Solutions to Exercises in Hestenes's Geometric Algebra Treatment of Constant-Acceleration (Parabolic) Motion

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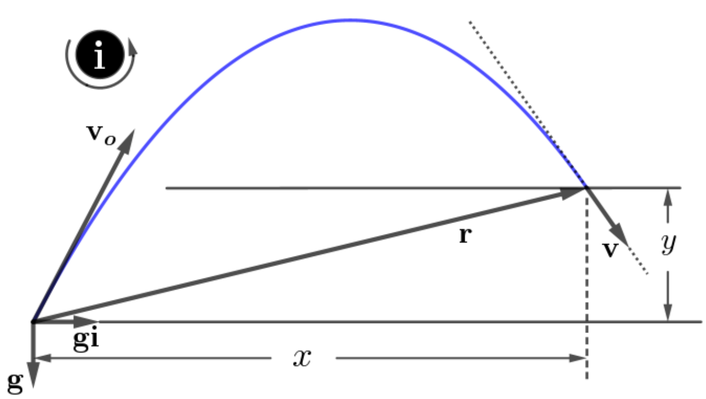
Abstract

As an aid to teachers and students who are learning to apply Geometric Algebra (GA) to high-school-level physics, this final installment in our guide to David Hestenes's treatment of constant-acceleration motion provides detailed solutions to the associated exercises in Hestenes's book *New Foundations for Classical Mechanics*.



Determine the maximum horizontal range x for a projectile with initial speed v_o fired at targets on a plateau with (vertical) elevation y above the firing pad.

Contents

1 Introduction

This document concludes our presentation and explanation of Hestenes's "GA" treatment of constant-acceleration motion. Like the previous installments in this series ([1], [2], [3]), the present one has been prepared in the spirit of Hestenes's observation that students will need "judicious guidance" to get through his book *New Foundations for Classical Mechanics*[4].[1] Therefore, this document is intended to be understandable by students and teachers who are still in the process of learning the basics of GA. For that same reason, our explanations and derivations are more detailed than is possible (because of length restrictions) in mass-market textbooks.

2 The Exercises and Their Solutions

2.1 Exercise 2.1

<u>Problem statement</u>

Derive the following expression for time of flight as a function of target location:

[1] "Though my book has been a continual best seller in the series for well over a decade, it is still unknown to most teachers of mechanics in the U.S. To be suitable for the series, I had to design it as a multipurpose book, including a general introduction to GA and material of interest to researchers, as well as problem sets for students. It is not what I would have written to be a mechanics textbook alone. Most students need judicious guidance by the instructor to get through it." [5]

$$t = \frac{\sqrt{2}}{g}\left\{v_o^2 + \mathbf{g}\cdot\mathbf{r} \pm \left[\left(v_o^2 + \mathbf{g}\cdot\mathbf{r}\right)^2 - r^2g^2\right]^{1/2}\right\}^{1/2}.$$

Solution

Note that the equation that we are asked to derive contains $\mathbf{g}\cdot\mathbf{r}$, but neither $\mathbf{g}\cdot\mathbf{v}_o$ nor $\mathbf{r}\cdot\mathbf{v}_o$. In addition, the requested equation contains v_o^2. These observations suggest that we might consider starting with Hestenes's Eq. 2.5:

Hestenes's Eq. 2.5, p. 128 of *NFCM*: $\overline{\mathbf{v}} = \frac{\mathbf{r}}{t} = \frac{\mathbf{1}}{2}\mathbf{g}t + \mathbf{v}_o$.

$$\overline{\mathbf{v}} = \frac{\mathbf{r}}{t} = \frac{\mathbf{1}}{2}\mathbf{g}t + \mathbf{v}_o.$$

We rearrange this as

$$\frac{\mathbf{r}}{t} - \frac{\mathbf{1}}{2}\mathbf{g}t = \mathbf{v}_o,$$

then square both sides,

$$\frac{r^2}{t^2} - \mathbf{r}\cdot\mathbf{g} + \frac{1}{4}g^2t^2 = v_o^2 \;\;;$$
$$\therefore g^2t^4 - 4\left(v_o^2 + \mathbf{r}\cdot\mathbf{g}\right)t^2 + 4r^2 = 0.$$

Now, we just solve for t.

2.2 Exercise 2.2

Problem statement

From Equation (2.4) one can get a quadratic equation for t:

$$t^2 + 2\mathbf{v}_o\cdot\mathbf{g}^{-1}t - 2\mathbf{r}\cdot\mathbf{g}^{-1} = 0.$$

Discuss the significance of the roots and how they are related to the result of Exercise 1.

