Euler's Identity and a Hidden Unit Assumption

Kshitiz Prabhakar
Independent student researcher (India)

September 13, 2025

Abstract

Euler's identity $e^{i\pi}+1=0$ is a central, rigorously proven result in complex analysis. This note does not dispute its correctness *inside* that formalism. Instead it isolates a single, widely overlooked modeling choice that arises when the analytic identity is applied to measurable angles in physics and engineering: the implicit treatment of angular measures as if they were plain, unitless real numbers. I (1) explain why the analytic/trigonometric power series force a particular angular normalization (radians); (2) show concretely, via a bradian renormalization and a kilogram counterexample, that the usual "divide-by-1 rad" maneuver is a convention that cannot be elevated to a general principle; (3) formalize the issue as a type/coercion error; and (4) show direct implications for phasor calculus and the Schrödinger plane-wave ansatz. I finish with minimal, implementable prescriptions (explicit coercion or typed wrappers) that preserve numerical results while restoring unit-aware rigor.

1 Introduction and motivation

Euler's formula

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

and its special case Euler's identity $e^{i\pi} + 1 = 0$ are ubiquitous across mathematics and applied science. In pure complex analysis these equalities are immediate consequences of the exponential and trigonometric power series and the algebra that separates even and odd terms. In physics and engineering the same equalities are used daily with measured angular quantities: phases ωt, wave-numbers kx, rotation angles, and so on.

Closely examining how analytic identities are moved into applied contexts reveals an implicit modeling step: *the analytic functions are defined on unitless complex numbers, so any unit-bearing angle must first be coerced to a number.* Practitioners do this everywhere (silently), which is why the formulas "work" numerically. The objective of this note is to expose that step, show why it is a convention (not a derivation), and demonstrate that the convention is fragile, indeed, if one treats it as a general rule one encounters immediate contradictions.

This is not a polemic against Euler or complex analysis. Rather it is a request for clarity and for making explicit the conversion (coercion) that is often left implicit when analysis meets units.

2 Analytic foundations and the role of radians

The power-series definitions are

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

valid for all $z \in \mathbb{C}$. Combining these series yields the algebraic identity $\exp(iz) = \cos z + i\sin z$ for complex z.

These series treat z as a pure complex number — there is no unit attached. When one wishes to interpret $\sin, \cos$ geometrically (as coordinates on the unit circle) one chooses a real parameter that maps a geometric angle to a real number. The classical calculus fact

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

and the associated derivative normalization $\sin'(0) = 1$ single out the radian parametrization: on the unit circle an infinitesimal change in the radian coordinate equals the corresponding infinitesimal arc length. In short: the analytic Taylor expansions coincide with geometric sine and cosine only when the chosen angular coordinate is the radian coordinate. That coincidence is a normalization choice linking geometry and pure analysis.

Therefore: the analytic identity $\exp(iz) = \cos z + i \sin z$ is a statement about the analytic variable z. Translating it into a statement about geometric angles requires a map (a coercion) from a unit-bearing geometric angle to the unitless analytic variable.

3 Bradians: a gentle build-up to normalization dependence

Before asserting any claim about units in exponents, distinguish clearly between:

- the *geometric angle* (an abstract object that locates a point on the circle), and
- the *numeric coordinate* used to represent that angle (radians, degrees, bradians, etc.).

Choosing a numeric coordinate is a *normalization*: it fixes a bijection Φ from the geometric angle set to $\mathbb{R}$. Different normalizations produce different numeric values for the same geometric angle.

Bradians (definition and algebra). Define *bradians* by declaring one full rotation equals 4π bradians (instead of 2π radians). If θ_r denotes the radian numeric coordinate and θ_b the bradian numeric coordinate for the same geometric angle, then

$$\theta_r = \tfrac{1}{2}\,\theta_b.$$

Thus bradians and radians are related by a simple scale factor.

