An essence of division by zero

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Abstract. We give several pairs of congruent circles arising from a generalized arbelos. A special case, in which the congruent circles are lines, is also considered using division by zero $1/0 = 0$.

Keywords. congruent circles, generalized arbelos, division by zero

Mathematics Subject Classification (2010). 51M04

1. Intoroduction

It has long been believed that division by zero is impossible, which has been keeping the defect of lack of division by zero left neglected. However as far as the author knows, all the logics asserting the impossibility are wrong. We will show their logical defects and give a natural definition of division by zero. However if division by zero occurs in a certain function, the definition does not give the appropriate value of the function. To improve this we have to give a generalization of the definition, which is called *division by zero calculus*.

Division by zero and division by zero calculus are entirely new concepts to our mathematics existing outside of our mathematics. Judgment as to whether we introduce them to our mathematics is totally depend on whether they make our mathematics richer. We will give several applications of division by zero and division by zero calculus to plane geometry, which show that the new concepts make our mathematics much richer.

Division by zero and division by zero calculus were founded by a professor emeritus at Gunma University Saburou Saitoh ([16]). The book [16] consists of about 200 pages, and an extended version consisting of more than 400 pages at this time is still making. On the other hand, we need a short document giving an outline of division by zero, since so many mathematicians believe its impossibility. This is the reason why this paper is made.

2. Division by zero and division by zero calculus

In this section, we consider a simple example of incorrect assertion of impossibility of division by zero. Then we give brief definitions of division by zero and division by zero calculus. We consider in the field **R** of real numbers.

2.1. **Incorrect assertion of impossibility of division by zero.** Saburou Saitoh and the author gave a lecture on division by zero $1/0 = 0$ in Kyoto University in 2018. Then Norio Adachi pointed out that our assertion is false in Twitter as follows ([1]):

> The point is that only well explained story is bringing up. For example, how do we calculate $1/2 + 1/0$? If we can reduce to a common denominator like usual fractions, $1/2 = 1/2 + 1/0 = 2/0 = 0$, a contradiction. Even if we try to redefine the four basic arithmetic operations, there will always be a contradiction. Therefore $1/0 = 0$ is not well-defined.

We show that Adachi's assertion is false. He proves by contradiction assuming division by zero $1/0 = 0$. But he misses one more assumption is left, i.e., reduction to common denominator. Hence there are two assumptions. Therefore after getting the contradiction, he should conclude that *division by zero and reduction to common denominator are not true at the same time.*

As in this example, many wrong proofs of impossibility of division by zero assume reduction of fraction, which is equivalent to the following equality:

$$\frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}. \tag{1}$$

Proposition 1. *Reduction of fraction and* (1) *are equivalent.*

Proof. If we assume reduction of fraction, we have $b/a = bc/ac$ and $d/c = ad/ac$. Adding the both sides we have (1). Conversely if (1) holds, then substituting $d = 0$ in (1), we have $b/a = bc/ac$. □

D. W. Dodge gave a similar wrong proof assuming (1) in [2]. Most wrong proofs asserting the impossibility of division by zero are assuming division by zero and something existing in our mathematics, which are (1) or some phenomena associated with division on the positive integers, and so on. We refer this here as P. Then after getting a contradiction, they conclude the impossibility of division by zero. But the correct conclusion should be division by zero and P are not true at the same time.

2.2. **Division by zero.** We give a definition of division by zero. We consider a canonical bijection $\psi : \mathbf{R} \to \mathbf{R}$ such that

$$\psi(a) = \begin{cases} a^{-1} & \text{if } a \neq 0 \ , \\ a & \text{if } a = 0. \end{cases}$$

With this bijection, we can define a noncommutative binary operation / or ÷ on $\mathbf{R}$ as follows:

$$a/b = a \div b \stackrel{\text{def}}{=} a \cdot \psi(b) \quad for \ a, b \in \mathbf{R}.$$

Indeed, this has been used in our mathematics and is called division, and a/b is represented by $\frac{a}{b}$, which is called a fraction. But there is an custom not to consider $a/0 = a \cdot \psi(0) = a \cdot 0 = 0$. However it is too unnatural to make a such exceptional case in our mathematics. Therefore we include the case, i.e., we use the following definition in this paper.

Definition 2.1 ([3]).

$$\frac{a}{0} = 0 \quad for\ a \in \mathbf{R}. \tag{2}$$

From now on $a/b = 0$ *if and only if* $a = 0$ *or* $b = 0$. Notice that $ac = bc$ does not imply $a = b$ if $c = 0$, and we have

$$\frac{a}{b} + \frac{c}{d} \neq \frac{ad + bc}{bd}$$

if $b = 0$ or $d = 0$.

Two numbers a and b are said to be inverse to each other if and only if $ab = 1$. However the equation $ab = 1$ in the definition should be updated to $a = 1/b$. Then by the updated definition, the conventional inverse is also inverse and 0 is also the inverse of 0.

2.3. **Division by zero calculus.** For the function

$$f_s(x) = \frac{\sin x}{x},$$

it is well-known that

$$\lim_{x \to 0} f_s(x) = 1$$

is the slope of the tangent line of $y = \sin x$ at $x = 0$. However (2) gives

$$f_s(x)\Big|_{x=0} = \frac{\sin 0}{0} = \frac{0}{0} = 0.$$

The fact shows that division by zero can not give an appropriate value for $f_s(x)$. Hence we have to improve the definition. The next definition is called division by zero calculus (2):

Definition 2.2 ([16]). For a function $f(z)$ of z having the Laurent expansion about $z = a$:

$$f(z) = \sum_{n=-\infty}^{n=-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n,$$

we define $f(a) = C_0$.

This is a generalization of (2), because if $f(z) = a/z$, then we have

$$f(z) = \cdots + \frac{0}{z^3} + \frac{0}{z^2} + \frac{a}{z} + 0 + 0z + 0z^2 + 0z^3 + \cdots ,$$

which gives $f(0) = 0$. Now division by zero calculus gives $f_s(0) = 1$, since

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots .$$

3. Applications of division by zero

Judgment, as to whether we introduce an entirely new concept like division by zero to our mathematics, totally depends on that it makes our mathematics richer. In this sense Adachi's criticism "only well explained story is bringing up." quoted in subsection 2.1 is totally missing the point. In this section, we show several examples which show that division by zero makes our mathematics richer.

3.1. **Radius of a line.** For lines and tangency, division by zero gives a totally new insight, which are essential to our consideration. From now on, we consider the plane with a Cartesian coordinate system. One of the fundamental results is stated as follows:

Theorem 1 ([16])**.** *A line can be considered as a circle of radius* 0*.*

Proof. Any circle in the plane has an equation

$$e(x^2 + y^2) - 2fx - 2gy + h = 0, \tag{3}$$

and has radius

$$\sqrt{\frac{f^2 + g^2 - eh}{e^2}}. \tag{4}$$

While the equation represents a line in the case $e = 0$, and (4) equals 0 in that case by (2). □

Since the center of the circle represented by (3) has coordinates $(f/e, g/e)$, a line can be considered as a circle of center at the origin $(0, 0)$ by (2).

