A framework to explore zeros of analytic functions

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September 15, 2025

Abstract

This paper explores the zeros of symmetric analytic functions, focusing on the Riemann zeta function. Using the Abel-Plana formula and an auxiliary function, we investigate their distribution. Our approach reformulates the problem algebraically and employs a proof by contradiction. We demonstrate this approach by applying it to the Riemann zeta function.

Keywords: Riemann zeta function, Abel-Plana formula, Symmetry of zeros

1 Separating zeta into real and imaginary parts

This section introduces the first step of our framework, which separates the function into its real and imaginary parts. In our case, we decompose the Riemann zeta function ζ using the Abel-Plana formula. This section lays the groundwork for our framework.

1.1 Abel-Plana Formula for the zeta function

We begin with the Abel-Plana representation of the zeta function (see DLMF, Eq. (25.5.11) [dlm10]):

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\int_0^\infty \frac{\sin(s\arctan(t))}{(1+t^2)^{s/2}(e^{2\pi t}-1)}dt \tag{1.1}$$

Denote:

$$\omega(s) = \int_0^\infty \frac{\sin(s\arctan(t))}{(1+t^2)^{s/2}(e^{2\pi t}-1)}dt, \tag{1.2}$$

then we have

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\omega(s). \tag{1.3}$$

We aim to decompose ζ into its real and imaginary parts. To do this, we first decompose $\omega(s)$ into its real and imaginary parts. We use the identity

$$\sin((\alpha+i\beta)\gamma) = \sin(\alpha\gamma)\cosh(\beta\gamma) + i\cos(\alpha\gamma)\sinh(\beta\gamma), \quad \alpha,\beta,\gamma\in\mathbb{R},$$

with $\alpha = x$, $\beta = y$, and $\gamma = \arctan(t)$, to obtain:

$$\begin{aligned}\omega(x+yi) &= \int_0^\infty \frac{\sin(x\arctan(t))\cosh(y\arctan(t))}{(e^{2\pi t}-1)(\sqrt{1+t^2})^{x+yi}}dt+ \\ & \quad i\int_0^\infty \frac{\cos(x\arctan(t))\sinh\big(y\arctan(t)\big)}{(e^{2\pi t}-1)(\sqrt{1+t^2})^{x+yi}}dt.\end{aligned}$$

Since $(1+t^2)^{(x+yi)/2} = (1+t^2)^{x/2}(1+t^2)^{iy/2}$, we define:

$$q_0(x,y,t) = \frac{\sin(x\arctan(t))\cosh(y\arctan(t))}{(e^{2\pi t}-1)(\sqrt{1+t^2})^{x}}$$

$$q_1(x,y,t) = \frac{\cos(x\arctan(t))\sinh\big(y\arctan(t)\big)}{(e^{2\pi t}-1)(\sqrt{1+t^2})^{x}}.$$

These definitions allow us to introduce the following functions:

$$\begin{aligned}\mathfrak{u}(x+yi) &= \int_0^\infty q_0(x,y,t)(\sqrt{1+t^2})^{-yi}dt \\ \mathfrak{v}(x+yi) &= \int_0^\infty q_1(x,y,t)(\sqrt{1+t^2})^{-yi}dt.\end{aligned}$$

Consequently, we can express $\omega(s)$ as follows:

$$\omega(x+yi) = \mathfrak{u}(x+yi) + i\mathfrak{v}(x+yi) \tag{1.4}$$

From the definition of powers with imaginary exponents, we have:

$$\begin{aligned}\left(\sqrt{1+t^2}\right)^{-yi} &= \exp\left(\ln\left(\left(\sqrt{1+t^2}\right)^{-yi}\right)\right) = \exp\left(-y\ln\left(\sqrt{1+t^2}\right)i\right) \\ &= \cos\left(y\ln(\sqrt{1+t^2})\right) - i\sin\left(y\ln(\sqrt{1+t^2})\right).\end{aligned}$$

For simplicity, we define the following functions:

$$\mathcal{C}_0(x,y) = \int_0^\infty q_0(x,y,t)\cos(y\ln(\sqrt{1+t^2}))dt$$

$$\mathcal{C}_1(x,y) = \int_0^\infty q_1(x,y,t)\cos(y\ln(\sqrt{1+t^2}))dt$$

$$\mathcal{S}_0(x,y) = \int_0^\infty q_0(x,y,t)\sin(y\ln(\sqrt{1+t^2}))dt$$

$$\mathcal{S}_1(x,y) = \int_0^\infty q_1(x,y,t)\sin(y\ln(\sqrt{1+t^2}))dt.$$

Hence, we have:

$$\mathfrak{u}(x+yi) = \mathcal{C}_0(x,y) - i\mathcal{S}_0(x,y) \quad \textbf{and} \quad \mathfrak{v}(x+yi) = \mathcal{C}_1(x,y) - i\mathcal{S}_1(x,y).$$

From this, it follows that:

$$\omega(x+yi) = \mathfrak{u}(x+yi) + i\mathfrak{v}(x+yi) = \Big(\mathcal{C}_0 + \mathcal{S}_1\Big)(x,y) + i\Big(\mathcal{C}_1 - \mathcal{S}_0\Big)(x,y).$$

For ease of notation, we define:

$$f(x,y) = (\mathcal{C}_0 + \mathcal{S}_1)(x,y) \quad \textbf{and} \quad g(x,y) = (\mathcal{S}_0 - \mathcal{C}_1)(x,y). \tag{1.5}$$

Using this notation, we have:

$$\omega(x+yi) = f(x,y) - ig(x,y). \tag{1.6}$$

To decompose ζ we first simplify the expression $\frac{1}{x+yi-1} + \frac{1}{2}$, we obtain:

$$\tfrac{1}{x+yi-1} + \tfrac{1}{2} = \tfrac{x-1-yi}{(x-1)^2+y^2} + \tfrac{1}{2} = \tfrac{x-1}{(x-1)^2+y^2} + \tfrac{1}{2} - i\tfrac{y}{(x-1)^2+y^2}$$

We define:

$$\mathfrak{a}(x,y) = \tfrac{x-1}{(x-1)^2+y^2} + \tfrac{1}{2} \quad \textbf{and} \quad \mathfrak{b}(x,y) = -\tfrac{y}{(x-1)^2+y^2}. \tag{1.7}$$