How the analytic identity behaves under rescaling. The analytic identity

$$\exp(iz) = \cos z + i \sin z \qquad (z \in \mathbb{C})$$

is written in terms of a numeric variable z. If we interpret z as the radian numeric coordinate we write

$$\exp\big(i\theta_r\big) = \cos(\theta_r) + i \sin(\theta_r).$$

If instead we substitute the bradian numeric coordinate without conversion:

$$\exp\big(i\theta_b\big) \stackrel{?}{=} \cos(\theta_b) + i \sin(\theta_b),$$

we have effectively changed the numeric input. Substituting $\theta_b = 2\theta_r$ yields

$$\exp\big(i\theta_b\big) = \exp\big(i(2\theta_r)\big), \qquad \cos(\theta_b) + i \sin(\theta_b) = \cos(2\theta_r) + i \sin(2\theta_r).$$

Thus the bradian-form identity becomes

$$\exp\big(i(2\theta_r)\big) = \cos(2\theta_r) + i \sin(2\theta_r),$$

which is not the same numeric identity as the radian-coordinate version unless one compensates for the factor 2 (either by converting coordinates or redefining sine/cosine relative to bradians).

Concrete numeric example. Take a geometric rotation of 45°. In radians $\theta_r = \pi/4$; in bradians $\theta_b = \pi/2$. Evaluating,

$$\exp\Big(i\frac{\pi}{4}\Big) = \cos\Big(\frac{\pi}{4}\Big) + i\sin\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

whereas

$$\exp\Big(i\frac{\pi}{2}\Big) = \cos\Big(\frac{\pi}{2}\Big) + i\sin\Big(\frac{\pi}{2}\Big) = 0 + i.$$

The two complex numbers differ. The computation shows that the *numeric* equality depends on the chosen numeric coordinate; the radian choice is a normalization that must be acknowledged when translating analytic identities into geometric statements.

Careful conclusion. This observation does not contradict the analytic identity $\exp(iz) = \cos z + i\sin z$ for unitless z. It shows that when z is obtained from a geometric angle one must specify the coordinate map Φ (i.e., which normalization is used). In applied practice this specification is usually hidden by the convention "angles in radians" or by silently applying the coercion $\theta \mapsto \theta/(1\ \text{rad})$. Making the coercion explicit is the conservative, formal step.

4 The kilogram counterexample: decisive inconsistency

A frequently offered defense is: just divide a unit-bearing exponent by the corresponding unit to get a pure number. For angles that reads $\theta \mapsto \theta/(1\ \text{rad})$. If this divide-by-unit maneuver were a principled universal rule it would apply to any unit. Consider the plainly absurd expression $3^{2\ \text{kg}}$. Applying the same operation produces

$$3^{2\ \text{kg}} = 3^{(2\ \text{kg})/(1\ \text{kg})} = 3^2,$$

which nonsensically renders mass-bearing exponents meaningful. Accepting divide-by-unit uniformly thus collapses dimensional discipline. Since dimensional analysis forbids arbitrary unit exponents, the divide-by-unit trick cannot be elevated to a general mathematical principle.

Hence one of three options must be taken:

1. Forbid unit exponents uniformly: exponents must be dimensionless (and therefore explicit coercions are required when angles appear in exponents).
2. Allow divide-by-unit universally: accept exponents with any unit after dividing by 1 of that unit (this destroys standard dimensional analysis).
3. Treat angular units as a special case by fiat (declare radians dimensionless in a way that is not extended to other units).

Options (2) and (3) are unacceptable for consistent unit-aware modeling. Therefore the only principled choice is (1): enforce that exponents be dimensionless, and make coercions explicit whenever applied quantities (angles) are used.

5 A typed formalization (coercion/unwrap)

A succinct way to present the issue is through a simple typed viewpoint. Introduce two types:

$$\texttt{Real}\quad\text{(pure numbers)},\qquad \texttt{Angle}\quad\text{(unit-bearing angles)}.$$

The analytic exponential is a function $\exp : \texttt{Real} \to \mathbb{C}$. Writing $\exp(i\cdot\theta)$ with $\theta : \texttt{Angle}$ is a type error. To make the expression well-typed one must supply an explicit coercion (unwrap)

$$U : \texttt{Angle} \longrightarrow \texttt{Real},\qquad U(\theta) = \frac{\theta}{1\ \text{rad}}.$$

Hiding U is equivalent to allowing unchecked implicit coercions; unit-aware languages and formal proof assistants explicitly forbid such behaviour. Making U explicit restores type correctness and prevents silent mistakes in formal derivations and code.