Solution

First, we will want to see how to obtain this quadratic equation from Hestenes's Eq 2.4, which is

$$\mathbf{r} = \frac{1}{2}\mathbf{g}t^2 + \mathbf{v}_o t.$$

Examining the quadratic, we see that both of the dot products are with $\mathbf{g}^{-1}$. We can also see that "dotting" both sides of Eq. 2.4 with $\mathbf{g}^{-1}$ not only produces the $\mathbf{v}_o\cdot\mathbf{g}^{-1}$ and $\mathbf{r}\cdot\mathbf{g}^{-1}$ terms, but eliminates the factor $\mathbf{g}$ in the term $\frac{1}{2}\mathbf{g}t^2$:

$$\mathbf{g}\cdot\mathbf{g}^{-1} = \mathbf{g}\mathbf{g}^{-1}(\text{ because } \mathbf{g}\wedge\mathbf{g}^{-1} = 0)$$
$$= 1.$$

Thus, we can obtain the requested quadratic by "dotting" both sides of Eq. 2.4 with $\mathbf{g}^{-1}$, then rearranging. We could also obtain that quadratic by "dotting" both sides of Eq. 2.4 with $\mathbf{g}$, then dividing both sides by g^2, and rearranging.

The roots of the quadratic are

$$t = -\mathbf{v}_o \cdot \mathbf{g}^{-1} \pm \left\{ \left(\mathbf{v}_o \cdot \mathbf{g}^{-1} \right)^2 + 2\mathbf{r} \cdot \mathbf{g}^{-1} \right\}^{1/2}.$$

Regarding the significance of the two roots, we will quote Hestenes's solution (*NFCM*, p. 682):

> *This expression has the drawback that* $\mathbf{r}$ *and* $\mathbf{v}_o$ *are not independent variables. For given* $\mathbf{r}$ *and* $\mathbf{v}_o$*, the two roots are times of flight to the same point by different paths. For given* $\mathbf{v}_o$*, they are times of flight to two distinct points on the same path equidistant from the vertical maximum.*

2.3 Exercise 2.3

Problem statement

The vertex of a parabolic trajectory is defined by the equation $\mathbf{v} \cdot \mathbf{g} = 0$*. Show that the time of flight to the vertex is given by* $-\mathbf{v}_o \cdot \mathbf{g}^{-1} = 0$*. Use this to determine the location of the vertex.*

Solution

Before beginning to solve this exercise, we should ask ourselves why (or indeed whether) it is true that at the vertex of the parabola, $\mathbf{v} \cdot \mathbf{g} = 0$. To respond to this doubt, we note that the vertex of the parabola is its highest point. Therefore, the upward component of $\mathbf{v}$ must be zero. In other words, when a projectile reaches the vertex, its $\mathbf{v}$ is purely horizontal. Consequently, $\mathbf{v} \cdot \mathbf{g} = 0$.

Now that we've accepted Hestenes's assertion, we'll begin by citing his Eq. 2.3:

$$\mathbf{v} = \mathbf{g}t + \mathbf{v}_o,$$

from which

$$\begin{aligned} \mathbf{v} \cdot \mathbf{g} &= \mathbf{g} \cdot \mathbf{g}t + \mathbf{v}_o \cdot \mathbf{g}, \\ \mathbf{v} \cdot \mathbf{g} &= g^2 t + \mathbf{v}_o \cdot \mathbf{g}, \end{aligned}$$

That result is true for any time t. Thus, for the time t^* when $\mathbf{v} \cdot \mathbf{g} = 0$,

$$\begin{aligned} t^* &= -\frac{\mathbf{v} \cdot \mathbf{g}}{g^2} \\ &= -\mathbf{v}_o \cdot \mathbf{g}^{-1}. \end{aligned}$$

Now, using Hestenes's Eq. 2.4, the vertex is at $\mathbf{r} = \frac{1}{2}\mathbf{g}\left(t^*\right)^2 + \mathbf{v}_o t^*$.

2.4 Exercise 2.4

Problem statement

Use equations (2.18) and (2.19) to determine the maximum horizontal range x for a projectile with initial speed v_o fired at targets on a plateau with (vertical) elevation y above the firing pad.

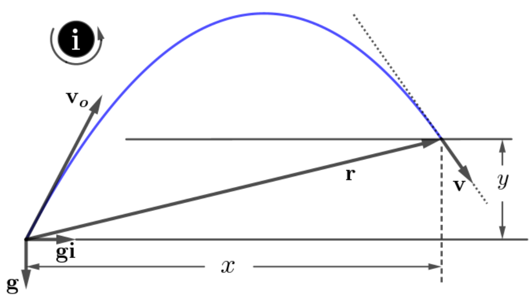


Figure 1: Exercise 2.4. Determine the maximum horizontal range x for a projectile with initial speed v_o fired at targets on a plateau with (vertical) elevation y above the firing pad.

