An experimental accessible inconsistency within hidden-variable theories for quantum optical phenomena

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Abstract

We study no-hidden-variable theorem for quantum optical phenomena. We consider the case that the uncertainty principle exists when we think both of commutativeness and non-commutativeness and then we derive a general and natural value. On the other hand, we consider the case that the uncertainty principle does not exist when we think only commutativeness and then we derive a specific and unnatural value. Thus, we are in the inconsistency within hidden-variable theories. We propose an experimental accessible inconsistency within hidden-variable theories for quantum optical phenomena in terms of the imperfect source and detector. As a result of our study, hidden-variable theories for quantum optical phenomena do not exist.

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I. INTRODUCTION

The great success of quantum mechanics (cf. [1–8]) is recognized by the scientific community of physical theories. We find researches concerning the mathematical formulations of quantum mechanics. For example, the mathematical foundations of quantum mechanics are discussed by Mackey [9]. On the quantum logic approach to quantum mechanics is also discussed by Gudder [10]. Conditional probability and the axiomatic structure of quantum mechanics are also reported by Guz [11].

The incompleteness argument to quantum mechanics itself is discussed by Einstein, Podolsky, and Rosen [12]. A hidden-variable interpretation of quantum mechanics is an interesting topic of research [3, 4]. The no-hidden-variable theorem is discussed by Bell, Kochen, and Specker [13, 14]. A strengthened Kochen–Specker theorem, i.e., the free will theorem is discussed by Conway and Kochen [15].

Recently, Nagata, Diep, and Nakamura discuss an inconsistency within hidden-variable theories, using two symmetric measurements. They are independent of the order of measurements themselves [16]. As the aim of this paper, we propose an experimental accessible inconsistency within hidden-variable theories for quantum optical phenomena in terms of the imperfect source and detector based on the argumentations [16]. We hope not only theoretical physicists but also experimental physicists understand our claim.

In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable in a real experimental situation. If we use the quantum predictions by even number $2N$ outcomes, then the degree of the inconsistency within hidden-variable theories increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

II. SOME CONSIDERATION OF USING THE HIDDEN-VARIABLE IN COMMUTING OBSERVABLES

The hidden-variable contradiction in this paper is explained this: If we allow to take both of commutativeness and non-commutativeness in consideration, there is the uncertainty principle, which fact seems to be likely to be quantum theory. And that seems to be natural to have the value of “0”.

On the other hand, we would discuss that the sum rule is equivalent to the product rule for commuting observables. First, we define the functional rule this:

$$f(g(O)) = g(f(O)), \tag{1}$$

where O is a Hermitian operator and f, g are appropriate functions to be used later. Second, the sum rule is defined this:

$$f(A + B) = f(A) + f(B), \tag{2}$$

where A and B are two commuting Hermitian operators. Finally, the product rule is defined this:

$$f(A \cdot B) = f(A) \cdot f(B). \tag{3}$$

This fact above (the sum rule is equivalent to the product rule) is based on the property of these two commuting Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two commuting Hermitian operators.

We may have [4, 16] the following relation between the three rules for commuting observables:

$$\begin{aligned}&\text{The functional rule}\\ \Leftrightarrow\ &\text{The sum rule}\\ \Leftrightarrow\ &\text{The product rule}\end{aligned} \quad (4)$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$ and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i, which are all different. Here O is assumed to be non-degenerate by construction. Let us define respectively functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$. Then we can see that if A and B commute, there exists a non-degenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned}&f(A\cdot B) = f(h(O)) = h(f(O)) = j(f(O))\cdot k(f(O))\\ &f(j(O))\cdot f(k(O)) = f(A)\cdot f(B),\end{aligned} \quad (5)$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned}&f(A+B) = f(l(O)) = l(f(O)) = j(f(O)) + k(f(O))\\ &f(j(O)) + f(k(O)) = f(A) + f(B),\end{aligned} \quad (6)$$

where we use the functional rule. In fact, the sum rule is equivalent to the product rule for commuting observables. Additionally, we may introduce a function m, if the following conditions satisfied, such that $m(l(O)) := h(O)$, $m(j(O)) := j(O)$, and $m(k(O)) := k(O)$. For example, let us define A and B this:

$$A = \begin{pmatrix}1-\epsilon & 0\\ 0 & -1+\epsilon\end{pmatrix} \text{ and } B = \begin{pmatrix}-1+\epsilon & 0\\ 0 & 1-\epsilon\end{pmatrix}, \quad (7)$$

where the meaning of ϵ is discussed later. Hence we have

$$A + B = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \text{ and } A\cdot B = \begin{pmatrix}-(1-\epsilon)^2 & 0\\ 0 & -(1-\epsilon)^2\end{pmatrix}. \quad (8)$$