Then the full decomposition of ζ may be written as:

$$\zeta(x+yi) = 2f(x,y) + \mathfrak{a}(x,y) + i\Big(\mathfrak{b}(x,y) - 2g(x,y)\Big). \tag{1.8}$$

1.2 Necessary and sufficient condition for zeros of ζ

From the symmetry of the zeros of ζ in the critical strip, we have:

$$\zeta(x+yi) = 0 \Longleftrightarrow \zeta(1-x+yi) = 0 \tag{1.9}$$

We define:

$$F_1 = \tfrac{-\mathfrak{a}(x,y)}{2} \quad , \quad G_1 = \tfrac{\mathfrak{b}(x,y)}{2} \quad , \quad \mathcal{A}_1 = \tfrac{-\mathfrak{a}(1-x,y)}{2} \quad , \quad \mathcal{B}_1 = \tfrac{\mathfrak{b}(1-x,y)}{2}. \tag{1.10}$$

Then we have:

Remark 1.1. The integrals defining $\omega(s)$ and hence $f(x,y)$ and $g(x,y)$, are convergent and define analytic functions for $s = x+iy$ in the critical strip $0 < x < 1$. See [dlm10] and surrounding commentary. All subsequent use of these functions occurs for values of x strictly in $(0,1)$, ensuring analyticity and well-defined evaluation of all expressions.

Lemma 1.2. $\zeta(x+yi)=0$ ***if and only if***

$$F_1 = f(x,y) \quad , \quad G_1 = g(x,y) \quad , \quad \mathcal{B}_1 = g(1-x,y) \quad , \quad \mathcal{A}_1 = f(1-x,y)$$

Proof. Let $x+yi$ be any complex number in the critical strip that is off the critical line such that $\zeta(x+yi)=0$. Since $\zeta(x+yi)=0$, the real part of $\zeta(x+yi)$ equals zero, so:

$$2f(x,y)+\mathfrak{a}(x,y)=0 \Longleftrightarrow f(x,y)=-\tfrac{\mathfrak{a}(x,y)}{2} \Longleftrightarrow f(x,y)=F_1. \tag{1.11}$$

Similarly, since $\zeta(x+yi)=0$, the imaginary part of $\zeta(x+yi)$ is also equal to zero, we have:

$$-2g(x,y)+\mathfrak{b}(x,y)=0 \Longleftrightarrow g(x,y)=\tfrac{\mathfrak{b}(x,y)}{2} \Longleftrightarrow g(x+yi)=G_1. \tag{1.12}$$

Additionally, property (1.9) implies that the real part of $\zeta(1-x+iy)$ is equal to zero, hence using Equation (1.11) we obtain:

$$f(1-x,y)=\mathcal{A}_1. \tag{1.13}$$

Similarly, property (1.9) also implies that the imaginary part of $\zeta(1-x+iy)$ is equal to zero, hence using Equation (1.12) we obtain:

$$g(1-x,y)=\mathcal{B}_1. \tag{1.14}$$

Now for the other direction, we have: $f(x,y)=F_1$ **and** $g(x,y)=G_1$.
From the definition of F_1 and G_1 we have: $f(x,y)=\frac{-\mathfrak{a}(x,y)}{2}$ and $g(x,y)=\frac{\mathfrak{b}(x,y)}{2}$.
Hence,

$$2f(x,y)+\mathfrak{a}(x,y)=0 \quad \textbf{and} \quad \mathfrak{b}(x,y)-2g(x,y)=0.$$

From the last two equations and Equation (1.8), we conclude that $\zeta(x+yi)=0$. ☐

2 The auxiliary function

In this section, we introduce an auxiliary function $\lambda(s)$ designed to incorporate the missing conditions of System (1.10). We define $\lambda(s)$) as follows:

$$\lambda(s)=\frac{4(1-s)}{s+1}\omega(s). \tag{2.1}$$

The function $\omega(s)$ in the above equality was obtained from the Abel–Plana representation of $\zeta(s)$ (see Equation 1.2).

Lemma 2.1. *Let s be any complex number in the critical strip.*

$$\zeta(s)=0 \Longleftrightarrow \lambda(s)=1. \tag{2.2}$$

Proof. From the Abel-Plana representation for ζ (Equation 1.2), we observe that,

$$\begin{aligned} \zeta(s) = 0 \quad & \Longleftrightarrow 2\omega(s) + \tfrac{1}{2} + \tfrac{1}{s-1} = 0 \Longleftrightarrow 2\omega(s) + \tfrac{(s-1)+2}{2(s-1)} = 0 \\ & \Longleftrightarrow 2\omega(s) = \tfrac{s+1}{2(1-s)} \Longleftrightarrow 4\omega(s)\tfrac{1-s}{s+1} = 1 \Longleftrightarrow \lambda(s) = 1. \end{aligned}$$

Hence, Equation (2.2) holds. ∎

Lemma (2.1) and Equation (1.9) imply that the condition $\lambda(s) = 1$ inherits the symmetry of ζ 's zeros, thus we have:

$$\zeta(x + yi) = 0 \Longleftrightarrow \lambda(x + yi) = 1 = \lambda(1 - x + yi). \tag{2.3}$$

Next, we proceed to decompose λ. Multiplying the numerator and denominator by the conjugate $(1 + x - yi)$, we obtain:

$$\frac{1 - x - yi}{1 + x + yi} = \frac{(1 - x - yi)(1 + x - yi)}{(1 + x)^2 + y^2} = \frac{1 - x^2 - y^2}{1 + 2x + x^2 + y^2} - \frac{2yi}{1 + 2x + x^2 + y^2}$$

We define:

$$d_0(x, y) = \frac{4(1 - x^2 - y^2)}{1 + 2x + x^2 + y^2} \quad , \quad d_1(x, y) = \frac{8y}{1 + 2x + x^2 + y^2}$$

From this, we have:

$$\begin{aligned} \lambda(x + yi) \quad &= (d_0 - id_1)\Big((\mathcal{C}_0 + \mathcal{S}_1) + i(\mathcal{C}_1 - \mathcal{S}_0)\Big)(x, y) \\ &= \left(\begin{array}{c} (\mathcal{C}_0 + \mathcal{S}_1)d_0 + (\mathcal{C}_1 - \mathcal{S}_0)d_1 + \\ \big((\mathcal{C}_1 - \mathcal{S}_0)d_0 - (\mathcal{C}_0 + \mathcal{S}_1)d_1\big)i \end{array} \right)(x, y) \end{aligned} \tag{2.4}$$

