$3D$ Horizons in the accelerated Natario warp drive vector using the ADM-MTW-Alcubierre formalism with motion over the x-axis

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector for constant speeds in Polar Coordinates but remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model so it must possesses variable speeds.We developed the extension for the original Natario warp drive vector that encompasses variable speeds.Also Polar Coordinates uses only two dimensions and we know that a real spaceship is a $3D$ object inserted inside a $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.In this work we present the new warp drive vector in $3D$ Spherical Coordinates for variable speeds.One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble.The behavior of a photon sent to the front of the warp bubble in the case of a Natario warp drive with variable velocity and a lapse function is also one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the 1+1 and 3+1 spacetimes with the lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and the ADM(Arnowitt-Dresner-Misner) formalism equations with the approach of MTW(Misner-Thorne-Wheeler) and Alcubierre.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).The final form of the original Natario warp drive vector is given by $nX = vs * d(r\cos\theta)$.However Polar Coordinates are not real tridimensional $3D$ coordinates since it uses only the two Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

The Hodge Star actually must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs * (dx) + x * (dvs)$ but due to a constant speed vs the term $x * d(vs) = 0$.In this work we examine what happens with the Natario vector when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The final form of the Hodge Star for this warp drive vector is calculated no longer over $*d(r\cos\theta)$ but instead over $*d(r\sin\phi\cos\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.

In this work we present the new warp drive vector in tridimensional $3D$ Spherical Coordinates for variable speeds $nX = vs * (dx) + x * (dvs)$.

The warp drive work that predates Natario by 7 years was written by Alcubierre in 1994.(see [16])

Alcubierre([18]) used the so-called $3+1$ original Arnowitt-Dresner-Misner(ADM) formalism using the approach of Misner-Thorne-Wheeler(MTW)([17]) to develop his warp drive theory.As a matter of fact the first equation in his warp drive paper is derived precisely from the original $3+1$ ADM formalism(see eq 2.2.4 pg 67 in [18],see also eq 1 pg 3 in [16]) and we have strong reasons to believe that Natario which followed the Alcubierre steps also used the original $3+1$ ADM formalism to develop the Natario warp drive spacetime.In this work concerning the ADM formalism we adopt the Alcubierre methodology.

The ADM equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq (21.40) pg 507 in [17])(see Appendix I).

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1}$$

In the equation above α is the so-called lapse function,γ_{ij} is the $3D$ diagonalized induced metric and β^i and β^j are the so-called shift vectors.

Combining the eqs (21.40),(21.42) and (21.44) pgs 507, 508 in [17]
with the eqs (2.2.4),(2.2.5) and (2.2.6) pg 67 in [18] using the signature $(-,+,+,+)$ we get the original matrices of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{2}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{3}$$

The Natario warp drive equation with signature $(-,+,+,+)$ that obeys the original $3+1$ ADM formalism is given below:(see eq 21.40 pg 507 in [17])

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{4}$$

Changing the signature from $(-,+,+,+)$ to $(+,-,-,-)$ making $\alpha = 1$ and inserting the components of the Natario vectors we have in Polar Coordinates with constant speeds:(see Appendix I).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{5}$$

And in $3D$ Spherical Coordinates also with constant speeds:(see also Appendix I).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{6}$$

The equations above dont have the lapse function.The equivalent equations using the lapse function and a variable velocity would then be:

Polar Coordinates:(see Appendix J).

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{7}$$

$3D$ Spherical Coordinates:(see also Appendix J).

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{8}$$

The difference between variable and constant velocity warp drives is due to the term α^2 that affect the geometrical structure of the whole spacetimes.The term α behaves as a lapse function.

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt} X^t + \gamma_{tt} X^t X^t) = (1 - 2X_t + X_t X^t) \tag{9}$$

In this work we also discuss the Horizon problem for both the Natario warp drive spacetime using polar and spherical coordinates in the $1+1$ and $3+1$ ADM formalisms with the lapse function at variable velocities and we arrive at the conclusion that while the equations without the lapse function at constant velocities in the $1+1$ spacetime suffers from the problem of the Horizon and cannot control the fromt of the warp bubble as depicted in the section 4 of the works [36],[37] and [43] the equations presented in this work(wether in the $1+1$ and $3+1$ formalisms in both polar or spherical coordinates with the lapse function and variable velocities)can circumvent the problem of the Horizon because in this case the warp bubble is totally connected due to the presence of the lapse function.