Solution

Hestenes's solution (which we present here) is based upon the idea that the launch direction $\hat{\mathbf{v}}_o$ that produces the maximum horizontal distance for a target at elevation y, is the same $\hat{\mathbf{v}}_o$ that causes $\mathbf{v}$ to be perpendicular to $\mathbf{v}_o$ when the projectile reaches that target. Because readers may question whether such a $\hat{\mathbf{v}}_o$ necessarily exists, we supplement Hestenes's solution by identifying that $\hat{\mathbf{v}}_o$.

The equations that Hestenes asks us to use as a starting point are:

$$\begin{aligned} 2\mathbf{g}\cdot\mathbf{r} &= v^2 - v_o^2 \qquad \text{(Eq. 2.18)}\,; \\ \mathbf{g}\wedge\mathbf{r} &= \mathbf{v}\wedge\mathbf{v}_o \qquad \text{(Eq. 2.19)}\,. \end{aligned}$$

Let's first consider the geometric significance of the inner and outer products of any two vectors $\mathbf{p}$ and $\mathbf{q}$. The significance of the inner product of $\hat{\mathbf{p}}$ and $\mathbf{q}$ is simple enough; it's the (algebraically-signed) length of the projection of $\mathbf{q}$ upon the direction $\mathbf{p}$. Thus, in the context of the present problem, $\hat{\mathbf{g}}\cdot\mathbf{r}$ is the signed length of the projection of $\mathbf{r}$ upon $\hat{\mathbf{g}}$. Because $\mathbf{r}$ and $\hat{\mathbf{g}}$ have opposite directions, $\hat{\mathbf{g}}\cdot\mathbf{r} = {}^{-}r$; $\mathbf{g}\cdot\mathbf{r} = {}^{-}gr$; and Hestenes's Eq. 2.18 becomes $-2gy = v^2 - v_o^2$. Thus,

$$v = \sqrt{v_o^2 - 2gy}. \tag{1}$$

To identify the relevance of the outer product to our present problem, we recall that the expression $\left(\mathbf{p}^{-1}\right)(\mathbf{p}\wedge\mathbf{q})$ evaluates to the "rejection" of $\mathbf{q}$ with respect to $\mathbf{p}$. Thus, $\left(\mathbf{g}^{-1}\right)(\mathbf{g}\wedge\mathbf{r})$ would be $x\,(\hat{\mathbf{g}}\mathbf{i})$, so the left-hand side of Eq.

2.19 would be $xg\mathbf{i}$:

$$\begin{aligned}
\left(\mathbf{g}^{-1}\right)(\mathbf{g}\wedge\mathbf{r}) &= x\left(\hat{\mathbf{g}}\mathbf{i}\right)\ ; \\
\mathbf{g}\left[\left(\mathbf{g}^{-1}\right)(\mathbf{g}\wedge\mathbf{r})\right] &= \mathbf{g}\left[x\left(\hat{\mathbf{g}}\mathbf{i}\right)\right]\ ; \\
\left[\mathbf{g}\left(\mathbf{g}^{-1}\right)\right](\mathbf{g}\wedge\mathbf{r}) &= x\left[\mathbf{g}\hat{\mathbf{g}}\right]\mathbf{i}\ ; \\
\mathbf{g}\wedge\mathbf{r} &= xg\mathbf{i}.
\end{aligned}$$

Making this substitution, Eq. 2.19 becomes $xg\mathbf{i} = \mathbf{v}\wedge\mathbf{v}_o$, from which

$$\begin{aligned}
xg\mathbf{i}\left(-\mathbf{i}\right) &= \left(\mathbf{v}\wedge\mathbf{v}_o\right)\left(-\mathbf{i}\right)\ ;\ \text{and} \\
x &= \frac{-\left(\mathbf{v}\wedge\mathbf{v}_o\right)\mathbf{i}}{g}.
\end{aligned} \tag{2}$$

A study of Fig. 1 reveals that the rotation from $\mathbf{v}$ to $\mathbf{v}_o$ is in the positive sense of $\mathbf{i}$. We can confirm this impression analytically: $\mathbf{v}\wedge\mathbf{v}_o = (\mathbf{v}_o + \mathbf{g}t)\wedge\mathbf{v}_o = \mathbf{g}\wedge\mathbf{v}_o t$. Clearly, the rotation from $\mathbf{g}$ to $\mathbf{v}_o$ is in the positive sense of $\mathbf{i}$; therefore, the rotation from $\mathbf{v}$ to $\mathbf{v}_o$ is in the positive sense as well.