Substituting the definitions of f and g (in the above equality), we have:

$$\lambda(x + yi) = \Big(f \cdot d_0 - g \cdot d_1 - \big(g \cdot d_0 + f \cdot d_1\big)i\Big)(x, y) \tag{2.5}$$

Substituting the definitions of d_0 and d_1 into the last equation, we obtain:

$$\begin{aligned} \lambda(x + yi) \quad = \quad & f(x, y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} - g(x, y)\tfrac{8y}{1+2x+x^2+y^2} - \\ & \Big(g(x, y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + f(x, y)\tfrac{8y}{1+2x+x^2+y^2}\Big)\, i \end{aligned} \tag{2.6}$$

3 Transforming the problem into algebra

In this section we transform the problem into an algebraic framework. To achieve this we first define a system of equations, denoted $\mathscr{S}$, as follows:

$$
\begin{aligned}
&f(x,y) = \frac{1}{2}\cdot\frac{1-x}{(x-1)^2+y^2} - \frac{1}{4}\\
&g(x,y) = -\frac{1}{2}\cdot\frac{y}{(1-x)^2+y^2}\\
&f(1-x,y) = \frac{1}{2}\cdot\frac{x}{x^2+y^2} - \frac{1}{4}\\
&g(1-x,y) = -\frac{1}{2}\cdot\frac{y}{x^2+y^2}\\
&\mathrm{Im}\ \lambda(x+yi) = 0\\
&\mathrm{Re}\ \lambda(1-x+yi) = 1.
\end{aligned}
\tag{3.1}
$$

Lemma 3.1. *Let x and y be any nonzero real numbers such that $x \in (0,1)$. Then,*

$$\zeta(x+yi) = 0 \quad \Longleftrightarrow \quad x \text{ and } y \text{ solve system } \mathscr{S}. \tag{3.2}$$

Proof.
$\Rightarrow$
Let x and y be any nonzero reals such that $x \in (0,1)$ and $\zeta(x+yi) = 0$. Lemma (1.2) implies that x and y satisfy the first four equations of system $\mathscr{S}$. From Equations (2.2), (2.3) and (2.6) we find that x and y satisfy the rest of the equations of system $\mathscr{S}$.
$\Leftarrow$
Let x and y be any nonzero real numbers such that $x \in (0,1)$ and x, y satisfy system $\mathscr{S}$. From the first four equations and Lemma (1.2) we have: $\zeta(x+yi) = 0$. $\square$

Remark 3.2. Note that the function λ is defined on $\mathbb{C} \setminus \{-1\}$ and does not generally differ from ζ by a constant c. To illustrate this, we explicitly assume that $\zeta(s) = \lambda(s) + c$. Hence,

$$
\begin{aligned}
\zeta(s) &= \lambda(s) - c\\
\frac{1}{s-1} + \frac{1}{2} + 2\omega(s) &= 4\cdot\frac{1-s}{1+s}\omega(s) - c\\
\omega(s)\left(2 + \frac{s-1}{1+s}\cdot 4\right) &= \frac{1}{1-s} - \frac{1}{2} - c\\
\omega(s) &= \frac{\frac{1}{1-s} - \frac{1}{2} - c}{2 + \frac{s-1}{1+s}\cdot 4}.
\end{aligned}
$$

Since the integral defining function $\omega(s)$ is not rational, the functions $\lambda(s) - c$ and $\zeta(s)$ must differ. Nevertheless, under the specific condition $\zeta(s) = 0$, we have precisely: $\zeta(s) = \lambda(s) - 1$.

To investigate the zeros of ζ, we construct a system of equations similar in structure and properties to $\mathscr{S}$. This system is built under the following reductio ad absurdum (RAA) assumption:

$$\exists a, b \in \mathbb{R}, \text{ such that } b \neq 0 \text{ and } a \in (0,1) \setminus \{1/2\} \text{ and } \zeta(a+bi) = 0 \tag{3.3}$$

Unless otherwise stated, we assume that Assumption (3.3) holds throughout this paper. Under this assumption, we introduce the following system of equations:

$$\begin{aligned}
&\hat{f}(x,y) = \tfrac{1}{2} \cdot \tfrac{1-x}{(x-1)^2+y^2} - \tfrac{1}{4} \\
&\hat{g}(x,y) = -\tfrac{1}{2} \cdot \tfrac{y}{(1-x)^2+y^2} \\
&\hat{f}(1-x,y) = \tfrac{1}{2} \cdot \tfrac{x}{x^2+y^2} - \tfrac{1}{4} \\
&\hat{g}(1-x,y) = -\tfrac{1}{2} \cdot \tfrac{y}{x^2+y^2} \\
&\hat{g}(x,y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + \hat{f}(x,y) \cdot \tfrac{8y}{1+2x+x^2+y^2} = 0 \\
&\hat{f}(1-x,y) \cdot \tfrac{4(1-(1-x)^2-y^2)}{1+2(1-x)+(1-x)^2+y^2} - \hat{g}(1-x,y) \cdot \tfrac{8y}{1+2(1-x)+(1-x)^2+y^2} = 1.
\end{aligned} \tag{3.4}$$

The system in (3.4) is similar in structure to the system $\mathscr{S}$. However, in system (3.4) we treat x, y, $\hat{f}(x,y)$, $\hat{f}(1-x,y)$, $\hat{g}(x,y)$, and $\hat{g}I(1-x,y)$ as variable placeholder. With this interpretation, the system (3.4) becomes redundant; it admits all (x,y) in the plane and therefore provides no new information about the zeros of the zeta function.