3.2. **Orthogonality and tangency.** We consider two orthogonal figures. We get the following notable relation by (2) ([4])

$$\tan \frac{\pi}{2} = 0. \tag{5}$$

Theorem 2. *The following statements are true.*
(i) *Two orthogonal figures can be considered to touch each other.*
(ii) *Lines perpendicular to the x-axis have slope* 0*.*
(iii) ([13]) *Two non-parallel lines of slopes m_1 and m_2 are perpendicular if and only if*

$$m_1 = -\frac{1}{m_2}. \tag{6}$$

Proof. Assume two figures have a point P in common and the angle between the tangents at P equals θ. Then the two figures are said to touch at P if and only if $\tan\theta = 0$. While the angle between the tangents at the point of intersection equals $\frac{\pi}{2}$ for two orthogonal figures. This proves (i) by (5). The part (ii) follows from (5). Since (6) holds in the case $m_1 = m_2 = 0$ by (2), (iii) is proved. □

3.2.1. *An application of Theorem 1.* We considering the following configuration (see Figure 1).

Let $\beta_1, \beta_2, \cdots, \beta_n$ be congruent circles with an external common tangent t such that β_1 and β_2 touch externally, $\beta_{i+1}(\neq \beta_{n-1})$ touches β_i for $i = 2, 3, 4, \cdots n-1$. Let α be the circle touching β_1 and β_n externally and t. We denote the figure consisting of α, β_1, β_2, $\cdots$, β_n and t by $\mathcal{A}(n)$, where α and β_1 have radii a and b, respectively. There are several problems involving the figure $\mathcal{A}(n)$ in Wasan geometry (old Japanese geometry in the Edo period) especially in the case $n = 4, 5$ ([10]).

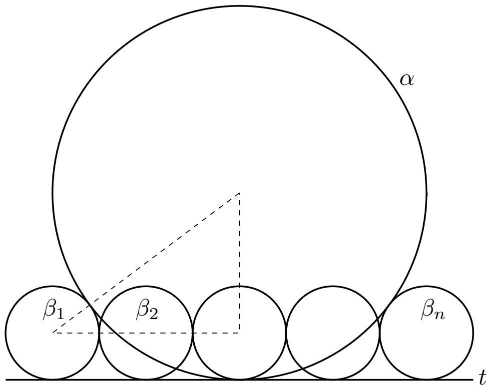

Figure 1: $\mathcal{A}(5)$.

Theorem 3 ([11])**.** *The following relation holds for* $\mathcal{A}(n)$ $(n \geq 2)$*.*

$$\frac{a}{b} = \left(\frac{n-1}{2}\right)^2. \tag{7}$$

Proof. We consider the right triangle formed by the perpendicular from the center of α to t, the line parallel to t passing through the center of β_1, and the line joining the centers of α and β_1. Then we have

$$(a-b)^2 + ((n-1)b)^2 = (a+b)^2.$$

Solving the equation for a, we have (7). □

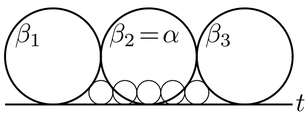

Figure 2: $\mathcal{A}(3)$ with $\mathcal{A}(5)$.

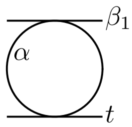

Figure 3: $\mathcal{A}(1)$.

The case $n = 1$ is not considered in the theorem. However we show that the theorem holds in the case $n = 1$ by (2) and Theorem 1. Figure 2 shows that we can construct the configuration $\mathcal{A}(5)$ from $\mathcal{A}(3)$ using the incircle of the curvilinear triangle made by α, β_1 and t. A similar construction of $\mathcal{A}(n+2)$ from $\mathcal{A}(n)$ can also be made ([11]). With this observation we can get the configuration $\mathcal{A}(1)$, in which β_1 is the line parallel to t (see Figure 3). Therefore we have $b = 0$ for $\mathcal{A}(1)$ by Theorem 1 and (7) holds by (2).

3.2.2. *Applications of Theorems 1 and 2.* We consider simple applications of Theorems 1 and 2. For a given two figure, each of which is a circle or a semicircle, if a circle touches one of the figures externally and the other internally, we say that the circle touches the two figures in the opposite sense, otherwise in the same sense. Generalizing a problem in Wasan geometry, we get the next theorem on a chain of circles, which is a simple application of Theorems 1 and 2 (see Figure 4).

Theorem 4 ([6])**.** *For the midpoint A of the line segment BO, let α and β be the semicircles of diameters AO and BO, respectively constructed on the same side of BO. γ_0 is the line BO and γ_1 is the circle touching α and β in the opposite sense and BO. If the circle γ_i has been defined for a positive integer i, then $\gamma_{i+1}(\neq \gamma_{i-1})$ is the circle touching α and β in the opposite sense and γ_i externally. If δ_0 is the line BO, and δ_n is the circle touching γ_n externally and BO at the point*

O for $n = 1, 2, 3, \cdots$, and d_n is the radius of δ_n for $n = 0, 1, 2, \cdots$, and $a = |AO|$, then for $n = 0, 1, 2, \cdots$ we have

$$d_n = \frac{a}{n}. \tag{8}$$

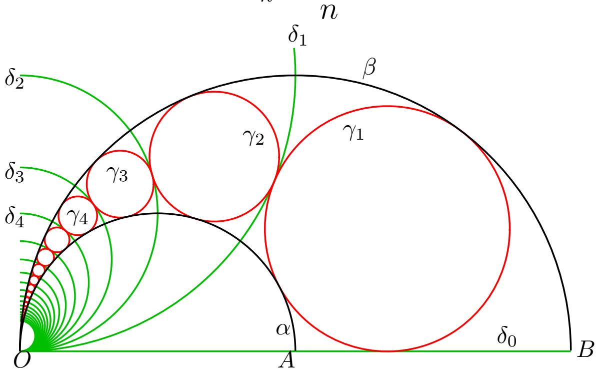

Figure 4: A chain of circles and circles touching them.

Proof. If $n = 0$, then $d_0 = 0$ by Theorem 1, i.e., (8) holds by (2). Assume $n \geq 1$. We use a Cartesian coordinate system with origin O so that the farthest point on β from BO has coordinates (a, a). Let (x_n^c, y_n^c) and r_n^c be the coordinates of the center of the circle γ_n and the radius of γ_n. If we invert the figure in the circle of center O orthogonal to δ_n, then α and β are inverted to the tangents of δ_n perpendicular to BO (see Figure 5). Therefore $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ are inverted to the circles congruent to γ_n touching the tangents. Therefore we have

$$y_n^c = (2n-1)r_n^c. \tag{9}$$

Considering the squares of the distances from the center of γ_n to the centers of α and β, we have

$$\left(\frac{a}{2} - x_n^c\right)^2 + ((2n-1)r_n^c)^2 = \left(\frac{a}{2} + r_n^c\right)^2$$

and

$$(a - x_n^c)^2 + ((2n-1)r_n^c)^2 = (a - r_n^c)^2.$$

Solving the equations for r_n^c and x_n^c, we have

$$r_n^c = \frac{4a}{4(n-1)n+9} \tag{10}$$

and

$$x_n^c = 3r_n^c. \tag{11}$$

Considering the square of the distance between γ_n and δ_n, we have

$$(x_n^c)^2 + (y_n^c - d_n)^2 = (r_n^c + d_n)^2.$$

Solving the last equation for d_n with (9), (10) and (11), we get (8). □

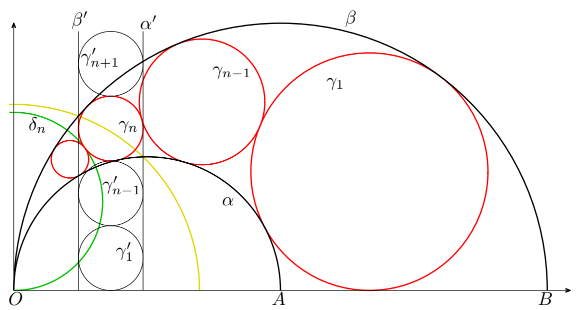

Figure 5: γ_k' is the inverse of γ_k.