Horizons were deeply covered in the warp drive literature but always for constant velocities and without lapse functions in the $1+1$ spacetime.(see pg 6 in [1],pg 34 in [34],pgs 268 in [35]).The behavior of a photon sent to the front of the warp bubble in the case of a warp drive with variable velocity and a lapse function is one of the main purposes of this work.We present the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ and $3+1$ spacetimes in polar and spherical coordinates with the lapse function at variable velocities using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provide here the step by step mathematical calculations in order to outline(or underline or reinforce) the final results found in our work which are the following ones:

- 1)-In the case of the Natario warp drive with variable velocities and a lapse functions in the $1+1$ spacetime due to a dimensional reduction from Polar(or Spherical) Coordinates the Horizon do not exists.
- 2)-In the case of the Natario warp drive with variable velocities and a lapse functions in the $3+1$ spacetime in Polar Coordinates the Horizon do not exists at all.
- 3)-In the case of the Natario warp drive with variable velocities and a lapse function in the $3+1$ spacetime in $3D$ Spherical Coordinates the Horizon do not exists at all.

In these solutions with variable velocities the whole spacetime geometries are affected by presence of the lapse functions and have different results when compared to the solutions without lapse functions presented in section 4 of the works [36],[37] and [43].

In the solutions with $3+1$ spacetimes the whole spacetime geometries are affected by presence of the $3+1$ spacetimes and have different results when compared to the solutions with only $1+1$ spacetimes.

Remember that we are presenting our results using step by step mathematics in order to better illustrate our point of view. For the solutions of the quadratic forms in $3+1$ spacetimes see Appendices K and L.These solutions are different than the ones obtained only in $1+1$ spacetimes.

We adopt here the Geometrized system of units in which $c = G = 1$ for geometric purposes.

In order to fully understand the main ideas presented in this work:a new Natario warp drive vector in tridimensional $3D$ Spherical Coordinates and the behavior of a photon sent to the front of the bubble in the Natario warp drive in $3+1$ spacetimes with the lapse function at variable velocities acquaintance or familiarity with the Natario original warp drive paper is required but we provide all the mathematical demonstration QED(Quod Erad Demonstratum) in the Appendices.

Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates so a photon sent to the front of the bubble fundamentally moves in a tridimensional spacetime.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical background needed to understand the process Natario used to generate warp drive vectors.

As a matter of fact if a novice,newcomer,beginner or intermediate student not familiarized with the Natario techniques reads the Natario warp drive paper in first place he(or she) will perhaps feel some difficulties.

We hope our paper is suitable to fill this gap.

Although this work was designed to be independent,self-consistent and self-contained it may be regarded as a companion work to our works in [8],[9],[10],[11][12],[13],[36],[37],[38] and [43].

2 The equation of the Natario warp drive vector in polar coordinates with a variable speed vs due to a constant acceleration a

The equation of the Natario vector nX in polar coordinates with a variable speed vs due to a constant acceleration a is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{10}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by(see Appendices A and B for pedagogical purposes and C for the final result):

$$X^t = 2f(r)rcos\theta a \tag{11}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{12}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{13}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).(see Appendix G in [8],[43]).

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^t e_t + X^r e_r \tag{14}$$

$$X^t = 2f(r)ra \tag{15}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \tag{16}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \tag{17}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $vs = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(r)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(r)$ of the Natario shape function $f(r)$ is zero and the shift vector $X^{rs} = 2[2f(r)^2]at$ with $X^r = 0$ inside the bubble and $X^r = 2[2f(r)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

3 The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{18}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result)

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{19}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{20}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{21}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{22}$$

Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs(t)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).(see Appendix G in [8],[43]).

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$(like Natario did) or the equatorial plane $x-y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components reduces to:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{23}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{24}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{25}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = 0 \rightarrow \cos\phi = 0 \tag{26}$$

The remaining contravariant components are:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \quad (27)$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \quad (28)$$

In a $1+1$ spacetime the equatorial plane we get¿:

$$nX = X^t e_t + X^r e_r \quad (29)$$

$$X^t = 2rf(r)a \quad (30)$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \quad (31)$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \quad (32)$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(r)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(r)$ of the shape function $f(r)$ is zero and the shift vector $X^{rs} = 2[2f(r)^2]at$ with $X^r = 0$ inside the bubble and $X^{rs} = 2[2f(r)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.