Remark 3.3 (Connection to zeta). using Lemma (1.2), we observe that for every $(x,y) \in \mathbb{R}^2$ such that $\zeta(x+yi) = 0$, the following hold:

$$\begin{aligned}
f(x,y) = \hat{f}(x,y) \quad &\text{and} \quad g(x,y) = \hat{g}(x,y) \\
f(1-x,y) = \hat{f}(1-x,y) \quad &\text{and} \quad g(1-x,y) = \hat{g}(1-x,y).
\end{aligned}$$

Moreover, when $\zeta(x+yi) = 0$, we also have:

$$\begin{aligned}
\text{Im } \lambda(x+yi) \quad &= \hat{g}(x,y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + \hat{f}(x,y) \cdot \tfrac{8y}{1+2x+x^2+y^2} = 0 \\
&= g(x,y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + f(x,y) \cdot \tfrac{8y}{1+2x+x^2+y^2} = 0 \\
\text{Re } \lambda(x+yi) \quad &= \hat{f}(1-x,y) \cdot \tfrac{4(1-(1-x)^2-y^2)}{1+2(1-x)+(1-x)^2+y^2} - \tfrac{\hat{g}(1-x,y)\cdot 8y}{1+2(1-x)+(1-x)^2+y^2} \\
&= f(1-x,y) \cdot \tfrac{4(1-(1-x)^2-y^2)}{1+2(1-x)+(1-x)^2+y^2} - \tfrac{g(1-x,y)\cdot 8y}{1+2(1-x)+(1-x)^2+y^2} = 1.
\end{aligned}$$

Hence, system (3.4) is strongly related to the problem of locating zeta zeros off the critical line. In this sense, it is unsurprising that system (3.4) turns out to be redundant. If this system were determined, then the value of the off-line zero $a+bi$ (from the RAA assumption) could be found by solving a simple algebraic system of equations.

We address the limitation of system (3.4) by introducing a family of systems, denoted $\mathscr{S}_t$, parameterized by a real number t. We define:

$$u_1(t) = \frac{a + tbi}{a + bi}$$

and

$$u_2(x + yi) = \frac{x + yi}{a + bi}.$$

The system $\mathscr{S}_t$ is defined as follows:

$$\begin{aligned}
&(1) \quad f^*(x, y) = \left(\tfrac{1}{2} \cdot \tfrac{1-x}{(x-1)^2+y^2} - \tfrac{1}{4}\right) \cdot u_1(t) \\
&(2) \quad g^*(x, y) = -\tfrac{1}{2} \cdot \tfrac{y}{(1-x)^2+(y)^2} \\
&(3) \quad f^*(1 - x, y) = \left(\tfrac{1}{2} \cdot \tfrac{x}{x^2+y^2} - \tfrac{1}{4}\right) \cdot u_1(t) \\
&(4) \quad g^*(1 - x, y) = -\tfrac{1}{2} \cdot \tfrac{y}{x^2+(y)^2} \cdot \tfrac{1}{u_2(x+yi)} \\
&(5) \quad g^*(x, y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + f^*(x, y) \cdot \tfrac{8y}{1+2x+x^2+y^2} = 0 \\
&(6) \quad f^*(1 - x, y) \cdot \tfrac{4(1-(1-x)^2-y^2)}{1+2(1-x)+(1-x)^2+y^2} - g^*(1 - x, y) \cdot \tfrac{8y}{1+2(1-x)+(1-x)^2+y^2} = 1 \\
&(7) \quad \text{If } b > 0, \text{ then } y = |y|. \qquad \text{If } b < 0, \text{ then } -y = |y|. \\
&(8) \quad y \neq 0,
\end{aligned} \tag{3.5}$$

We now define the notion of limit of system.

Definition 3.4. Let c be a real constant and $\mathfrak{S}_t$ be any family of systems of equations. The system $\lim_{t\to c} \mathfrak{S}_t$ is obtained by taking the limit $\lim_{t\to c}$ from both sides of each equation in $\mathfrak{S}_t$.

Using Definition (3.4) we will analyze the system $\lim_{t\to 1} \mathscr{S}_t$. From the definition of limits of system (definition 3.4) we find that the system $\lim_{t\to 1} \mathscr{S}_t$ is equivalent to the following system:

$$\begin{aligned}
&(1) \quad f^*(x, y) = \tfrac{1}{2} \cdot \tfrac{1-x}{(x-1)^2+y^2} - \tfrac{1}{4} \\
&(2) \quad g^*(x, y) = -\tfrac{1}{2} \cdot \tfrac{y}{(1-x)^2+y^2} \\
&(3) \quad f^*(1 - x, y) = \tfrac{1}{2} \cdot \tfrac{x}{x^2+y^2} - \tfrac{1}{4} \\
&(4) \quad g^*(1 - x, y) = -\tfrac{1}{2} \cdot \tfrac{y}{x^2+y^2} \cdot \tfrac{a+bi}{x+yi} \\
&(5) \quad g^*(x, y) \cdot \tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + f^*(x, y) \cdot \tfrac{8y}{1+2x+x^2+(y)^2} = 0 \\
&(6) \quad f^*(1 - x, y) \cdot \tfrac{4(1-(1-x)^2-y^2)}{1+2(1-x)+(1-x)^2+y^2} - g^*(1 - x, y) \cdot \tfrac{8y}{1+2(1-x)+(1-x)^2+y^2} = 1. \\
&(7) \quad \text{If } b > 0, \text{ then } y = |y|. \qquad \text{If } b < 0, \text{ then } -y = |y|. \\
&(8) \quad y \neq 0,
\end{aligned} \tag{3.6}$$

Definition 3.5. Let $D = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), y \neq 0\}$.

- $(x_0, y_0) \in D$ is an *analytic solution* of $\lim_{t\to 1} S_t$ if substituting $f^* \mapsto f$, $g^* \mapsto g$ (the analytic functions from §1) satisfies all equations in (3.6).
- $(x_0, y_0) \in \mathbb{R}^2$ is an *algebraic (symbolic) solution* of $\lim_{t\to 1} S_t$ if the system (3.6) is satisfied when $f^*(x, y), g^*(x, y), f^*(1-x, y), g^*(1-x, y)$ are treated as independent algebraic placeholders subject only to the algebraic relations in (3.6).