Let us take the function m as $m(0) = -(1-\epsilon)^2$, $m(1-\epsilon) = 1-\epsilon$, and $m(-1+\epsilon) = -1+\epsilon$. Therefore we can define the function m that satisfies the requirement conditions. Then we have

$$\begin{aligned}&f(A)\cdot f(B) = f(A\cdot B) = f(h(O)) = f(m(l(O)))\\ &= f(l(m(O))) = l(f(m(O)))\\ &= j(f(m(O))) + k(f(m(O)))\\ &= f(m(j(O))) + f(m(k(O))) = f(A) + f(B),\end{aligned} \quad (9)$$

where we use the functional rule and the product rule. Note we do not directly use the sum rule. Therefore we have

$$f(A)\cdot f(B) = f(A) + f(B). \quad (10)$$

Thus we may introduce a supposition that the operation Addition $(f(A) + f(B))$ is equivalent to the operation Multiplication $(f(A)\cdot f(B))$ when $A = \begin{pmatrix}1-\epsilon & 0\\ 0 & -1+\epsilon\end{pmatrix}$, $B = \begin{pmatrix}-1+\epsilon & 0\\ 0 & 1-\epsilon\end{pmatrix}$, and the quantum state is a simultaneous eigenstate of A, B.

And then, we can create a novel algebra that might be called commuting observable algebra. The space for commuting observable algebra is different from our common space in which Newton's mechanics is created. In the space holding commuting observable algebra, the operation Addition and the operation Multiplication are the same as each other. In the normal space where Newton's mechanics is held, the operation Addition and the operation Multiplication are different from each other.

The space holding commuting observable algebra in considering only commutativeness is opposite to quantum theory. The reason of the hidden-variable inconsistency is because there is not the uncertainty principle because of not being noncommutative. Besides, holding commuting observable algebra allows any unnatural value to be "-1". Therefore, we have the hidden-variable contradiction. The point of the hidden-variable contradiction is based on our consideration that we have non-commutative property in quantum operations. This fact is based on the uncertainty principle. Here we must notice theoretically that the space holding commuting observable algebra thinking of only commutativeness is different from the normal space holding Newton's mechanics.

The scenario of this paper tells us that Newton's mechanics is not held when thinking of only commutativeness. The other case is in the space holding commuting observable algebra. On the other hand, in case of holding both of commutativeness and non-commutativeness, the space not permitting commuting observable algebra is allowed, and quantum theory is permitted, and then the uncertainty principle exists.

III. NO-HIDDEN-VARIABLE THEOREM BASED ON AN IMPERFECT EXPERIMENT

Suppose now that C and D are two commuting Hermitian operators. We define C, D this:

$$C = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \text{ and } D = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \quad (11)$$

Let $|\uparrow\rangle$ be a simultaneous eigenstate of C, D such that $C|\uparrow\rangle = +1|\uparrow\rangle$ and $D|\uparrow\rangle = -1|\uparrow\rangle$. The predetermined hidden results for quantum measurement outcomes are either $+1$ or -1 in the ideal case.

Let $|\downarrow\rangle$ be the other simultaneous eigenstate of C, D such that $C|\downarrow\rangle = -1|\downarrow\rangle$ and $D|\downarrow\rangle = +1|\downarrow\rangle$. When we consider a quantum optical experiment, we have the following relations with the photon polarization states:

$$|\uparrow\rangle \leftrightarrow |H\rangle,$$
$$|\downarrow\rangle \leftrightarrow |V\rangle, \quad (12)$$

where $|H\rangle$ is a quantum state interpreted by a horizontally polarized photon and $|V\rangle$ is a quantum state interpreted by a vertically polarized photon.

Let us introduce the random noise admixture $\rho_{\text{noise}}(= \frac{1}{2}I)$ into the quantum state $|\uparrow\rangle$, where I is the two-dimensional identity operator. We consider the noisy quantum state emerged from an imperfect source this:

$$\rho = (1-\epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\text{noise}}. \quad (13)$$

The value of $\epsilon(< 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\text{tr}[\rho C] = +1-\epsilon$ and $\text{tr}[\rho D] = -1+\epsilon$.

However the quantum state ρ is not a simultaneous eigenstate of C, D. Thus we change the definition of the two measured observables and of the quantum state ρ this:

$$A = \begin{pmatrix} 1-\epsilon & 0 \\ 0 & -1+\epsilon \end{pmatrix}, B = \begin{pmatrix} -1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}, \quad (14)$$

and

$$\rho = |\uparrow\rangle\langle\uparrow|. \quad (15)$$

Then the quantum state ρ becomes a simultaneous eigenstate of A, B. Further we have $\text{tr}[\rho A] = +1-\epsilon$ and $\text{tr}[\rho B] = -1+\epsilon$.