This result (Eq. (2)) is true for any distance x. But we want to find the maximum x for the given initial speed v_o. So, what is the maximum value of the right-hand side? Hestenes's solution notes that the magnitude of $\mathbf{v}\wedge\mathbf{v}_o$ is greatest when $\mathbf{v}$ and $\mathbf{v}_o$ are perpendicular, in which case $\mathbf{v}\wedge\mathbf{v}_o = v_o v\mathbf{i}$. Making this substitution in EQ. (2),

$$\begin{aligned}
x_{max} &= \frac{-\left(v_o v\mathbf{i}\right)\mathbf{i}}{g} \\
&= \frac{v_o v}{g}\ .
\end{aligned}$$

Now, using EQ.'s expression for v, we obtain

$$x_{max} = \frac{v_o}{g}\sqrt{v_o^2 - 2gy}\ . \tag{3}$$

At this point, the reader may be wondering how we can be certain that there will necessarily be a launch direction $\hat{\mathbf{v}}_o$ such that $\mathbf{v}$ will be perpendicular to $\mathbf{v}_o$ at altitude y. To address that qualm, let's determine the necessary $\hat{\mathbf{v}}_o$, by determining its components $\hat{\mathbf{v}}_o\cdot\hat{\mathbf{g}}$ and $\hat{\mathbf{v}}_o\cdot\left(\hat{\mathbf{g}}\mathbf{i}\right)$. We'll start by finding the relationship between $\hat{\mathbf{v}}_o\cdot\hat{\mathbf{g}}$ and the time at which $\mathbf{v}\perp\mathbf{v}_o$; that is, when $\mathbf{v}\cdot\mathbf{v}_o = 0$. From the equation $\mathbf{v} = \mathbf{v}_o + \mathbf{g}t$, we find that $\mathbf{v}\cdot\mathbf{v}_o = v_o^2 + \mathbf{v}_o\cdot\mathbf{g}t$. For the case of $\mathbf{v}\cdot\mathbf{v}_o = 0$, we obtain

$$t = -\left[\frac{v_o}{g\left(\hat{\mathbf{v}}_o\cdot\hat{\mathbf{g}}\right)}\right]. \tag{4}$$

Next, from the equation $\mathbf{r} = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2$, we find that $\mathbf{r}\cdot\hat{\mathbf{g}} = \mathbf{v}_o\cdot\hat{\mathbf{g}}t + \frac{1}{2}gt^2$. When the vertical component of $\mathbf{r}$ is y, $\mathbf{r}\cdot\hat{\mathbf{g}} = -y$, from which

$$\begin{aligned}
-y &= \mathbf{v}_o\cdot\hat{\mathbf{g}}t + \frac{1}{2}gt^2 \\
&= v_o\left[\hat{\mathbf{v}}_o\cdot\hat{\mathbf{g}}\right]t + \frac{1}{2}gt^2\ .
\end{aligned}$$

Substituting the expression for t that was presented in Eq. (4),

$$-y = \left[v_o\left(\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}}\right)\right]\left\{-\left[\frac{v_o}{g\left(\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}}\right)}\right]\right\} + \frac{1}{2}g\left\{-\left[\frac{v_o}{g\left(\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}}\right)}\right]\right\}^2 .$$

Thus,

$$\left(\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}}\right)^2 = \frac{v_o^2}{2\left(v_o^2 - gy\right)} .$$

The vertical component of $\mathbf{v}_o$ is upward (contrary to $\mathbf{g}$), so

$$\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}} = -\sqrt{\frac{v_o^2}{2\left(v_o^2 - gy\right)}} .$$

Then,

$$\begin{aligned}\hat{\mathbf{v}}_o \cdot \left(\hat{\mathbf{g}}\mathbf{i}\right) &= \sqrt{1 - \left(\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}}\right)^2} \\ &= \sqrt{\frac{v_o^2 - 2gy}{2\left(v_o^2 - gy\right)}} .\end{aligned}$$

We have now shown that there does indeed exist a $\hat{\mathbf{v}}_o$ such that $\mathbf{v} \perp \mathbf{v}_o$ at the altitude y. Therefore, Hestenes was justified in using $\mathbf{v} \wedge \mathbf{v}_o = v_o v \mathbf{i}$ at altitude y to find x_{max}.

2.5 Exercise 2.5

Problem statement

Find the minimum initial speed v_o needed for a projectile to reach a target with horizontal range x and elevation y. Determine also the firing angle θ, the time of flight t, and the final speed v of the projectile. Specifically, show that

$$v = \left[g\left(r - y\right)\right]^{1/2} \;\; ; v_o = \left[g\left(r + y\right)\right]^{1/2} \;\; ; \; \textit{and} \;\; \tan\theta = \left[\frac{r+y}{r-y}\right]^{1/2} .$$

Solution

Our first reaction to this problem might be to apply Hestenes's Eqs. 2.10-2.14, which Hestenes developed in order to determine the maximum range that a projectile with launch speed v_o can reach along a given direction r. However—is it true that the v_o in said equations is the minimum that is needed to reach a given point (x, y)? Yes —as is explained in Fig. 2.

Hestenes's Eq. 2.12, p. 128 of *NFCM*: $r_{max} = \frac{v_o^2}{g}\left[\frac{\mathbf{1}}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}}\right]$.