Since the circle δ_k is inverted to a line parallel to BO, it touches γ_k at the point of tangency of γ_k and γ_{k+1}. Notice that γ_0 touches α and β by Theorem 2(i). By Theorem 1 and the remark at the end of subsection 3.1, we also have $x_0^c = 0 = 3r_0^c$, i.e., the equation (11) is also true in the case $n = 0$.

3.3. **Arbelos.** For a point C on the segment AB, let α, β and γ be the semicircles of diameters BC, CA and AB constructed on the same side of AB, respectively, where $|AB| = 2c$, $|BC| = 2a$ and $|CA| = 2b$. The area surrounded by the three semicircles is called an arbelos (see Figure 6). We consider using a Cartesian coordinate system with origin C so that the farthest point on α from AB has coordinates (a, a). Notice that $c = a + b$.

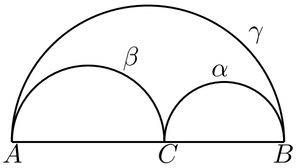

Figure 6: arbelos.

3.3.1. *The twin circles of Archimedes.* The radical axis of α and β is called the axis. The circle touching α (resp. β) and γ in the opposite sense and the axis from the side opposite to A (resp. B) has radius ab/c. It has been believed that the two circles were studied by Archimedes, and are called the twin circles of Archimedes. The circles of radius ab/c are called Archimedean circles.

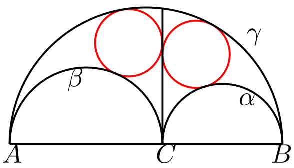

Figure 7: The twin circles of Archimedes.

The semicircle of center of coordinates $(ma, 0)$ (resp. $(-nb, 0)$) and passing through the point C constructed on the same side of AB as γ is denoted by α_m (resp. β_n) for a real numbers m (resp. n). A circle touching γ internally and touching α_m and β_n is said to touch the three semicircles appropriately if the points of tangency of this circle and each of α_m, β_n and γ lie clockwise (see Figure 8). We consider the next theorem.

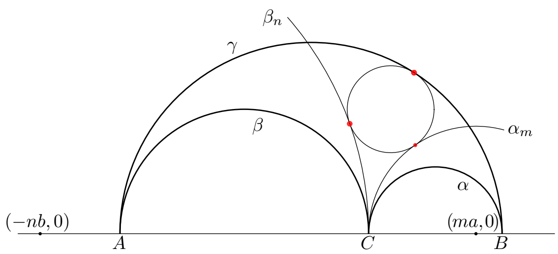

Figure 8: A circle touching α_m, β_n and γ appropriately.

Theorem 5 ([15]). *Assume $(m,n) \neq (0,1),(1,0)$. A circle touching α_m, β_n and γ appropriately is Archimedean if and only if*

$$\frac{1}{m} + \frac{1}{n} = 1. \tag{12}$$

The theorem characterizes the Archimedean circles touching γ internally, but the twin circles of Archimedes are not considered. We show that the twin circles can be included in the theorem by (2). We consider the case $(m,n) = (1,0)$. The circle β_n has an equation $(x+nb)^2 + y^2 = (nb)^2$ or

$$x^2 + y^2 + 2nbx = 0. \tag{13}$$

Therefore we get $x^2 + y^2 = 0$ if $n = 0$, i.e., β_0 coincides with the origin. On the other hand (13) implies

$$\frac{x^2 + y^2}{n} + 2bx = 0.$$

Therefore we get $x = 0$ if $n = 0$ by (2). Therefore $n = 0$ implies that β_0 is the origin or the y-axis. Since the origin is a part of the y-axis, we can consider that β_0 *is the* y*-axis.* While α_1 coincides with the semicircle α. Hence $(m,n) = (1,0)$ satisfies (12) and we get one of the twin circles of Archimedes touching $\alpha_1 = \alpha$ and the axis ([9]). Similarly α_0 coincides with the y-axis. Hence $(m,n) = (0,1)$ satisfies (12), and we get the other Archimedean circle touching $\beta_1 = \beta$ and the axis. Therefore we have

Theorem 6. *A circle touching α_m, β_n and γ appropriately is Archimedean if and only if*

$$\frac{1}{m} + \frac{1}{n} = 1.$$

A similar result for two congruent semicircles can be found in [8].

3.3.2. *Twin circles in a generalized arbelos.* For points P and Q on the line AB, we denote the semicircle of diameter PQ constructed on the same side of AB as γ by (PQ). We consider the next theorem (see Figures 9 and 10).

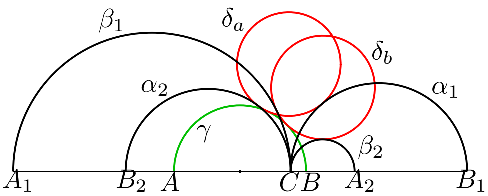

Figure 9: δ_a touches α_i and γ externally.

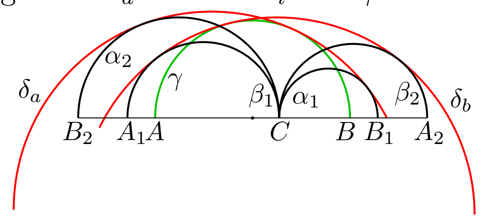

Figure 10: δ_a touches α_i and γ internally.

Theorem 7 ([12]). *For three concentric semicircles $\gamma = (AB)$, (A_1B_1) and (A_2B_2), the semicircles $\alpha_i = (CB_i)$ and $\beta_i = (CA_i)$ have radii a_i and b_i, respectively, where the common center has x-coordinate $a - b = a_1 - b_1 = b_2 - a_2$. If a circle δ_a touches α_1, α_2 and γ externally and a circle δ_b touches β_1, β_2 and*

γ externally, or δ_a touches α_1, α_2 and γ internally and δ_b touches β_1, β_2 and γ internally, then δ_a and δ_b are congruent.