We might be in the inconsistency within hidden-variable theories when the first predetermined hidden result is $+1-\epsilon$ by the measured observable A in the quantum state ρ, the second predetermined hidden result is $-1+\epsilon$ by the measured observable B in the same quantum state ρ, and then we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. This means the uncertainty principle does not exist because we think only commutativeness.

In general, the physical situation is either $[A, B] \neq 0$ or $[A, B] = 0$. This means the uncertainty principle exists because we think both of commutativeness and non-commutativeness.

We consider a value V which is the sum of the two predetermined hidden results in a thought optical experiment. The predetermined hidden results for quantum measurement outcomes are either $+1-\epsilon$ or $-1+\epsilon$. We suppose the number of $-1+\epsilon$ is equal to the number of $+1-\epsilon$. If the number of trials is two, then we have

$$V = (+1-\epsilon) + (-1+\epsilon) = 0. \quad (16)$$

We derive a general and natural necessary condition of the value of V. In the general and natural case, the uncertainty principle exists because we think both of commutativeness and non-commutativeness. This is the general and natural necessary condition because we consider both of the propositions $[A, B] \neq 0$ and $[A, B] = 0$ and we assign simultaneously the different two values ("0" and "1") for the propositions $[A, B] \neq 0$ and $[A, B] = 0$. We assign the value "0" for the proposition $[A, B] \neq 0$.

On the other hand, we can depict the optical predetermined hidden results using v_1, v_2 this: $v_1 = +1-\epsilon$ and $v_2 = -1+\epsilon$. Let us write V this:

$$V = v_1 + v_2. \quad (17)$$

In the following, we evaluate another value of V and derive a specific and unnatural necessary condition under the supposition that the two measured observables are commuting (that is, $[A, B] = 0$ and we assign the value "1" for the commutator) and we do not consider the existence of the following proposition $[A, B] \neq 0$. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We may introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have, using commuting observable algebra,

$$V = v_1 + v_2 = v_1 \times v_2 = -(+1-\epsilon)^2, \quad (18)$$

where $f(A) + f(B) = f(A) \cdot f(B)$ is used. Here $f(\cdot) = \text{tr}[\rho(\cdot)]$. This is possible for the specific and unnatural case that we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two propositions (16) and (18) when we consider only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. We derive the hidden-variable inconsistency when we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator.

In summary, we have been in the inconsistency within hidden-variable theories when the first predetermined hidden result is $+1-\epsilon$ by measuring the observable A in the quantum state ρ, the second predetermined hidden result is $-1+\epsilon$ by measuring the observable B in the same quantum state ρ, and then we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. This has meant the uncertainty principle does not exist because we think only commutativeness and then we derive the specific and unnatural value to be "-1".

IV. INCOMPLETENESS IN A REAL EXPERIMENT

In a real experiment, there are no perfect detectors, but the good ones with some errors. There is an unforeseen effect that an imperfect detector does not count even though the particle indeed passes through the detector (the quantum efficiency). There is also an unforeseen effect that an imperfect detector counts even though the particle does not pass through the detector (the dark count). In this case, we increase measurement outcomes to even number $2N(\gg 1)$ and then we change such errors into trivial things. If we use the quantum predictions by even number $2N$ trials, then the degree of the inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

Again, we consider the noisy quantum state emerged from an imperfect source this:

$$\rho = (1-\epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\rm noise}. \tag{19}$$

The value of $\epsilon(<1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\mathrm{tr}[\rho C] = +1-\epsilon$ and $\mathrm{tr}[\rho D] = -1+\epsilon$. Thus the predetermined hidden results for quantum measurement outcomes are either $+1-\epsilon$ or $-1+\epsilon$.

Again, however, the quantum state ρ is not a simultaneous eigenstate of C, D. Thus we change the definition of the two measured observables and of the quantum state ρ this:

$$A = \begin{pmatrix} 1-\epsilon & 0 \\ 0 & -1+\epsilon \end{pmatrix}, B = \begin{pmatrix} -1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}, \tag{20}$$

and

$$\rho = |\uparrow\rangle\langle\uparrow|. \tag{21}$$

Then the quantum state ρ becomes a simultaneous eigenstate of A, B. Further we have $\mathrm{tr}[\rho A] = +1-\epsilon$ and $\mathrm{tr}[\rho B] = -1+\epsilon$.

The odd number predetermined hidden results are $+1-\epsilon$ by measuring the observable A in the quantum state ρ and the even number predetermined hidden results are $-1+\epsilon$ by measuring the observable B in the same quantum state ρ. We suppose the number of outcomes of obtaining the predetermined hidden result $-1+\epsilon$ is N that is equal to the number (N) of outcomes of obtaining the predetermined hidden result $+1-\epsilon$. That is, the number of outcomes is even number $2N$.