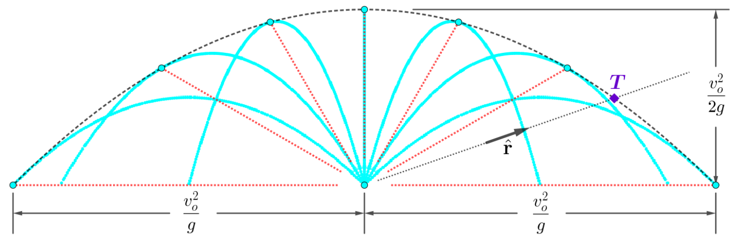


Figure 2: The dotted black line is outline of the paraboloid within which a target can be reached by a projectile that is launched with initial velocity v_o. The distance attainable along any given direction (e.g, $\hat{\mathbf{r}}$) is proportional to v_o^2. Thus, the target T can be reached only if the launch velocity is at least as great as the v_o corresponding to this paraboloid. In other words, the $\mathbf{v}_o$ corresponding to this specific paraboloid is the minimum launch speed for reaching T.

Solving EQ. 2.12 for v_o, we obtain

$$v_o = \sqrt{(1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}})\, g\, r_{max}},$$

which for our present purposes can be interpreted as

> *The (minimum) launch velocity v_o needed to reach a target at distance r along direction $\hat{\mathbf{r}}$ is*
>
> $$v_o = \sqrt{(1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}})\, g\, r}\,. \qquad (5)$$

In the problem at hand, $r = \sqrt{x^2 + y^2}$, and $\hat{\mathbf{r}} = \frac{-y\hat{\mathbf{g}} + x\hat{\mathbf{g}}\mathbf{i}}{r}$. Making this substitution in Eq. (5), then simplifying,

$$v_o = \sqrt{g\,(r + y)}\,.$$

Now, to find the velocity of the projectile upon reaching the target, we use our expression for v_o from Eq. (2.5), plus Hestenes's Eq. 2.18:

$$\begin{aligned} v^2 &= v_o^2 + 2\mathbf{g} \cdot \mathbf{r} \qquad \text{(Eq. 2.18)} \\ &= \underbrace{g\,(r + y)}_{v_o^2} + 2\mathbf{g} \cdot \left(\underbrace{-y\hat{\mathbf{g}} + x\hat{\mathbf{g}}\mathbf{i}}_{\mathbf{r}} \right) \\ &= g\,(r - y) \\ \therefore\ v &= \sqrt{g\,(r - y)}\,. \end{aligned}$$

To find $\tan\theta$, we use

$$\tan\theta = \frac{\textit{vertical component of } \hat{\mathbf{v}}_o}{\textit{horizontal component of } \hat{\mathbf{v}}_o} = \frac{-\hat{\mathbf{v}}_o \cdot \hat{\mathbf{g}}}{\hat{\mathbf{v}}_o \cdot (\hat{\mathbf{g}}\mathbf{i})}.$$

For the $\hat{\mathbf{v}}_0$ that gives the maximum range (and therefore the $\hat{\mathbf{v}}_0$ for the minimum speed for reaching the point (x, y)), we use Hestenes's Eq. 2.11:

$$\hat{\mathbf{v}}_o = \frac{\hat{\mathbf{r}} - \hat{\mathbf{g}}}{\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\|}\,.$$

Therefore,

$$\begin{aligned}\tan\theta &= \frac{\left[-\dfrac{\hat{\mathbf{r}}-\hat{\mathbf{g}}}{\|\hat{\mathbf{r}}-\hat{\mathbf{g}}\|}\right]\cdot\hat{\mathbf{g}}}{\left[\dfrac{\hat{\mathbf{r}}-\hat{\mathbf{g}}}{\|\hat{\mathbf{r}}-\hat{\mathbf{g}}\|}\right]\cdot(\hat{\mathbf{g}}\mathbf{i})} \\ &= \frac{-\hat{\mathbf{r}}\cdot\hat{\mathbf{g}}+1}{\hat{\mathbf{r}}\cdot(\hat{\mathbf{g}}\mathbf{i})}\ .\end{aligned}$$

Now, we substitute $\dfrac{-y\hat{\mathbf{g}}+x\hat{\mathbf{g}}\mathbf{i}}{r}$ for $\hat{\mathbf{r}}$, thus obtaining

$$\begin{aligned}\tan\theta &= \frac{r+y}{x} \\ &= \frac{r+y}{\sqrt{r^2-y^2}} \\ &= \frac{\left(\sqrt{r+y}\right)^2}{\sqrt{(r+y)\,(r-y))}} \\ &= \frac{\sqrt{r+y}}{\sqrt{r-y}}\ .\end{aligned}$$

Regarding the time of flight, Hestenes's Eq. 2.14 provides a simple solution:

$$t^2 = \frac{2r_{max}}{g}\ ; \quad \therefore t = \sqrt{\frac{2\sqrt{x^2+y^2}}{g}}\ .$$

2.6 Exercise 2.6

Problem statement

From Equations 2.16 and 2.16 (in NFCM) obtain

Eq. 2.15: $\mathbf{v}-\mathbf{v}_o = \mathbf{g}t$.