Lemma 3.6 (Analytic solution for system $\lim_{t\to 1} \mathscr{S}_t$)**.** *The only analytic solution that the system $\lim_{t\to 1} \mathscr{S}_t$ admits is $x = a$ and $y = b$.*

Proof. To prove our Lemma we need to show that (a, b) must satisfy all the equations in $\lim_{t\to 1} \mathscr{S}_t$ and that the solution align with the analytic interpretation of the functions f, g, and λ. Since $\zeta(a + bi) = 0$ (RAA assumption 3.3) and since $\frac{a+bi}{a+bi} = 1$, we can apply Lemma (1.2) to conclude that $f(a, b) = f^*(a, b)$, $g(a, b) = g^*(a, b)$, $f(1-a, b) = f^*(1-a, b)$, and $g(1-a, b) = g^*(1-a, b)$. Hence, we can use Equation (2.6) to conclude that (a, b) must satisfy equations (1-6) of system (3.6). This holds analytically when we replace f^* with f and g^* with g. Therefore, $(x, y) = (a, b)$ is an analytic solution for $\lim_{t\to 1} \mathscr{S}_t$. Suppose that we have another analytic solution $(x_0, y_0c) \neq (a, b)$ for the system $\lim_{t\to 1} \mathscr{S}_t$, then we have $f(x_0, y_0) = f^*(x_0, y_0)$ and $g(x_0, y_0) = g^*(x_0, y_0)$. We can therefore use (1.7) and (1.8) we conclude that $\zeta(x_0 + y_0 i) = 0$. Applying Lemma (1.2) we have $g(1 - x_0, y_0) = -\frac{1}{2} \cdot \frac{y}{x^2+y^2}$. But $x_0 + y_0 i \neq a + bi$ therefore $\frac{1}{u_2(x_0+y_0 i)} \neq 1$. From the system $\lim_{t\to 1} \mathscr{S}_t$ we have

$$\begin{aligned} g^*(1 - x_0, y_0) \quad &= -\tfrac{1}{2} \cdot \tfrac{y}{x^2+y^2} \cdot \tfrac{1}{u_2(x_0+y_0 i)} \\ &\neq -\tfrac{1}{2} \cdot \tfrac{y}{x^2+y^2} = g(x_0, y_0). \end{aligned}$$

Thus the assumption that there exists $(x_0, y_0) \neq (a, b)$ that analytically solve the system $\lim_{t\to 1} \mathscr{S}_t$ is false. □

Lemma 3.7 (Algebraic solution for system $\lim_{t\to 1} \mathscr{S}_t$)**.** *The only algebraic solution that the system $\lim_{t\to 1} \mathscr{S}_t$ admits is $x = a$ and $y = b$.*

Proof. When we treat $f^*(x, y)$, $g^*(x, y)$, $f^*(1 - x, y)$, and $g^*(1 - x, y)$ as variable placeholders and replace (x, y) by (a, b) the system $\lim_{t\to 1} \mathscr{S}_t$ (or equivalently system 3.6) become exactly the same as the system (3.4). Since (3.4) admits all (x, y) in the plane it also admits (a, b). Thus, the system $\lim_{t\to 1} \mathscr{S}_t$ (algebraically) admits (a, b). Now, suppose that the system $\lim_{t\to 1} \mathscr{S}_t$ admits another algebraic solution $(x_0, y_0) \neq (a, b)$. From (3.6) we have:

$$f^*(1 - x_0, y_0) = \hat{f}(1 - x_0, y_0) \quad , \quad g^*(1 - x_0, y_0) = \hat{g}(1 - x_0, y_0) \cdot \frac{1}{u_2(x_0 + y_0 i)}. \tag{3.7}$$

Since $(x_0, y_0) \neq (a, b)$ we have:

$$\frac{1}{u_2(x_0 + y_0 i)} \neq 1.$$

Since the system (3.4) admits all points in the plane we have:

$$\hat{f}(1-x_0, y_0) \cdot \frac{4(1-(1-x_0)^2 - y_0^2)}{1 + 2(1-x_0) + (1-x_0)^2 + y_0^2} - \hat{g}^*(1-x_0, y_0) \cdot \frac{8y_0}{1 + 2(1-x_0) + (1-x_0)^2 + y_0^2} = 1.$$

Since we assume that the system $\lim_{t \to 1} \mathscr{S}_t$ admits (x_0, y_0), we have:

$$f^*(1-x_0, y_0) \cdot \frac{4(1-(1-x_0)^2 - y_0^2)}{1 + 2(1-x_0) + (1-x_0)^2 + y_0^2} - g^*(1-x_0, y_0) \cdot \frac{8y_0}{1 + 2(1-x_0) + (1-x_0)^2 + y_0^2} = 1$$

The right-hand side of the last two equations is 1, so we can equate their left hand-side. We obtain

$$f^*(1-x_0, y_0) \cdot \tfrac{4(1-(1-x_0)^2 - y_0^2)}{1+2(1-x_0)+(1-x_0)^2+y_0^2} - g^*(1-x_0, y_0) \cdot \tfrac{8y_0}{1+2(1-x_0)+(1-x_0)^2+y_0^2} =$$
$$\hat{f}(1-x_0, y_0) \cdot \tfrac{4(1-(1-x_0)^2 - y_0^2)}{1+2(1-x_0)+(1-x_0)^2+y_0^2} - \hat{g}(1-x_0, y_0) \cdot \tfrac{8y_0}{1+2(1-x_0)+(1-x_0)^2+y_0^2}$$

Using (3.7) we substitute $f^*(1-x_0, y_0)$ and $g^*(1-x_0, y_0)$ in the last equation. We obtain:

$$\hat{f}(1-x_0, y_0) \cdot \tfrac{4(1-(1-x_0)^2 - y_0^2)}{1+2(1-x_0)+(1-x_0)^2+y_0^2} - \hat{g}(1-x_0, y_0) \cdot \tfrac{8y_0}{1+2(1-x_0)+(1-x_0)^2+y_0^2} =$$
$$\hat{f}(1-x_0, y_0) \cdot \tfrac{4(1-(1-x_0)^2 - y_0^2)}{1+2(1-x_0)+(1-x_0)^2+y_0^2} - \tfrac{1}{u_2(x_0+y_0 i)} \hat{g}(1-x_0, y_0) \cdot \tfrac{8y_0}{1+2(1-x_0)+(1-x_0)^2+y_0^2}$$

which equiv to:

$$-\frac{\hat{g}(1-x_0, y_0) \cdot 8y_0}{1 + 2(1-x_0) + (1-x_0)^2 + y_0^2} = -\frac{\hat{g}(1-x_0, y_0) \cdot 8y_0}{1 + 2(1-x_0) + (1-x_0)^2 + y_0^2} \cdot \frac{1}{u_2(x_0 + y_0 i)}.$$