Proof. Let r_a and (x_a, y_a) be the radius of the circle δ_a and the coordinates of the center of δ_a. Assume δ_a touches γ externally (see Figure 9). Since the centers of α_1, α_2 and γ have coordinates $(a_1, 0)$, $(-a_2, 0)$ and $(a-b, 0)$, respectively, the squares of the distances from the center of δ_a to the centers of α_i and γ are represented by

$$(x_a - a_1)^2 + y_a^2 = (r_a + a_1)^2, \tag{14}$$

$$(x_a + a_2)^2 + y_a^2 = (r_a + a_2)^2, \tag{15}$$

and

$$(x_a - (a-b))^2 + y_a^2 = (r_a + a + b)^2. \tag{16}$$

Eliminating x_a and y_a from (14), (15) and (16), and solving the resulting equation for r_a, we get

$$r_a = -\frac{ab(a_1 + a_2)}{aa_2 + a_1(b - a_2)}. \tag{17}$$

If r_b is the radius of δ_b, then similarly we have

$$r_b = -\frac{ab(b_1 + b_2)}{bb_2 + b_1(a - b_2)}.$$

Substituting $a_1 = a - b + b_1$ and $a_2 = b - a + b_2$ in (17), we have $r_a = r_b$. If δ_a touches γ internally, we similarly have

$$r_a = \frac{ab(a_1 + a_2)}{aa_2 + a_1(b - a_2)} = \frac{ab(b_1 + b_2)}{bb_2 + b_1(a - b_2)} = r_b.$$

□

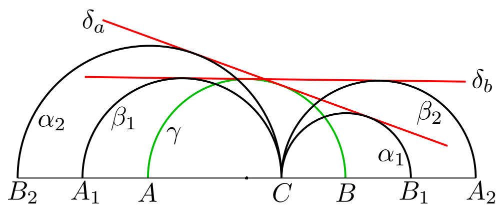

Figure 11: $r_a = r_b = 0$.

We consider the case in which the circle δ_a degenerates to a line. Hence we assume $r_a = 0$ by Theorem 1. Since the numerator of the right side of (17) does not equal 0, we have $aa_2 - a_1(a_2 - b) = 0$ by the remark after (2). While $a_1 = a - b + b_1$ and $a_2 = b - a + b_2$ imply $aa_2 + a_1(b - a_2) = bb_2 + b_1(a - b_2)$. Hence we also have $r_b = 0$, i.e.,

$$aa_2 - a_1(a_2 - b) = ab_1 - b_2(b_1 - b) = 0. \tag{18}$$

Solving (18) for a and b, we have

$$(a, b) = \left(\frac{a_1 b_2 (b_1 - a_2)}{a_1 b_1 - a_2 b_2}, \frac{a_2 b_1 (a_1 - b_2)}{a_1 b_1 - a_2 b_2} \right),$$

which enable us to get the semicircle γ from the semicircles (A_1B_1) and (A_2B_2) so that δ_a and δ_b are lines (see Figure 11).

4. Applications of division by zero calculus

In this section, we show several examples which show that division by zero calculus makes our mathematics richer. We now consider some families of circles in the plane, each of the members is represented by a Cartesian equation $f_z(x,y)=0$ with a parameter $z \in \mathbf{R}$. Here, we assume that if x and y are fixed, $f_z(x,y)$ has the Laurent expansion about $z=a$, then the corresponding coefficient $C_n(a;x,y)$ depends on x and y.

In this setting we will see some mysterious relation with the equation

$$f_z(x,y)=0$$

and the equations

$$C_n(a;x,y)=0,$$

for a fixed a. Then we will see that the equation $C_n(a;x,y)=0$ implies some meaningful things even for an integer $n \neq 0$ also even in the case in which $z=a$ is not a singular case. Moreover we will show that the equation $C_n(a;x,y)=0$ gives some notable and meaningful figures, which have never been considered before. On the other hand, we have no idea why the coefficients of the Laurent expansion show such meaningful and marvelous facts at the present time of writing. We will show several such facts with little explanations in the next two subsections.

4.1. A circle touching a circle and its secant line.

Let α be a circle of diameter AB and center O with $|AO|=a$, and assume that a line parallel to AB meets α at points P and Q. We use a Cartesian coordinate system with origin O such that A has coordinates $(a,0)$ and P lies in the first quadrant. For a point Z on the line PQ, δ_Z is the circle touching PQ at Z and the minor arc of α cut by PQ if Z lies between P and Q (see Figure 12), otherwise δ_Z is the circle touching α externally and the line PQ at Z from the same side as A (see Figure 13).

Assume that b is the distance between PQ and AB, r is the radius of δ_Z, (z,b) and (z,w) are the coordinates of the point Z and the center of δ_Z, respectively.

If Z lies between P and Q, then we have

$$z^2+w^2=(a-r)^2. \tag{19}$$

Substituting $w=b+r$ in (19) and solving the resulting equation for r, we have

$$r=\frac{a^2-b^2-z^2}{2(a+b)}. \tag{20}$$

Therefore if

$$\delta_Z(x,y)=(x-z)^2+(y-(b+r))^2-r^2, \tag{21}$$

then the circle δ_Z is represented by the equation $\delta_Z(x,y)=0$ with (20) using parameter z.

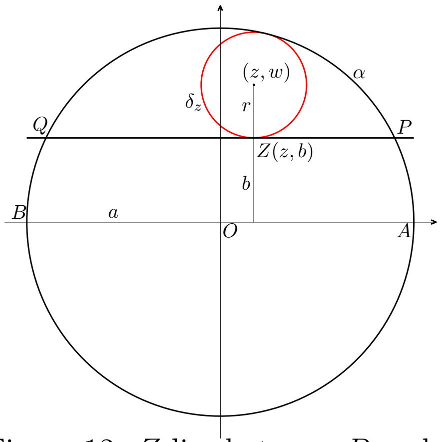

Figure 12: Z lies between P and Q

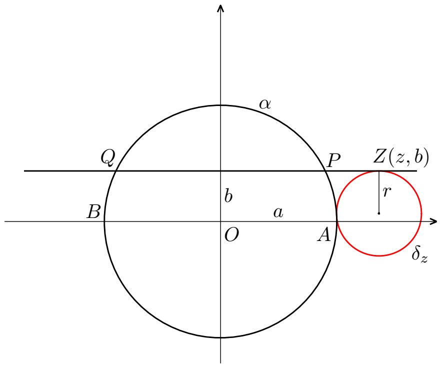

Figure 13: Z does not lie between P and Q

Assume Z does not lie between P and Q. If we consider that r has minus sign in this case, then δ_Z can also be represented by (21) with (20). Therefore in any case δ_Z can be represented by the equation $\delta_Z(x, y) = 0$ with (20).

4.1.1. *The circle* δ_L. Let L be the midpoint of PQ. The point L has coordinates $(0, b)$. We consider the circle δ_L.

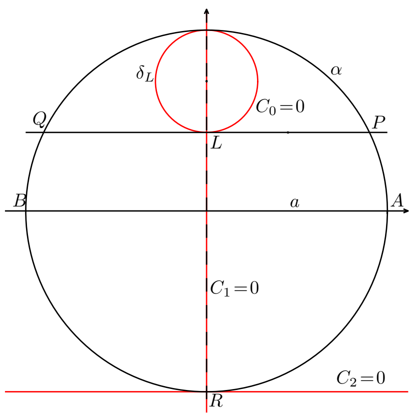

Figure 14: The y-axis touches α and PQ

The Laurent expansion of $\delta_Z(x,y)$ about $z=0$ is

$$\begin{aligned}\delta_Z(x,y) &= \sum_{n=-\infty}^{\infty} C_n z^n \\ &= \left(x^2+\left(y-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2\right) \\ &- 2xz+\left(\frac{y+a}{a+b}\right)z^2.\end{aligned}$$

Therefore we have
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$,
(ii) $C_0 = x^2+\left(y-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$,
(iii) $C_1 = -2x$,
(iv) $C_2 = \frac{y+a}{a+b}$,
(v) $C_3 = C_4 = C_5 = \cdots = 0$.