Eq. 2.16: $\mathbf{v}+\mathbf{v}_o = \frac{2\mathbf{r}}{t}$.

$$\begin{aligned}\mathbf{v}\wedge\mathbf{g} &= \mathbf{v}_o\wedge\mathbf{g}, \\ \mathbf{v}\wedge\mathbf{r} &= \mathbf{r}\wedge\mathbf{v}_o\ .\end{aligned}$$

Solve these equations to get

$$\mathbf{v} = \left[\frac{\mathbf{v}_o\wedge\mathbf{r}}{\mathbf{r}\wedge\mathbf{g}}\right]\mathbf{g} + \left[\frac{\mathbf{v}_o\wedge\mathbf{g}}{\mathbf{r}\wedge\mathbf{g}}\right]\mathbf{r}\ .$$

Solution

In his solutions for the "Constant Acceleration" section, Hestenes suggests "Use the Jacobi identity for $\mathbf{g}$, $\mathbf{v}$, and $\mathbf{r}$, and the fact that the vectors are coplanar." The Jacobi identity to which he refers is (*NFCM*, p. 47)

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}\cdot(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}\cdot(\mathbf{a}\wedge\mathbf{b}) = 0. \tag{6}$$

We'll come back to that equation later, after first obtaining the requested equations $\mathbf{v}\wedge\mathbf{g} = \mathbf{v}_o\wedge\mathbf{g}$ and $\mathbf{v}\wedge\mathbf{r} = \mathbf{r}\wedge\mathbf{v}_o$. From Hestenes's Eq. 2.15,

$$\begin{aligned}\mathbf{v}-\mathbf{v}_o &= \mathbf{g}t \\ (\mathbf{v}-\mathbf{v}_o)\wedge\mathbf{g} &= (\mathbf{g}t)\wedge\mathbf{g} \\ (\mathbf{v}-\mathbf{v}_o)\wedge\mathbf{g} &= 0 \\ \therefore\ \ \mathbf{v}\wedge\mathbf{g} &= \mathbf{v}_o\wedge\mathbf{g}\ .\end{aligned}$$

From Hestenes's Eq. 2.16,

$$\begin{aligned}
\mathbf{v}+\mathbf{v}_o &= \frac{2\mathbf{r}}{t} \\
(\mathbf{v}+\mathbf{v}_o)\wedge\mathbf{r} &= \left(\frac{2\mathbf{r}}{t}\right)\wedge\mathbf{r} \\
(\mathbf{v}+\mathbf{v}_o)\wedge\mathbf{r} &= 0 \\
\therefore\quad \mathbf{v}\wedge\mathbf{r} &= \mathbf{r}\wedge\mathbf{v}_o \ .
\end{aligned}$$

Now, back to the Jacobi identity, and the fact that the problem is 2-D. The significance of that fact is twofold. First, in a 2-D problem all of the vectors are coplanar. Therefore, the outer product of any given vector with the outer product of two other vectors is zero. For example, $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=0$. Therefore, from the definition of the geometric product of a vector with a bivector,

$$\begin{aligned}
\mathbf{a}\,(\mathbf{b}\wedge\mathbf{c}) &= \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) + \underbrace{\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})}_{=0}\ \text{, and} \\
\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) &= \mathbf{a}\,(\mathbf{b}\wedge\mathbf{c})\ .
\end{aligned}$$

Consequently, the Jacobi identity for 2-D vectors can be rewritten as

$$\mathbf{a}\,(\mathbf{b}\wedge\mathbf{c})+\mathbf{b}\,(\mathbf{c}\wedge\mathbf{a})+\mathbf{c}\,(\mathbf{a}\wedge\mathbf{b})=0.$$

A third significance of the 2-D nature is that in 2-D, every bivector is some scalar multiple of the unit bivector $\mathbf{i}$ for the plane that "contains" the vectors that are involved. That is, for any bivector $\mathbf{B}$,

$$\mathbf{B}=B\mathbf{i},$$

where B is some scalar. As a result, the product of any two bivectors is a scalar:

$$\begin{aligned}
\mathbf{BN} &= [B\mathbf{i}]\,[N\mathbf{i}] \\
&= BN\mathbf{i}^2 \\
&= {}^{-}BN\ .
\end{aligned}$$