From this we conclude that $\frac{1}{u_2(x_0+y_0 i)} = 1$. This is equivalent to $\frac{a+bi}{x_0+y_0 i} = 1$, hence $a + bi = x_0 + y_0 i$. But we assumed that $(x_0, y_0) \neq (a, b)$. Thus we arrive at a contradiction. Therefore, the only valid solution for the system $\lim_{t \to 1} \mathscr{S}_t$ is (a, b). □

Lemma 3.8. *The system* $\lim_{t \to 1} \mathscr{S}_t$ *admits only one solution* $x = a$ *and* $y = b$. *This solution holds both algebraically and analytically.*

Proof. From Lemma (3.7) we learn that the only algebraic solution of the system $\lim_{t \to 1} \mathscr{S}_t$ is (a, b). From Lemma (3.6) we learn that the only analytic solution of the system $\lim_{t \to 1} \mathscr{S}_t$ is (a, b). Thus, the system $\lim_{t \to 1} \mathscr{S}_t$ admits only the solution $(x, y) = (a, b)$. □

Lemma 3.9. *Let $t \neq 1$ be any real number.*

1. *If $b > 0$, then the algebraic solution for the system $\mathscr{S}_t$ in the domain $(0,1) \times \mathbb{R}$ is $x = 1/2$ and $y = \sqrt{3}/2$.*
2. *If $b < 0$, then the algebraic solution for the system $\mathscr{S}_t$ in the domain $(0,1) \times \mathbb{R}$ is $x = 1/2$ and $y = -\sqrt{3}/2$.*

Proof. We refer only to the case where $b > 0$, the proof for the other case is similar. To prove this lemma we will solve the system $\mathscr{S}_t$ symbolically.
Plug in the right hand sides of equations (1)-(2) into equation (5) of system $\mathscr{S}_t$ we have:

$$\Big(-\tfrac{1}{2}\frac{y}{(1-x)^2+y^2}\Big)\frac{4(1-\big(x^2+y^2\big))}{1+2x+x^2+y^2}+\Big(\tfrac{1}{2}\frac{1-x}{(1-x)^2+y^2}-\tfrac{1}{4}\Big)u_1(t)\,\frac{8y}{1+2x+x^2+y^2}=0.$$

Factor $\dfrac{y}{1+2x+x^2+y^2}$ and simplify the coefficients:

$$\frac{y}{1+2x+x^2+y^2}\left[-\frac{2\big(1-(x^2+y^2)\big)}{(1-x)^2+y^2}+u_1(t)\left(\frac{4(1-x)}{(1-x)^2+y^2}-2\right)\right]=0.$$

since $\frac{y}{1+2x+x^2+y^2} \neq 0$, we have:

$$-\frac{2\big(1-(x^2+y^2)\big)}{(1-x)^2+y^2}+u_1(t)\left(\frac{4(1-x)}{(1-x)^2+y^2}-2\right)=0.$$

Multiply both sides by $(1-x)^2+y^2 \neq 0$, we have:

$$-2\Big(1-(x^2+y^2)\Big)+u_1(t)\left(4(1-x)-2\Big((1-x)^2+y^2\Big)\right)=0.$$

expanding and rearranging, we have:

$$-2\Big(1-(x^2+y^2)\Big)+u_1(t)\Big(2-2(x^2+y^2)\Big)=0.$$

Thus, we have:

$$2\big(1-(x^2+y^2)\big)\Big(u_1(t)-1\Big)=0.$$

If $t \neq 1$ then $u_1(t) \neq 1$, so

$$x^2+y^2=1. \tag{3.8}$$

Applying (3.8) on equations (3)-(4) of system $\mathscr{S}_t$ we have:

$$f^*(1-x,y)=\Big(\frac{x}{2}-\frac{1}{4}\Big)u_1(t), \qquad g^*(1-x,y)=\frac{-y}{2u_2(x+iy)}.$$

Plugging into (6) and multiplying by $1+2(1-x)+(1-x)^2+y^2$ gives

$$\frac{2x-1}{4}u_1(t)\cdot 4\big(2x-(x^2+y^2)\big)\;-\;\Big(-\frac{-y}{2u_2(x+yi)}\Big)\cdot 8y\;=\;1+2(1-x)+(1-x)^2+y^2.$$

From Equation (3.8) we have $1+2(1-x)+(1-x)^2+y^2 = 1+2-2x+1-2x+x^2+y^2 = 5-4x)$ and $2x-(x^2+y^2) = 2x-1$, hence:

$$u_1(t)\,(2x-1)^2 \;+\; \frac{4y^2}{u_2(x+iy)} \;=\; 5-4x. \tag{3.9}$$

Using the identity

$$5-4x = (2x-1)^2 + 4(1-x^2) = (2x-1)^2 + 4y^2,$$

we can rearrange (3.9) as

$$\Big(u_1(t)-1\Big)\,(2x-1)^2 \;+\; 4y^2\Big(\frac{1}{u_2(x+iy)}-1\Big) \;=\; 0. \tag{3.10}$$

Since the right-hand side of (3.9) is real, taking imaginary parts in (3.9) (equivalently in (3.10)) yields

$$\operatorname{Im}\big(u_1(t)\big)\,(2x-1)^2 \;+\; 4y^2\,\operatorname{Im}\Big(\frac{1}{u_2(x+iy)}\Big) \;=\; 0. \tag{3.11}$$

Likewise, the real parts give

$$\operatorname{Re}\big(u_1(t)\big)\,(2x-1)^2 \;+\; 4y^2\,\operatorname{Re}\Big(\frac{1}{u_2(x+iy)}\Big) \;=\; 5-4x. \tag{3.12}$$

From the definition of $u_2(x+yi)$ and (3.8) we have:

$$\frac{1}{u_2(x+yi)} = \frac{(a+bi)(x-yi)}{x^2+y^2} = ax+by+(bx-ay)i.$$

Thus,

$$\begin{aligned}
\left|\frac{1}{u_2(x+yi)}\right|^2 &= (ax+by)^2 + (bx-ay)^2 \\
& a^2x^2 + 2axby + b^2y^2 + b^2x^2 - 2bxay + a^2y^2 \\
&= (a^2+b^2)x^2 + (a^2+b^2)y^2 \\
&= (a^2+b^2)(x^2+y^2) \\
&= a^2+b^2
\end{aligned}$$

Hence

$$\left|\frac{1}{u_2(x+yi)}\right| \;=\; \sqrt{a^2+b^2} \;=\; \text{Real Constant}. \tag{3.13}$$

Under $t \neq 1$ we have $u_1(t) \neq 1$, and by (8) also $y \neq 0$.