The equation $C_0 = 0$ represents the circle of radius $(a-b)/2$ and the center of coordinates $(0,(a+b)/2)$. This circle coincides with δ_L. The equation $C_1 = 0$ represents the y-axis. Let R be the midpoint of the major arc of α cut by PQ. The equation $C_2 = 0$ is the tangent of α at the point R. The figures obtained by $C_0 = 0$, $C_1 = 0$ and $C_2 = 0$ are indicated in Figure 14 in red. They touch both the circle α and the line PQ by Theorem 2(i).

4.1.2. *The circle* δ_M. Let M be the midpoint of LP, which has coordinates $(\sqrt{a^2-b^2}/2, b)$. We consider the circle δ_M (see Figure 15).

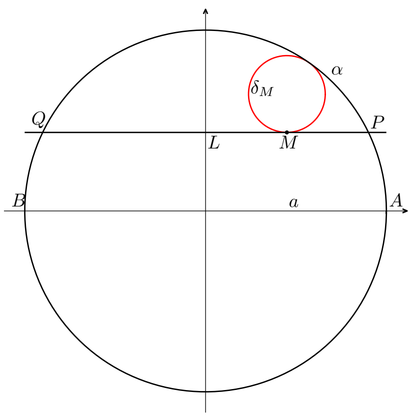

Figure 15: The circle δ_M

Let $z_m = \sqrt{a^2 - b^2}/2$. The Laurent expansion of $\delta_Z(x,y)$ about $z = z_m$ is

$$\begin{aligned}\delta_Z(x,y) &= \sum_{n=-\infty}^{\infty} C_n (z - z_m)^n \\ &= (x - z_m)^2 + \left(y - \frac{3a+5b}{8}\right)^2 - \left(\frac{3(a-b)}{8}\right)^2 \\ &+ \left(-2x + \sqrt{\frac{a-b}{a+b}}(y+a)\right)(z - z_m) \\ &+ \left(\frac{y+a}{a+b}\right)(z-z_m)^2.\end{aligned}$$

Therefore we have
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$,
(ii) $C_0 = (x - z_m)^2 + \left(y - \frac{3a+5b}{8}\right)^2 - \left(\frac{3(a-b)}{8}\right)^2$,
(iii) $C_1 = -2x + \sqrt{\frac{a-b}{a+b}}(y+a)$,
(iv) $C_2 = \frac{y+a}{a+b}$,
(v) $C_3 = C_4 = C_5 = \cdots = 0$.

The equation $C_0 = 0$ represents the circle of radius $3(a-b)/8$ and the center of coordinates $(z_m, (3a+5b)/8)$. Since

$$(z_m)^2 + \left(\frac{1}{8}(3a+5b)\right)^2 = \left(a - \frac{3(a-b)}{8}\right)^2,$$

this circle coincides with the circle δ_M. Let ε be the circle of center R passing through the point P. If we invert the figure by the inversion in the circle ε, the line PQ is inverted to the circle α. Hence δ_M is fixed by the inversion. Therefore the line MR passes through the point of tangency of the circles α and δ_M. The equation $C_1 = 0$ represents the line MR. The equation $C_2 = 0$ represents the tangent of α at the point R. The figures obtained by $C_0 = 0$, $C_1 = 0$ and $C_2 = 0$ are described in Figure 16 in red. The line MR does not touch the circle α and the line PQ. But it forms the same angle with the two figures.

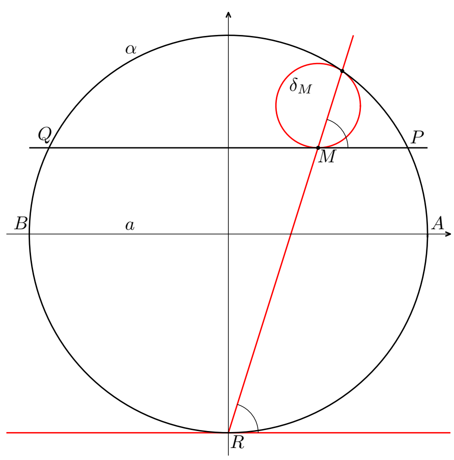

Figure 16: MR forms the same angle with α and PQ

4.1.3. *The circle δ_P.* Let $z_p = \sqrt{a^2 - b^2}$. We consider the circle δ_P, where P has coordinates (z_p, b). The Laurent expansion of $\delta_z(x, y)$ about $z = z_p$ is

$$\begin{aligned}\delta_Z(x,y) &= \sum_{n=-\infty}^{\infty} C_n (z - z_p)^n \\ &= \left((x - z_p)^2 + (y - b)^2\right) \\ &+ 2\left(-x + \sqrt{\frac{a-b}{a+b}}(y + a)\right)(z - z_p) \\ &+ \left(\frac{y+a}{a+b}\right)(z - z_p)^2.\end{aligned}$$

Therefore we have
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$,
(ii) $C_0 = (x - z_p)^2 + (y - b)^2$,
(iii) $C_1 = 2\left(-x + \sqrt{\dfrac{a-b}{a+b}}(y + a)\right)$,
(iv) $C_2 = \dfrac{y+a}{a+b}$,
(v) $C_3 = C_4 = C_5 = \cdots = 0$.

The equation $C_0 = 0$ represents the point P. The equation $C_1 = 0$ represents the line PR. The equation $C_2 = 0$ represents the tangent of α at the point R. The figures obtained by $C_0 = 0$, $C_1 = 0$ and $C_2 = 0$ are described in Figure 17 in red.

The point P and the tangent of α at R can be considered to touch both α and PQ, but the line PR can not. But it forms the same angle with α and PQ. The same result using different coordinate system can be found in [9].

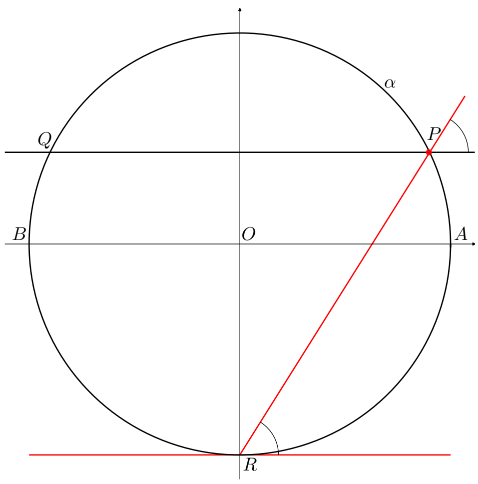

Figure 17: δ_P is the point P

4.1.4. *The circle δ_N.* Let N be the point on the line PQ such that P is the midpoint of the segment MN. The point N has coordinates (z_n, b), where $z_n = 3\sqrt{a^2 - b^2}/2$. We consider the circle δ_N (see Figure 18).