Note that in the dimensional reduction from $3+1$ to a $1+1$ spacetime both spherical coordinates $3D$ and polar coordinates vectors produces the same result.

4 Natario Vectors and Natario vectors

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{33}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by:
(see Appendices F and G for pedagogical purposes and H for the final result)

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{34}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{35}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{36}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{37}$$

The equation of the Natario warp drive vector nX in polar coordinates with a variable speed vs due to a constant acceleration a is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{38}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by(see Appendices A and B for pedagogical purposes and C for the final result):

$$X^t = 2f(r)rcos\theta a \tag{39}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{40}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{41}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates.

The equation of the Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nX is given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{42}$$

With the contravariant shift vector components X^r, X^θ and X^ϕ given by:
(see Appendices F and G for details)

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{43}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{44}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{45}$$

The equation of the Natario warp drive vector nX in polar coordinates with a constant speed is given by(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{46}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details)

$$X^r = 2v_s f(r)\cos\theta \tag{47}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{48}$$

The equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates

In the dimensional reduction from $3+1$ to a $1+1$ spacetime the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi=1$ and $cos\phi=0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates and both spherical coordinates $3D$ and polar coordinates vectors produces the same and identical result.This is due to the fact that Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$ only (like Natario did) or the equatorial plane $x-y$ in r where $\theta=0$ $\sin(\theta)=0$ and $\cos(\theta)=1$.(see pgs 4,5 and 6 in [1])

The remaining Natario warp drive vector in polar coordinates $1+1$ spacetime with variable velocities is:

$$nX = X^t e_t + X^r e_r \tag{49}$$

$$X^t = 2rf(r)a \tag{50}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at \tag{51}$$

The remaining Natario warp drive vector nX in polar coordinates $1+1$ spacetime with a constant speed is:

$$nX = X^r e_r \tag{52}$$

$$X^r = 2v_s f(r) \tag{53}$$

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).

The Hodge Star actually must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs*(dx) + x*(dvs)$ but due to a constant speed vs the term $x*d(vs)=0$.In this work we examine what happens with the Natario vector when the velocity is variable and then the term $x*d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

In this work we already presented new warp drive vectors for variable speeds $nX = vs*(dx) + x*(dvs)$.

The warp drive vector in polar coordinates with constant speed was presented in the Appendices A and B and the warp drive vector in $3D$ spherical coordinates with constant speed was presented in the Appendices F and G. The warp drive vector in polar coordinates with variable speeds speed was presented in the Appendix C and the warp drive vector in $3D$ spherical coordinates with variable speeds was presented in the Appendix H.

When in the warp drive vector whether in polar or spherical coordinates the velocity becomes constant the term $x*d(vs)$ disappears and the remaining term is $vs*(dx)$.Note that this term $vs*(dx)$ exists in the constant speed and in the variable speeds warp drive vectors.

In this section we demonstrated the possibility of a dimensional reduction from $3D$ spherical coordinates to polar coordinates in the geometry of warp drive vectors.

We also pointed out that a variable warp drive vector $vs * (dx) + x * (dvs)$ can be reduced to a constant speed warp drive vector $vs * (dx)$ because for constant velocities the term $x * d(vs)$ disappears.

The Appendices M,N,O,P and Q outlines the problem of the negative energy density distribution for the Natario warp drive in polar coordinates with constant speeds.

This negative energy is in front of the ship able to deflect incoming hazardous objects from the interstellar space avoiding dangerous collisions between the ship and the Interstellar Medium IM.

We dont have the negative energy density distribution for the $3D$ spherical or accelerated warp drive vectors but since the Natario warp drive in polar coordinates with constant speeds is a particular case of these new warp drive vectors we hope that in these warp drive spacetimes the negative energy density also remains in front of the ship.

Otherwise we would need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors if these calculations are made "by the hand".

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [17], pg 276 in [30],pgs $454, 457, 560$ in [31] pg 98 in [32],pg 178 in [33]).