6 Why the "rotation in the complex plane" argument does not resolve the issue

The geometric intuition that $\exp(i\theta)$ parametrizes the unit circle and multiplication by it effects a rotation — is compelling visually. However, as a defense against the unit objection it is insufficient:

- If θ is taken as an *angle* (unit-bearing), then writing $\exp(i\theta)$ raises the unit-exponent problem we have described.
- If θ is taken as a *pure number* (to make exp defined), then asserting that $\exp(i\theta)$ *means* rotation is an extra semantic mapping from numbers to rotations that must be defined and justified. One cannot simultaneously treat θ as a number and assume it carries geometric meaning without explaining the identification map.

In short: the rotation story either presupposes the coercion or hides it. It does not eliminate the need to make the coercion explicit.

7 Implications for applied contexts

AC circuits and phasors

Phasor notation writes voltages/currents as $V(t) = \Re\{\tilde{V}e^{i\omega t}\}$. Here ω (rad/s) and t (s) produce ωt with angular units. Practitioners silently normalize by dividing by 1 rad; numerically the method works because the same convention is used consistently. For unit-aware models, simulation code, and formal proofs however, one should either make the normalization explicit (write $e^{i(\omega t/(1\ \mathrm{rad}))}$) or use typed wrappers so that no hidden coercion takes place.

Quantum plane waves and Schrödinger's ansatz

The plane-wave ansatz $\Psi(x,t) = Ae^{i(kx-\omega t)}$ uses k (rad/m) and ω (rad/s) so the phase carries angular units. Quantum mechanics as a predictive theory remains unaffected because the same implicit normalization is applied throughout derivations; but attempts to formalize quantum mechanics in unit-aware theorem provers or static-analyzed code must insert explicit normalization steps or otherwise declare the coercion in function signatures.

8 Objections and brief replies

"Radians are dimensionless by definition." SI treats radians as a dimensionless derived quantity because they are ratios of lengths. That bookkeeping cancels base units but does not eliminate that a numeric coordinate depends on a chosen normalization. The bradian demonstration shows that different normalizations produce different numeric parameters; thus the statement "radians are dimensionless" is a convention that should be exposed where it matters.

"Pure mathematics ignores units." Indeed in pure analysis the identity is correct. The present critique targets the *application* of analytic identities to unit-bearing situations. When identities from pure analysis are used in physics and engineering, a coercion from unit-bearing quantities to unitless variables is required and should be documented.

9 Minimal prescriptions for clarity and safety

To preserve numerical utility while restoring formal rigor, adopt one of the following equivalent strategies:

1. **Explicit coercion.** Always write $\exp\big(i(\theta/(1\ \mathrm{rad}))\big)$ for unit-bearing angles θ before applying analytic power-series definitions.

2. **Typed interfaces.** In code and formal proofs, distinguish `Angle` from `Real` and require an explicit coercion U : `Angle` $\to$ `Real`.

3. **Wrapper functions.** Provide convenience functions, e.g. $\sin_{\mathrm{Angle}}(\theta) := \sin(\theta/(1\ \mathrm{rad}))$, so that the coercion is documented in code and signatures while notation remains compact.

10 Conclusion

Euler's identity is an elegant, correct theorem of complex analysis. The point of this note is narrower: when Euler's formula is applied to measurable angles in physics and engineering, it relies on a hidden convention — the normalization/coercion that converts unit-bearing angles to unitless analytic inputs. The bradian and kilogram examples demonstrate that this conversion is a modeling choice and that treating angles as a special exception without explicit coercion is inconsistent. The remedy is straightforward: make the coercion explicit, or adopt typed/wrapper definitions that incorporate it. Doing so preserves all practical utility and restores formal transparency in unit-aware modeling, proofs, and code.

Acknowledgements. I thank the online community for feedback during early drafts and welcome constructive criticism and suggestions for formalization in theorem provers.I arrived at these conclusions independently while studying Euler's identity. Only afterwards did I discover that related concerns about radians and units have been raised in metrology literature (e.g. Mohr 2014, Quincey 2020). My argument differs in applying this to the mathematical consistency of Euler's formula itself."

References

[1] Paul Quincey, "Angles in the SI: a detailed proposal for solving the problem," arXiv:2108.05704 [physics.class-ph], 2021. Available at: https://arxiv.org/abs/2108.05704

[2] Peter J. Mohr and William D. Phillips, "Dimensionless Units in the SI," arXiv:1409.2794 [physics.data-an], 2014. Available at: https://arxiv.org/abs/1409.2794