Case A: If $2x-1=0$, then $x=\frac{1}{2}$. From (3.8),

$$y^2 = 1-\left(\tfrac{1}{2}\right)^2 = \frac{3}{4} \quad\Longrightarrow\quad y = \pm\frac{\sqrt{3}}{2}.$$

This pair satisfies (3.9), and (7) selects the sign of y.

Case B: If $2x - 1 \neq 0$, then from (3.11)–(3.12) we can solve

$$\mathrm{Im}\Big(\tfrac{1}{u_2(x+iy)}\Big) = -\tfrac{\mathrm{Im}(u_1(t))}{4y^2}\,(2x-1)^2,$$

$$\mathrm{Re}\Big(\tfrac{1}{u_2(x+iy)}\Big) = \tfrac{5-4x-\mathrm{Re}(u_1(t))(2x-1)^2}{4y^2}.$$

But by (3.13) we find that $\big|\tfrac{1}{u_2(x+iy)}\big|^2 = \mathrm{Re}(\cdot)^2 + \mathrm{Im}(\cdot)^2$ is a fixed positive constant (independent of x). The right-hand sides above vary with x unless $2x - 1 = 0$, which contradicts the constancy of the modulus. Hence this case is impossible.

Combining the two cases, any solution with $t \neq 1$, $y \neq 0$, and $u_2(x + iy) \neq 0$ must be

$$x = \frac{1}{2}, \qquad y = \pm\frac{\sqrt{3}}{2},$$

with the sign of y determined by (7). □

Remark 3.10 (Dropping the RAA assumption)*.* Suppose that we drop the RAA assumption. What would happen to system $\mathscr{S}_t$ in this case? All of our derivations still hold, even though $a = 1/2$. Since we dropped the assumption, we are now assuming that all zeros of zeta are in the critical, so a valid solution (x, y) for system $\mathscr{S}_t$ must have the property $x = 1 - x$ and of course $x + yi = a + bi$. Thus, dropping the assumption reduce $\mathscr{S}_t$ to a family of systems where each system consists of four equations with four variables. The system $\mathscr{S}_t$ becomes:

$$\begin{aligned}
&f^*(x,y) = \Big(\tfrac{1}{2}\cdot\tfrac{1-x}{(x-1)^2+y^2} - \tfrac{1}{4}\Big)\cdot u_1(t)\\
&g^*(x,y) = -\tfrac{1}{2}\cdot\tfrac{y}{(1-x)^2+y^2}\\
&g^*(x,y)\cdot\tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} + f^*(x,y)\cdot\tfrac{8y}{1+2x+x^2+y^2} = 0\\
&f^*(x,y)\cdot\tfrac{4(1-x^2-y^2)}{1+2x+x^2+y^2} - g^*(x,y)\cdot\tfrac{8y}{1+2x+x^2+y^2} = 1.
\end{aligned} \tag{3.14}$$

Similar to the proof of Lemma (3.9) we can use the first three equations of system (3.14) to obtain

$$\Big(u_1(t) - 1\Big)\Big(x^2 + y^2 - 1\Big) = 0. \tag{3.15}$$

Now if $(u_1(t) - 1 = 0$, then the third equation of system (3.14) reduced to identity, so $\mathscr{S}_t$ becomes undetermined. Thus, we shall continue with the case where $(u(t) - 1 \neq 0$. From Equation (3.15) we obtain

$$x^2 + y^2 = 1 \tag{3.16}$$

Applying Equation (3.16) on the fourth equation of system (3.14), we obtain

$$f^*(x,y)\cdot\frac{4(1-x^2-y^2)}{1+2x+x^2+y^2} - g^*(x,y)\cdot\frac{8y}{1+2x+x^2+y^2} = 1$$

$$f^*(x,y)\cdot\frac{4(1-1)}{1+2x+x^2+y^2}-g^*(x,y)\cdot\frac{8y}{1+2x+x^2+y^2}=1.$$

$$-g^*(x,y)\cdot\frac{8y}{1+2x+x^2+y^2}=1.$$

$$\left(\frac{1}{2}\cdot\frac{y}{(1-x)^2+y^2}\right)\cdot\frac{8y}{1+2x+x^2+y^2}=1$$

$$\frac{4y^2}{(1-2x+x^2+y^2)(1+2x+x^2+y^2)}=1$$

$$\frac{4y^2+4x^2-4x^2}{(2-2x)(2+2x)}=1$$

$$\frac{y^2+x^2-x^2}{(1-x)(1+x)}=1$$

$$\frac{1-x^2}{(1-x)(1+x)}=1$$

$$1=1.$$

Hence, the fourth equation of system (3.14) is reduced to identity providing no additional information. Therefore, when we drop the RAA assumption system (3.5) becomes a non determined system.
Note that without the RAA assumption the system $\mathscr{S}_t$ is not only under determined, without the RAA assumption the system $\lim_{t\to 1}\mathscr{S}_t$ can not represent zeros of zeta. Without the RAA assumption equation 6 of the system $\lim_{t\to 1}\mathscr{S}_t$ will be missing. So, we would not have any representation for the real part of $\lambda(1-x+yi)$. In this case the analytic interpretation of $a+bi$ becomes meaningless.

Lemma 3.11. *Let S_t be the parameterize system*

$$\varphi_1(\overline{x},t)=0$$
$$\vdots$$
$$\varphi_n(\overline{x},t)=0$$

and let c be any real number or ∞. Where $\varphi_1,\ \dots\ ,\ \varphi_n$ are continuous with respect to $\overline{x}$ and for each integer $j\in[1,n]$ the limit $\lim_{t\to c}\varphi_j(\overline{u},t)$ converged to a finite value $\varphi_j^*(\overline{u})$.
If the system S_t has a constant solution $\overline{u}$ for all $t\neq c$ and $\varphi_1,\dots,\varphi_n$ continuous in $\overline{u}$, then $\overline{u}$ is a solution of $\lim_{t\to c}S_t$.