Now that these preambles are out of the way, let's compare the 2-D version of the Jacobi identity to the third equation that we are to derive:

$$\begin{aligned}
\mathbf{a}\,(\mathbf{b}\wedge\mathbf{c})+\mathbf{b}\,(\mathbf{c}\wedge\mathbf{a})+\mathbf{c}\,(\mathbf{a}\wedge\mathbf{b}) &= 0\ \text{, or equivalently} \\
\mathbf{a}\,(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}\,(\mathbf{a}\wedge\mathbf{c})+\mathbf{c}\,(\mathbf{b}\wedge\mathbf{a}) &= 0\ \text{, versus} \\
\mathbf{v} = \left[\frac{\mathbf{v}_o\wedge\mathbf{r}}{\mathbf{r}\wedge\mathbf{g}}\right]\mathbf{g} &+ \left[\frac{\mathbf{v}_o\wedge\mathbf{g}}{\mathbf{r}\wedge\mathbf{g}}\right]\mathbf{r}\ .
\end{aligned}$$

A study of these equations suggests that the equation for $\mathbf{v}$ came from an equation of the form (ignoring algebraic signs, for the moment)

$$\mathbf{v}\,(\mathbf{r}\wedge\mathbf{g})=(\mathbf{v}_o\wedge\mathbf{r})\,\mathbf{g}+(\mathbf{v}_o\wedge\mathbf{g})\,\mathbf{r}\ .$$

This equation contains four vectors, but the Jacobi identity contains only three. So, something doesn't quite "fit". To see how we might proceed, let's first recall that the purpose of the equation $\mathbf{v} = \left[\frac{\mathbf{v}_o \wedge \mathbf{r}}{\mathbf{r} \wedge \mathbf{g}}\right]\mathbf{g} + \left[\frac{\mathbf{v}_o \wedge \mathbf{g}}{\mathbf{r} \wedge \mathbf{g}}\right]\mathbf{r}$ is to express $\mathbf{v}$ in terms of the other vectors. Let's also review the first two equations that we were asked to derive: $\mathbf{v} \wedge \mathbf{g} = \mathbf{v}_o \wedge \mathbf{g}$ and $\mathbf{v} \wedge \mathbf{r} = \mathbf{r} \wedge \mathbf{v}_o$. These equations show that if we write the Jacobi identity in terms of $\mathbf{v}$, $\mathbf{r}$, and $\mathbf{g}$, we'll be able, later, to substitute $\mathbf{v}_o$ for $\mathbf{v}$ in outer products of $\mathbf{v}$ with $\mathbf{r}$ and $\mathbf{g}$. That procedure should lead to the requested "$\mathbf{v}$" equation.

We now have a plan, so let's follow it, and see where it leads. For clarity, we'll present the term-by-term correspondence between the generic identity (which is written in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$) and our specific version (which will be written in terms of $\mathbf{v}$, $\mathbf{r}$, and $\mathbf{g}$):

$$\begin{aligned}
&\mathbf{a}\,(\mathbf{b} \wedge \mathbf{c}) + \mathbf{b}\,(\mathbf{c} \wedge \mathbf{a}) + \mathbf{c}\,(\mathbf{a} \wedge \mathbf{b}) = 0\ ; \\
&\mathbf{v}\,(\mathbf{r} \wedge \mathbf{g}) + \mathbf{r}\,(\mathbf{g} \wedge \mathbf{v}) + \mathbf{g}\,(\mathbf{v} \wedge \mathbf{r}) = 0\ .
\end{aligned}$$

Now, we solve for $\mathbf{v}$. To simplify the process, recall that the multiplicative inverse of any bivector $\mathbf{B}$ is itself a bivector. Specifically, it is a scalar multiple of $\mathbf{B}$: $(-\mathbf{B})/\|\mathbf{B}\|^2$. Therefore, in a 2-D problem, the product of $\mathbf{B}^{-1}$ with any other bivector is a scalar. Proceeding,

$$\mathbf{v}\,(\mathbf{r} \wedge \mathbf{g}) = -\mathbf{r}\,(\mathbf{g} \wedge \mathbf{v}) + -\mathbf{g}\,(\mathbf{v} \wedge \mathbf{r})\ .$$

Using our first two equations ($\mathbf{v} \wedge \mathbf{g} = \mathbf{v}_o \wedge \mathbf{g}$, and $\mathbf{v} \wedge \mathbf{r} = \mathbf{r} \wedge \mathbf{v}_o$), this becomes