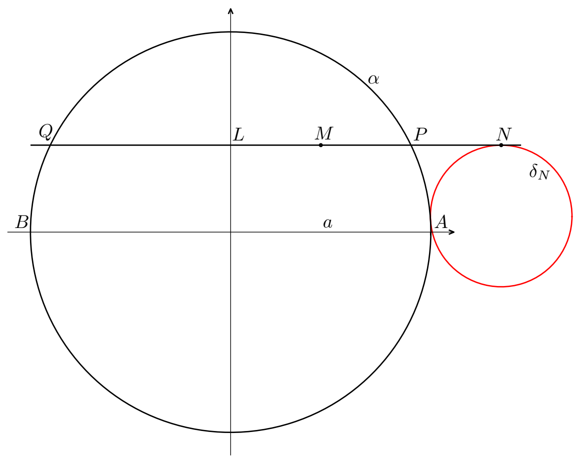

Figure 18: The circle δ_N

The Laurent expansion of $\delta_Z(x,y)$ about $z = z_n$ is

$$\begin{aligned}\delta_Z(x,y) &= \sum_{n=-\infty}^{\infty} C_n(z-z_n)^n \\ &= (x-z_n)^2 + \left(y - \frac{-5a+13b}{8}\right)^2 - \left(\frac{5(a-b)}{8}\right)^2 \\ &+ \left(-2x + 3\sqrt{\frac{a-b}{a+b}}(y+a)\right)(z-z_n) \\ &+ \left(\frac{y+a}{a+b}\right)(z-z_n)^2.\end{aligned}$$

Therefore we have
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$,
(ii) $C_0 = (x-z_n)^2$
$+ \left(y - \frac{-5a+13b}{8}\right)^2 - \left(\frac{5(a-b)}{8}\right)^2$,
(iii) $C_1 = -2x + 3\sqrt{\frac{a-b}{a+b}}(y+a)$,
(iv) $C_2 = \frac{y+a}{a+b}$,
(v) $C_3 = C_4 = C_5 = \cdots = 0$.

The equation $C_0 = 0$ represents the circle of radius $5(a-b)/8$ and the center of coordinates $(z_n, (-5a+13b)/8)$. Since

$$z_n^2 + \left(\frac{1}{8}(-5a+13b)\right)^2 = \left(a + \frac{5(a-b)}{8}\right)^2,$$

this circle coincides with the circle δ_N. The circle δ_N is fixed by the inversion in the circle ε. Therefore the line NR passes through the point of tangency of the circles α and δ_N. The equation $C_1 = 0$ represents the line NR. The equation $C_2 = 0$ represents the tangent of α at the point R. The figures obtained by $C_0 = 0$, $C_1 = 0$ and $C_2 = 0$ are described in Figure 19 in red. The line NR does not touch the circle α and the line PQ. But it forms the same angle with the two figures.

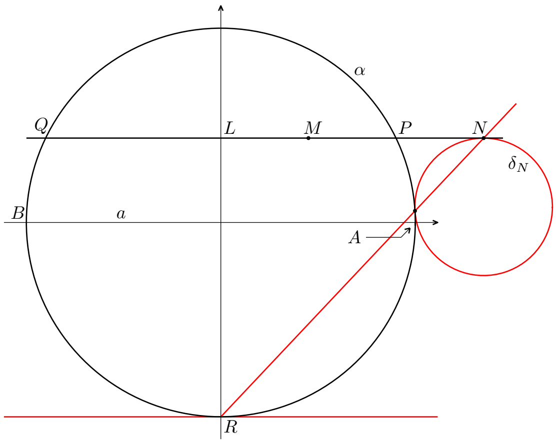

Figure 19: The circle δ_N

4.1.5. *Cosine of two circles.* We consider one more case. However it is not simple to consider the angle between two circles in that case. Hence we introduce the cosine of two circles before considering the case. For two circles δ_1 and δ_2 of radii r_1 and r_2, we define the cosine of the two circles by

$$\cos(\delta_1, \delta_2) = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}, \tag{22}$$

where d is the distance between the centers of the two circles. If the two circles intersect, it is actually the cosine of the angle between them (see Figure 20).

For a circle δ of radius r and a line l, whose distance from the center of δ equals d, we define the cosine of l and δ by

$$\cos(l, \delta) = \frac{d}{r}.$$

If they intersect, it is actually the cosine of the angle between them (see Figure 21).

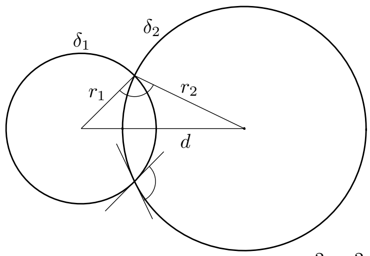

Figure 20: $\cos(\delta_1, \delta_2) = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}$

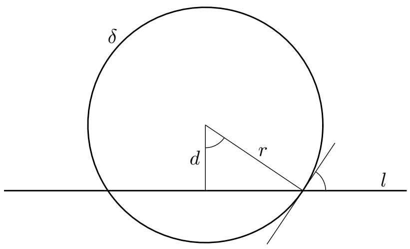

Figure 21: $\cos(l, \delta) = \frac{d}{r}$

4.1.6. *The case δ_z and PQ being parallel.* At the end of this subsection, we consider the case in which the circle δ_Z touches the circle α at the point R. To consider the case, we consider the circle δ_z using another parameter. We denote the origin and center of δ_Z by O and D, respectively. Let θ be the angle such that if we rotate $\overrightarrow{OA}$ through an angle of θ radian counterclockwise about O then it overlaps with $\overrightarrow{OD}$ (see Figure 22). The point D has coordinates $((a-r)\cos\theta, (a-r)\sin\theta)$, where recall that r has minus sign if it touches α externally. From $(a-r)\sin\theta = b+r$, we have

$$r = \frac{-b + a\sin\theta}{1+\sin\theta}. \tag{23}$$

Using (23), the circle δ_Z is represented by the equation

$$\delta_\theta = 0, \tag{24}$$

where

$$\delta_\theta(x,y) = (x-(a-r)\cos\theta)^2 + (y-(a-r)\sin\theta)^2 - r^2.$$

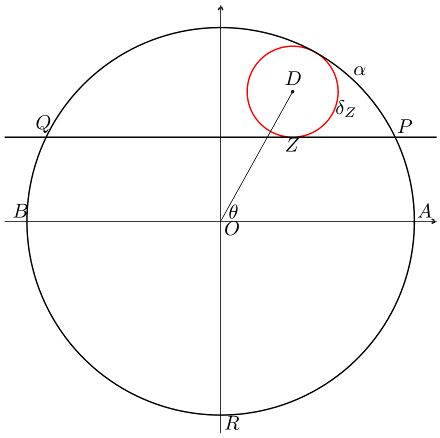

Figure 22: Representing δ_z by the parameter θ

We consider the case $\theta = -\pi/2$, and the Laurent expansion of $\delta_\theta(x,y)$ about $\theta = -\pi/2$:

$$\delta_\theta(x,y) = \sum_{n=-\infty}^{\infty} C_n \left(\theta + \frac{\pi}{2}\right)^n.$$

However we can not obtain all the coefficients C_n collectively. Therefore we consider only the cases $n = 0, \pm1, \pm2$. Then we have
(i) $C_{-2} = 4(a+b)(y+a)$,
(ii) $C_{-1} = -4(a+b)x$,
(iii) C_0
$$= x^2 + \left(y - \frac{5(a+b)}{6}\right)^2 - \left(\frac{\sqrt{49a^2+38ab+25b^2}}{6}\right)^2,$$ (iv) $C_1 = \dfrac{(a+b)x}{3}$,
(v) $C_2 = \dfrac{(a+b)(y+a)}{60}$.