Proof. Since $\overline{u}$ is a constant solution of S_t for all $t\neq c$, we have:

$$\varphi_1(\overline{u},t)=0$$
$$\vdots$$
$$\varphi_m(\overline{u},t)=0$$

for all $t \neq c$. Since $\varphi_1, \ldots, \varphi_m$ continuous in $\lim_{t\to c} \overline{u}_t = \overline{u}$, we can take the limit $\lim\limits_{t\to c}$ from each of the above equations. We obtain

$$\begin{aligned}\lim_{t\to c} \varphi_1(\overline{u}, t) &= \lim_{t\to c} 0 \\ &\vdots \\ \lim_{t\to c} \varphi_m(\overline{u}, t) &= \lim_{t\to c} 0.\end{aligned}$$

For each integer $j \in [1, n]$ the limit $\lim\limits_{t\to c} \varphi_j(\overline{u}, t)$ converged to a finite value $\varphi_j^*(\overline{u})$. Thus,

$$\begin{aligned}\varphi_1^*(\overline{u}) &= 0 \\ &\vdots \\ \varphi_m^*(\overline{u}) &= 0.\end{aligned}$$

or equivalently,

$$\begin{aligned}\lim_{t\to c} \varphi_1(\overline{u}, t) &= 0 \\ &\vdots \\ \lim_{t\to c} \varphi_m(\overline{u}, t) &= 0.\end{aligned}$$

So, from the definition of limits of systems (Definition 3.4) we conclude that $\overline{u}$ is a solution of $\lim\limits_{t\to c} S_t$. □

3.1 The contradiction

When analyzing the system $\lim\limits_{t\to 1} \mathscr{S}_t$ we have analyzed it in two different paths. The algebraic (or symbolic) path and the analytic path. From Lemmas (3.7), (3.6), and (3.8) we have concluded that in both of these paths the system $\lim\limits_{t\to 1} \mathscr{S}_t$ must have a unique solution. In particular, the solution of the system $\lim\limits_{t\to 1} \mathscr{S}_t$ in both paths (algebraic and analytic) coincide.

Lets now analyze the algebraic path. From Lemmas (3.9) we find that for every real $t \neq 1$ the system $\mathscr{S}_t$ admits the solution

$$(x, y) = \begin{cases} (1/2, \sqrt{3}/2) & b > 0 \\ (1/2, -\sqrt{3}/2) & b < 0 \end{cases}$$

Denote

$$b_0 = \begin{cases} \sqrt{3}/2) & b > 0 \\ -\sqrt{3}/2 & b < 0 \end{cases}$$

For any $t \neq 1$, we denote the algebraic (symbolic) solution for $\mathscr{S}_t$ by $x(t) = 1/2$ and $y(t) = b_0$. Clearly all equations in system $\mathscr{S}_t$ are continuous in $t = 1$. So we can apply Lemma (3.11) to conclude that $(1/2, b_0)$ is a solution for the limiting system $\lim\limits_{t\to 1} \mathscr{S}_t$. From Lemma (3.9) we find that for every $t \neq 1$ the solution $\big(x(t), y(t)\big) = (1/2, b_0)$ for $\mathscr{S}_t$ is unique. Clearly, all expressions in the

systems $\mathscr{S}_t$ are continuous with respect to t and the solution function (with respect to t) is a constant and therefore must be continuous as well (see also Lemma 3.11). Therefore, we can use Lemma (3.8) and the definition of limits of systems of equations (definition 3.4) to conclude that $(1/2, b_0)$ is also a solution for system $\lim_{t\to 1} \mathscr{S}_t$. From Lemmas (3.7) we find that there exists only one unique algebraic solution to system $\lim_{t\to 1} \mathscr{S}_t$ and that this solution is (a, b). Hence, $(1/2, b_0) = (a, b)$. So, $a = 1/2$, but this contradicts the RAA assumption (3.3). This contradiction live inside the algebraic path. However, the solution for $\lim_{t\to 1} \mathscr{S}_t$ in the algebraic path and the analytic path agrees. Indeed, in Lemma (3.8) we have found that $\lim_{t\to 1} \mathscr{S}_t$ has one unique solution in both the algebraic and analytic paths. Thus, the solution $(1/2, b_0) = (a, b)$ must be a valid solution to the system $\lim_{t\to 1} \mathscr{S}_t$ in both the algebraic and analytic paths. Hence, in both the algebraic and analytic paths we get a contradiction.
Since the zeta function has no nontrivial zeros on the real line, our contradiction support the Riemann Hypothesis. Furthermore, Hardy and Littlewood [HL21] demonstrated that ζ has infinitely many zeros on the critical line. Thus, combining Hardy and Littlewood result with the above provides further evidence for the validity of the Riemann Hypothesis.

Remark 3.12 (Bridges and passages). We can picture the reasoning in terms of two "bridges":

Analytic bridge: the setting where (x, y) satisfy the full Abel–Plana integral equations.

Algebraic bridge: the setting where (x, y) satisfy the auxiliary algebraic system $\mathscr{S}_t$ obtained by replacing the integrals with closed-form expressions.

The RAA assumption (3.3) creates the first passage, letting us move from the analytic bridge to the algebraic bridge.
Once on the algebraic bridge, we take the limit $t \to 1$ and solve $\lim_{t\to 1} \mathscr{S}_t$ explicitly, obtaining a unique algebraic solution (a, b) that satisfy the system $\lim_{t\to 1} \mathscr{S}_t$ both algebraically and analytically.
Lemma (3.8) gives us the second passage. This lemma is valid on both bridges: it asserts that $\lim_{t\to 1} \mathscr{S}_t$ has exactly one solution, regardless of whether we view it analytically or algebraically.
Thus, for the limit system, we are effectively standing on both bridges at once — the unique algebraic solution (a, b) must also be the analytic solution, without needing to follow an analytic family $(x(t), y(t))$ for t near 1.

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