$$\begin{aligned}
\mathbf{v}\,(\mathbf{r} \wedge \mathbf{g}) &= -\mathbf{r}\,(\mathbf{g} \wedge \mathbf{v_o}) + -\mathbf{g}\,(\mathbf{r} \wedge \mathbf{v}_o) \\
&= \mathbf{r}\,(\mathbf{v_o} \wedge \mathbf{g}) + \mathbf{g}\,(\mathbf{v}_o \wedge \mathbf{r})\ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{v} &= [\mathbf{r}\,(\mathbf{v_o} \wedge \mathbf{g})]\,(\mathbf{r} \wedge \mathbf{g})^{-1} + [\mathbf{g}\,(\mathbf{v}_o \wedge \mathbf{r})]\,(\mathbf{r} \wedge \mathbf{g})^{-1} \\
&= \mathbf{r}\left[\underbrace{(\mathbf{v_o} \wedge \mathbf{g})\,(\mathbf{r} \wedge \mathbf{g})^{-1}}_{a\ scalar}\right] + \mathbf{g}\left[\underbrace{(\mathbf{v}_o \wedge \mathbf{r})\,(\mathbf{r} \wedge \mathbf{g})^{-1}}_{a\ scalar}\right] \\
&= \left[(\mathbf{v_o} \wedge \mathbf{g})\,(\mathbf{r} \wedge \mathbf{g})^{-1}\right]\mathbf{r} + \left[(\mathbf{v}_o \wedge \mathbf{r})\,(\mathbf{r} \wedge \mathbf{g})^{-1}\right]\mathbf{g}\ .
\end{aligned}$$

This is the result that Hestenes writes as

$$\mathbf{v} = \left[\frac{\mathbf{v}_o \wedge \mathbf{r}}{\mathbf{r} \wedge \mathbf{g}}\right]\mathbf{g} + \left[\frac{\mathbf{v}_o \wedge \mathbf{g}}{\mathbf{r} \wedge \mathbf{g}}\right]\mathbf{r}\ .$$

2.7 Exercise 2.7

Problem statement

Determine the area swept out in time t by the displacement vector of a particle with constant acceleration $\mathbf{g}$ and initial velocity $\mathbf{v}_o$.

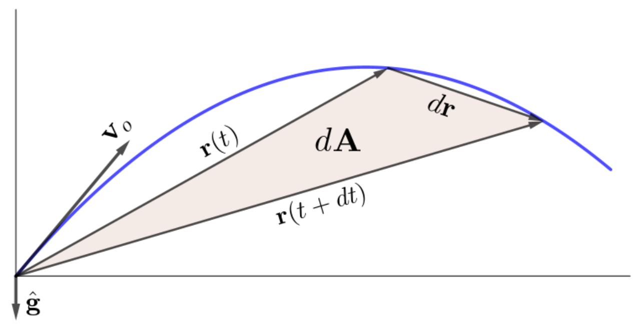


Figure 3: The incremental bivector area used in Exercise 2.6 is $d\mathbf{A} = \frac{1}{2}\mathbf{r}\left(t\right) \wedge (d\mathbf{r})$. (Recall that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, the geometric interpretation of the product $\mathbf{a} \wedge \mathbf{b}$ —without the factor $\frac{1}{2}$— is the oriented area of the parallelogram whose sides are $\mathbf{a}$ and $\mathbf{b}$.)

Solution

To clarify, we interpret the problem statement as meaning "the area swept out during the interval of time from launch to the instant t". Rather than use Hestenes's results from previous chapters to determine that area, we'll start from basic principles, with reference to Fig. 3. The requested area (in the form of a bivector) is

$$\begin{aligned} \mathbf{A}\left(t\right) &= \int_0^t d\mathbf{A} \\ &= \int_0^t \left[\frac{1}{2}\mathbf{r}\left(t\right) \wedge (d\mathbf{r})\right] \ , \end{aligned}$$

where $\mathbf{r}\left(t\right) = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2$. Because $\mathbf{g}$ and $\mathbf{v}_o$ are constant, $d\mathbf{r} = (\mathbf{v} + \mathbf{g}t)\,dt$, and

$$\begin{aligned} \frac{1}{2}\mathbf{r}\left(t\right) \wedge (d\mathbf{r}) &= \frac{1}{2}\left[\mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2\right] \wedge \left[(\mathbf{v} + \mathbf{g}t)\,dt\right] \\ &= \frac{1}{2}\left[\mathbf{v}_o \wedge \mathbf{g}t^2 + \frac{1}{2}\mathbf{g} \wedge \mathbf{v}_o t^2\right] dt \\ &= \frac{1}{2}\left[\mathbf{v}_o \wedge \mathbf{g}t^2 - \frac{1}{2}\mathbf{v}_o \wedge \mathbf{g}t^2\right] dt \\ &= \left[\frac{1}{4}\mathbf{v}_o \wedge \mathbf{g}t^2\right] dt. \end{aligned}$$

Therefore (again because $\mathbf{g}$ and $\mathbf{v}_o$ are constant),

$$\begin{aligned} \mathbf{A}\left(t\right) &= \int_0^t \left[\frac{1}{4}\mathbf{v}_o \wedge \mathbf{g}t^2\right] dt \\ &= \frac{1}{12}\mathbf{v}_o \wedge \mathbf{g}t^3 \ . \end{aligned}$$

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