The equations $C_{-2} = C_2 = 0$ represent the tangent of the circle α at the point R. The equations $C_{-1} = C_1 = 0$ represent the y-axis. The equation $C_0 = 0$ represents the circle of radius $\sqrt{49a^2+38ab+25b^2}/6$ and center of coordinates $(0, 5(a+b)/6)$. We denote this circle by δ_∞. It does not touch the circle α and the line PQ, but intersects the two figures at the same angle, because

$$\cos(PQ, \delta_\infty) = \cos(\alpha, \delta_\infty) = \frac{5a-b}{\sqrt{49a^2+38ab+25b^2}}.$$

Since the circle ε has center of coordinates $(0, -a)$ and radius $\sqrt{2a(a+b)}$, the circles δ_∞ and ε are orthogonal. The figures represented by $C_{\pm 2} = 0$, $C_{\pm 1} = 0$ and $C_0 = 0$ are described in Figure 23 in red.

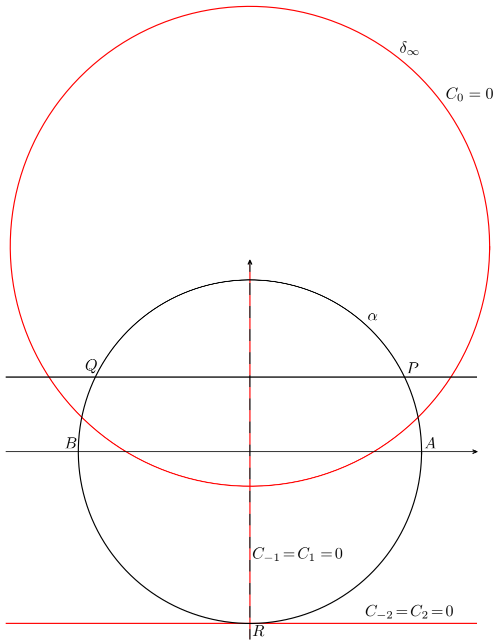

Figure 23: The circle δ_∞

4.2. **Arbelos with division by zero calculus.** We have considered the arbelos with division by zero in section 3. We now consider the arbelos with division by zero calculus. We consider three circles instead of the three semicircles forming the arbelos. For a point C on the segment AB, α, β and γ are the circles of diameters BO, AO and AB, respectively, where $|BO| = 2a$, $|AO| = 2b$, $|AB| = 2c$. We use the following theorem with the same coordinate system. Recall $d_t = \sqrt{ab}/c$.

Theorem 8 ([9]). *The following statements hold.*
(i) *A proper circle touches the circles α and β in the same sense if and only if its has radius r_z^γ and center of coordinates (x_z^γ, y_z^γ) given by*

$$q_z^\gamma = \frac{abc}{c^2z^2 - ab}, \quad r_z^\gamma = |q_z^\gamma|$$

$$\text{and} \quad (x_z^\gamma, y_z^\gamma) = \left(\frac{b-a}{c} q_z^\gamma, 2zq_z^\gamma\right)$$

for a real number $z \neq \pm d_t$.
(ii) *A circle touches the circles β and γ in the opposite sense if and only if it has radius r_z^α and center of coordinates (x_z^α, y_z^α) given by*

$$r_z^\alpha = \frac{abc}{a^2z^2 + bc}, \quad (x_z^\alpha, y_z^\alpha) = \left(-2b + \frac{b+c}{a} r_z^\alpha, 2zr_z^\alpha\right)$$

for a real number z.

The circle given by (i) is denoted by γ_z and represented by the equation

$$\gamma_z(x,y) = (x - x_z^\gamma)^2 + (y - y_z^\gamma)^2 - r_z^\gamma = 0. \tag{25}$$

Similarly the circle given by (ii) is denoted by α_z and represented by the equation

$$\alpha_z(x,y) = (x - x_z^\alpha)^2 + (y - y_z^\alpha)^2 - r_z^\alpha = 0. \tag{26}$$

4.2.1. *The circle* $\overline{\gamma}$. Since

$$\frac{1}{d_t} = \frac{c}{\sqrt{ab}} = \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \geq 2\sqrt{\frac{a}{b}}\sqrt{\frac{b}{a}} = 2,$$

we have $d_t \leq 1/2$. The external common tangents of α and β are represented by the equations ([14])

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0, \tag{27}$$

which are denoted by $\gamma_{\pm d_t}$.

We consider the circle γ_z in the singular case $z = d_t$. From the Laurent expansion of $\gamma_z(x,y)$ about $z = d_t$

$$\gamma_z(x,y) = \sum_{n=-\infty}^{\infty} C_n (z - d_t)^n,$$

we get
(i) $\cdots = C_{-4} = C_{-3} = C_{-2} = 0$,
(ii) $C_{-1} = d_t((a-b)x - 2\sqrt{ab}y + 2ab)$,
(iii) $C_0 = \left(x - \dfrac{a-b}{4}\right)^2 + \left(y - \dfrac{\sqrt{ab}}{2}\right)^2 - \left(\dfrac{\sqrt{a^2+18ab+b^2}}{4}\right)^2$,
(iv) $C_n = -\dfrac{1}{2}\left(\dfrac{-1}{2d_t}\right)^n ((a-b)x + 2\sqrt{ab}y + 2ab)$ for $n = 1, 2, 3, \cdots$.

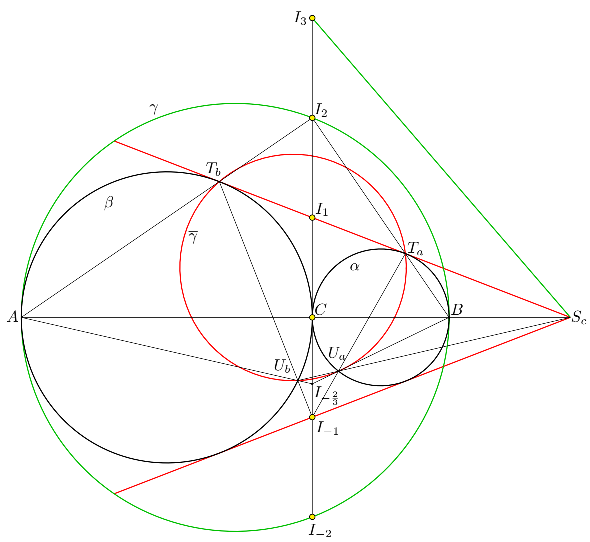

Figure 24: The circle $\overline{\gamma}$ with evenly spaced six points

Therefore the equation $C_{-1} = 0$ represents the line γ_{d_t} given by (27). The equation $C_0 = 0$ represents a circle, whose radius and the coordinates of its center are given by

$$\frac{\sqrt{a^2+18ab+b^2}}{4}, \quad \left(\frac{a-b}{4}, \frac{\sqrt{ab}}{2}\right).$$

We denote this circle by $\overline{\gamma}$. The equations $C_1 = C_2 = C_3 = \cdots = 0$ represent the line γ_{-d_t} given by (27). The figures obtained by $C_n = 0$ are described in Figure 24 in red.

