number"? (Does Pythagorean Theorem "forces" us to accept it? A: No! The "correct concept of "distance" is the metric itself (see *quadratic differentials*".
• Whzat is π? A: an algebraic period ...
• But we all think such a "number" require and has approximations: "that's the number they are"! (Is it?); so "Cauchy was right" ...

So, as usual, it depends of context and author: 1) axiomatic viewpoint; 2) Tool. But these approximations $\{x_n\}_{n\in N}$ are akin to "time trajectories" (time as an order parameter of "causal computation" - QC + Phil.).
So ...

What is "Time", as an Order Parameter?

[From "Lie Theory-16 / Varna 2025 talk": for the "big picture" - see [3]]. The role of Quantum Phase ($U(1)$-Gauge Theory) is that of a local cyclic order parameter interpreted as an *Einstein-Feynman Relativistic Time* (see Feynmann QED).

• Linear Time is NOT a physical dimension / observable; even more, Lorentz Frames are a *linear approximation* as in Special Relativity.

• Similarly regarding "Space": local frames are defined by nucleon field principal directions we call "quarks" (fields of EM-type in a spinorial formulation; not just "electron" $U(1)$-gauge theory).

Newtonian Flow and Graph: basics

• *Newtonian vector field* $N_P = P(z)/P'(z)$; its Euler Method to compute integral lines, with stepsize $h = 1$, is *Newton's Method* for finding roots;
• Its *sources/sinks* define the vertices of Newtonian Graph, while the *unstable manifold* define the edges, with the boundary regions as *basins*; each basin contains a root, a fixed point that is a sink;

From "Computing the Newtonian Graph" [4], figures 1,3, p.3 and 4:

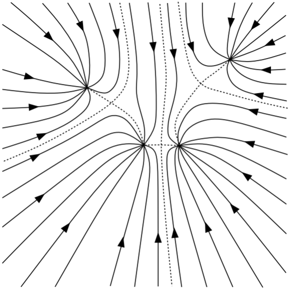

Figure 1. The Newtonian vector field of a polynomial of degree four. Every curve of flow is directed to a root, except the basin boundaries (dotted lines). There is a root of f' on every basin boundary, and a curve of flow from there to "adjacent" roots (also dotted lines).

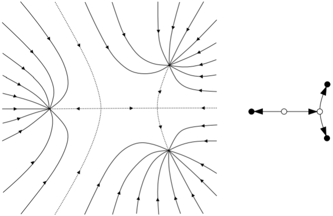

Figure 3. The vector field and the graph of a degree three (real) polynomial with an edge between two derivative roots.

Example: quadratic case $X^2 - D$

- Newton flow $\mathcal{N}_P = (z^2 - D)/(2z) = \frac{1}{2}(z - D/z)$;
- Newton Map $N_P = z - P(z)/P'(z) = \frac{1}{2}(z + D/z)$;
- It is conjugate to the map $z \to z^2$; or alternatively, rescaling, we may assume $P(z) = z^2 - 1$ (This is a general fact: the two roots are finite fixed points of the double cover, and can be mapped to 0 and ∞; alternatively, can be checked by a direct computation);

N. B. This disregards algebraic aspects, i.e. field extensions; see Part III.

- The case $P(z) = z^2 - 1$:
  1) Fixed points: $z = \pm 1$ attractors and $z = \infty$ is repelling;
  2) The basins of the roots are the left/right half-planes;
  3) Unstable manifold of ∞ is the imaginary axis.

Part III: Relation to Number Fields and Function Fields

What is a "Root"!?

• Original Galois Theory studies the general polynomial $P(a_1, ...a_n; x)$ and its associated symmetry group: Galois group, e.g. S_2 for quadratic case. *Galois roots* are extensions *rational functions*, i.e. elements of *Function Fields*!

• The E. Artin's modern formulation in terms of field extensions focuses on abstract vector spaces with automorphisms having the polynomial as a minimal polynomial of a linear operator: *Artin root*; e.g. "$X^2 - D = 0$" and $M_r = [0\ 2//1\ 0]$;

• Newton's (Numeric) Method is in fact a *Finite Difference Equation*, with the solution of the recursive equation, a Cauchy sequence (hopefully :).

• How is this FDE/Newton Method related to Dirichlet Unit Theorem?

• How we can interpret this in relation to the *Algebraic Variety*!? ... and concept of *uniformizer*?

These are Problems for UG :)

Conclusions

Real numbers as Cauchy completion of the rationals, is a moduli space of dynamics trajectories, with subsets *Newtonian roots* of polynomials.
One should extend to $C = R[i]$ first, as the complex plane has much more structure (*Kahler triangle*). Then the roots of a polynomial and critical points define a divisor and *Newtonian Graph* starts to resamble a *Dessin d'Enfant* associated with an algebraic curve over a number field.
These DdEs are pull-backs of the "standard cut" of S^2 via a Belyi map, an AG-Tool for Particle Physics [2].

Therefore the "real numbers content" of C are in fact "not numbers", but rather objects belonging to *Complex Dynamics* ...

Further Developments I

The next goal at this stage is to relate Galois roots and Newtonian Roots, by applying Dirichlet Unit Theorem, with embeddings defined by uniformizers, and connecting with the punctured Riemann Surfaces of a Belyi map as the underlying structure.
Then *periods* as complex algebraic integrals will be periods of algebraic flows on algebraic cycles, associated to the Hodge-de Rham Isomorphism in an appropriate homological basis, derived from a DdE, defined by a Belyi Map. Then the Galois group will have a dynamical interpretation and be related to the "Belyi solids" / DdEs of this framework ...

Further Developments II

• Current work with Jeremy Tobolaski: relate the Artin roots (Field theory) and numerical roots (Newton dynamics), in the context of Dirichlet Unit Theorem.

• Past work with Anurag Kurumbail: Continuous Fractions representations of the real numbers can be expressed using $SL(2, Z)$ the generators $S(z) = 1/z$ and $T(z) = z + 1$ as "digits" - see [4].

This can be used for quadratic extensions, like $\sqrt{2}$, in order to see the period of CF as a dynamics trajectory under the Newton flow (Future work).

Feedback and suggestions are welcome!

Thank You!

References

- E. Wigner, The Unreasonable Effectiveness of Mathematics in the Natural Sciences", 1960. https://webhomes.maths.ed.ac.uk/ v1ranick/papers/wigner.pdf
- L. M. Ionescu: "Beauty and the Beast", vixra.org and ISU web site; see also "AG-Tools for Particle Physics", https://vixra.org/abs/2305.0001
- L. M. Ionescu, Natural Units, Pi-Groups and Period Laws, https://vixra.org/abs/2309.0086
- L. M. Ionescu, "Real numbers: a new quantum look", PAMS Oct. 2023; "Rethinking real numbers" PAMS April 2024, https://about.illinoisstate.edu/lmiones/research/
- L. M. Ionescu, MAT 320 History of Mathematics - Lecture Notes, https://about.illinoisstate.edu/lmiones/teaching/

References (cont)

- Minhyong Kim, "History, identity and ownership", Notices of AMS, Vol. 72, Nr. 8.
- L. M. Ionescu, "On some programs in Math-Physics: parts I and II", 2024, https://about.illinoisstate.edu/lmiones/research/
- L. M. Ionescu, "Emergence of Space-Time Physics from Quantum Computing Formalism", International Workshop on Lie Theory and Applications to Physics, June 13, Varna, Bulgaria.
- D. Kozen and K. Stefansson, Computing the Newtonian Graph, 1995, https://www.cs.cornell.edu/ kozen/Papers/newton.pdf