We consider a following value $V_i(i = 1, 2, ..., N)$ which is the sum of the two predetermined hidden results in the thought optical experiment:

$$V_i = (+1-\epsilon) + (-1+\epsilon) = 0. \tag{22}$$

We introduce the following function $S(N)$ which is the sum over i of V_i in a thought optical experiment:

$$S(N) = V_1 + V_2 + ... + V_N. \tag{23}$$

If the number of outcomes is even number $2N$, then we have

$$S(N) = V_1 + V_2 + ... + V_N = N\Big((+1-\epsilon) + (-1+\epsilon)\Big)$$
$$= N(0) = 0, \tag{24}$$

where we use $V_i = (+1-\epsilon) + (-1+\epsilon) = 0$. We derive a general and natural necessary condition of the function value $S(N)$. In the general and natural case, the uncertainty principle exists because we think both of commutativeness and non-commutativeness. In more detail, we consider both of the propositions $[A, B] \neq 0$ and $[A, B] = 0$ and we assign simultaneously the different two values ("0" and "1") for the two propositions. We assign the value "0" for the proposition $[A, B] \neq 0$.

On the other hand, we can depict the predetermined hidden results using $v_1, v_2, v_3, ..., v_{2N}$ this: $v_1 = +1-\epsilon$, $v_2 = -1+\epsilon$, $v_3 = +1-\epsilon$,..., $v_{2N} = -1+\epsilon$. We can write V_i this:

$$V_i = r_{2i-1} + r_{2i}, \tag{25}$$

where

$$r_{2i-1} = +1-\epsilon,\ r_{2i} = -1+\epsilon. \tag{26}$$

Thus, we have

$$V_i = (+1-\epsilon) + (-1+\epsilon). \tag{27}$$

In the following, using commuting observable algebra, we evaluate another value to V_i and derive a specific and unnatural necessary condition for the function value $S(N)$ under the supposition that the two measured observables are commuting (that is, $[A, B] = 0$ and we assign the value "1" for the commutator) and we do not consider the existence of the following proposition $[A, B] \neq 0$. In the specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We may introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have, using commuting observable algebra,

$$V_i = (+1-\epsilon) + (-1+\epsilon)$$
$$= (+1-\epsilon) \times (-1+\epsilon) = -(+1-\epsilon)^2. \tag{28}$$

Thus, we have

$$S(N) = V_1 + V_2 + ... + V_N$$
$$= -(+1-\epsilon)^2 - (+1-\epsilon)^2 - ... - (+1-\epsilon)^2$$
$$= -N(+1-\epsilon)^2. \tag{29}$$

This is possible for the specific and unnatural case that we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. In this specific and unnatural case, the uncertainty principle does not exist because we think only commutativeness.

We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two propositions (24) and (29) when we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. We derive the inconsistency within hidden-variable theories when we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator.

If we use the quantum predictions by even number $2N$ trials, then the degree of the inconsistency increases by an amount that grows linearly with N. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We note our argumentations here agree with the discussions in Section III when $N = 1$. From the relation (24), we have the following general and natural value:

$$S(1) = V_1 = 0. \quad (30)$$

From the relation (29), using commuting observable algebra, we have the following specific and unnatural value:

$$S(1) = V_1 = -(+1 - \epsilon)^2. \quad (31)$$

Thus, our discussions here are a natural expansion of Section III.

In summary, we have been in the inconsistency within hidden-variable theories when quantum measurement outcomes are even number $2N(\gg 1)$, the odd number predetermined hidden results are $+1-\epsilon$ by measuring the observable A in the quantum state ρ, the even number predetermined hidden results are $-1+\epsilon$ by measuring the observable B in the same quantum state ρ, and then we consider the existence of only the following proposition $[A, B] = 0$ and we assign the value "1" for the commutator. This has meant the uncertainty principle does not exist because we think only commutativeness and then we derive the specific and unnatural value to be "-1".

V. CONCLUSIONS

In conclusions, we have studied no-hidden-variable theorem for quantum optical phenomena. We have considered the case that the uncertainty principle exists when we think both of commutativeness and non-commutativeness and then we derive a general and natural value. On the other hand, we have considered the case that the uncertainty principle does not exist when we think only commutativeness and then we derive a specific and unnatural value. Thus, we have been in the inconsistency within hidden-variable theories. We have proposed an experimental accessible inconsistency within hidden-variable theories for quantum optical phenomena in terms of the imperfect source and detector. As a result of our study, hidden-variable theories for quantum optical phenomena have not existed.

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The authors are in an applicable thought to ethical approval.

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The authors state that there is no conflict of interest.

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