Let I_k be the point of coordinate $(0, k\sqrt{ab})$. The axis meets γ_{d_t} and γ at the points I_1 and $I_{\pm 2}$, respectively. Let T_a (resp. T_b) be the point of tangency of γ_{d_t} and α (resp. β), and let $U_a(\neq T_a)$ (resp. $U_b(\neq T_b)$) be the point of intersection of the line $I_{-1}T_a$ and α (resp. $I_{-1}T_b$ and β). Then the circle $\overline{\gamma}$ passes through the points T_a, T_b, U_a and U_b. For the points have coordinates

$$T_a : \left(\frac{2ab}{c}, \frac{2ab}{c}\sqrt{\frac{a}{b}}\right), \quad T_b : \left(-\frac{2ab}{c}, \frac{2ab}{c}\sqrt{\frac{b}{a}}\right),$$

$$U_a : \left(\frac{2ab}{8a+c}, -\frac{6ab}{8a+c}\sqrt{\frac{a}{b}}\right),$$

$$U_b : \left(-\frac{2ab}{8b+c}, -\frac{6ab}{8b+c}\sqrt{\frac{b}{a}}\right).$$

Let S_c be the external center of similitude of the circles α and β. Then S_c has coordinates $(-2ab/(a-b), 0)$ and the points U_a, U_b and S_c are collinear. The equation $(x-(a-b))^2 + y^2 = c^2$ represents the circle γ. From

$$\begin{aligned} &\left(x - \frac{a-b}{4}\right)^2 + \left(y - \frac{\sqrt{ab}}{2}\right)^2 - \left(\frac{\sqrt{a^2+18ab+b^2}}{4}\right)^2 \\ &-((x-(a-b))^2 + y^2 - c^2) \\ &= 3(a-b)x - 2\sqrt{ab}y + 6ab, \end{aligned}$$

we see that the radical axis of the circles γ and $\overline{\gamma}$ is represented by the equation $3(a-b)x - 2\sqrt{ab}y + 6ab = 0$. Since the coordinates of the points I_3 and S_c satisfy the equation, the line I_3S_c is the radical axis of the two circles. The fact that the lines BT_a and AT_b meet at the point I_2 is well known. Similarly the lines BU_a and AU_b meet at the point $I_{-\frac{2}{3}}$.

Since

$$\cos(\alpha, \overline{\gamma}) = \cos(\beta, \overline{\gamma}) = \frac{c}{\sqrt{a^2+18ab+b^2}},$$

the circle $\overline{\gamma}$ does not touch the circles α and β, but intersects them at the same angle.

4.2.2. *The internal center of similitude of the circles β and γ.* We give an example that we can still get a meaningful results from a non-singular case by division by zero calculus. We consider the circle α_z represented by the equation (26). If we consider $\alpha_z(x,y)$ as a function of z, there is no singular case. We consider the Laurent expansion of $\alpha_z(x,y)$ about $z=0$:

$$\alpha_z(x,y) = \sum_{n=-\infty}^{\infty} C_n z^n.$$

Then we get
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$,
(ii) $C_0 = (x-a)^2 + y^2 - a^2$,
(iii) $C_{2n-1} = (-1)^n \dfrac{4a^{2n-1}}{(bc)^{n-1}} y$ for $n = 1, 2, 3, \cdots$,
(iv) $C_{2n} = (-1)^{n-1} \dfrac{2a^{2n}(b+c)}{(bc)^n} \left(x + \dfrac{2b^2}{b+c} \right)$ for $n = 1, 2, 3, \cdots$.
Therefore $C_0 = 0$ represents the circle α. The equations $C_1 = C_3 = C_5 = \cdots = 0$ represent the line AB. The equations $C_2 = C_4 = C_6 = \cdots = 0$ represent the line given by $x = -2b^2/(b+c)$, which is denoted by s_a. The three figures represented by $C_n = 0$ are described in Figure 25 in red. Notice that the circle α can be obtained in the usual way from (26), but the lines s_a and AB can not.

The line AB touches the circles β and γ by Theorem 2(i), i.e., AB is eligible to be a figure touching β and γ in the opposite sense, where we ignore the words "in the opposite sense", since AB is not a proper circle and the words have no sense. Recall that we have considered a similar figure γ_0 in Theorem 4.

Let S_a be the point of intersection of the lines s_a and AB. The point S_a is the internal center of similitude of the circles β and γ. Though the line s_a does not touch the circles β and γ, it intersects them at the same angle, because $\cos(s_a, \beta) = \cos(s_a, \gamma) = a/(b+c)$ ([9]). Therefore it is a figure similar to the line MR in 4.1.2, the line PR in 4.1.3, the line NR in 4.1.4, the circle δ_∞ in 4.1.6, and the circle $\overline{\gamma}$ in 4.2.1.

Let σ_a be the circle of center S_a passing through the point A. Then σ_a has radius $2bc/(b+c)$. Since

$$\frac{2b^2}{b+c}\left(2a + \frac{2b^2}{b+c}\right) = \frac{2b^2}{b+c}\left(2(c-b) + \frac{2b^2}{b+c}\right)$$

$$= \frac{2b^2(2(c^2-b^2)+2b^2)}{(b+c)^2} = \frac{4b^2c^2}{(b+c)^2},$$

the points B and C are inverses by the inversion in the circle σ_a. Therefore the circles β and γ are also inverses by the inversion. This implies that if a circle α_k touches β and γ in the opposite sence, then the line joining the point of tangency of α_k and β and the point of tangency of α_k and γ passes through the point S_a.

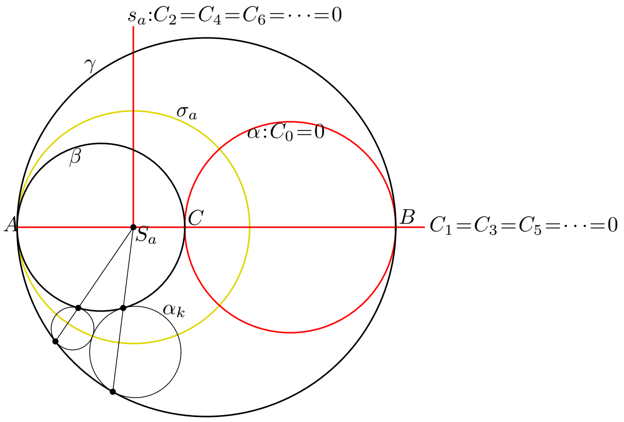

Figure 25: The circles α and σ_a are orthogonal

5. Conclusion

Since division by zero and division by zero calculus are definitions, it has no sense to consider that they are true or not. Thereby the discussion in 2.1 is unnecessary. Alternatively we have to consider that they makes our mathematics richer. In sections 3 and 4 we have shown several examples which show that division by zero calculus makes our mathematics much richer. Saburou Saitoh has a list of more than a thousand such examples, some of which can be found in [16]. For an extensive reference of division by zero calculus, see [16].

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