Appendix C pgs $551 - 555$ in [31] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.[1]

[1] Unfortunately we dont have access to anyone of these programs so we have our hands "tied up"

5 Natario Warp Drive and Natario warp drive

The Natario warp drive equation for variable velocities in the original $3+1$ ADM formalism in real $3D$ spherical coordinates is given by:(see Appendix J for details)

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (54)$$

$$ds^2 = ((1-2X_t+X_tX^t) - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (55)$$

The equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the original $3+1$ ADM formalism in polar coordinates is given by:(see Appendix J for details)

$$ds^2 = (1 - 2X_t + X_tX^t - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_{rs} dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (56)$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (57)$$

The equation of the Natario warp drive spacetime in $3D$ spherical coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:(see Appendix I for details)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (58)$$

The equation of the Natario warp drive spacetime in polar coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:(see Appendix I for details)

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \quad (59)$$

The difference between variable and constant velocity warp drives is due to the term α^2 that affect the geometrical structure of the whole spacetimes.The term α behaves as a lapse function.

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^tX^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^tX^t) = (1 - 2X_t + X_tX^t) \quad (60)$$

In the dimensional reduction from $3+1$ to a $1+1$ spacetime the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components in $3D$ spherical coordinates reduces to the equivalent counterparts in polar coordinates and both spherical coordinates $3D$ and polar coordinates vectors produces the same and identical result.This is due to the fact that Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$ only (like Natario did) or the equatorial plane $x-y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1])

The equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the $1+1$ spacetime is:

$$ds^2 = (1 - 2X_t + X_t X^t - X_r X^r)dt^2 + 2(X_r dr)dt - dr^2 \tag{61}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^t X^t) = (1 - 2X_t + X_t X^t) \tag{62}$$

$$ds^2 = (\alpha^2 - X_r X^r)dt^2 + 2(X_r dr)dt - dr^2 \tag{63}$$

The equation of the Natario warp drive spacetime in polar coordinates with a constant speed vs in the $1+1$ spacetime is:

$$ds^2 = (1 - X_r X^r)dt^2 + 2(X_r dr)dt - dr^2 \tag{64}$$

Since $X^t = X_t$ and $X^r = X_r$ and $\gamma_{tt} = 1$(see Appendices I and J)[2] the equations above are better written as:

$$ds^2 = (1 - 2X_t + X_t^2 - X_r^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{65}$$

$$ds^2 = (\alpha^2 - X_r^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{66}$$

$$ds^2 = (1 - X_r^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{67}$$

The remaining Natario warp drive vector in polar coordinates $1+1$ spacetime with variable velocities is:

$$nX = X^t e_t + X^r e_r \tag{68}$$

$$X^t = 2rf(r)a = X_t \tag{69}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at = X_r \tag{70}$$

[2]geometrized units $c = G = 1$

6 Horizons(causally disconnected portions of spacetime geometry in the equation of the Natario warp drive spacetime metric with a variable speed vs and a constant acceleration a in the original $1+1$ ADM formalism) in Polar Cordinates

Like the section 4 in [36],[37] and [43] the mathematical discussions of this section also uses mainly quadratic equations.We choose quadratic equations to outline the problem of the Horizons in the Natario warp drive spacetime because and although quadratic equations are often regarded as being elementary forms of mathematics these quadratic equations can illustrate very well the problem of the Horizons.(Unlike the section 4 in [36],[37] and [43] where from the geometrical point of view the photon stopped in the Horizon and the outermost layers of the bubble were causally disconnected from the observer in the center of the bubble for the case of constant velocity vs) in this section and still from a geometrical point of view we will demonstrate that in the case of variable velocities the photon do not stops in the Horizon and the Horizon do not exists and the outermost layers of the bubble are causally connected to the observer in the center of the bubble.All the mathematical calculations are presented step by step.

Examining the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime:

$$ds^2 = (1 - 2X_t + (X_t)^2 - (X_r)^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{71}$$

The covariant shift vector components X_t and X_r are then:

$$X_t = 2f(r)ra \tag{72}$$

$$X_r = 2[2f(r)^2 + rf'(r)]at \tag{73}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \tag{74}$$

The term $1 - 2X_t + (X_t)^2$ in the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime can be simplified as:

$$1 - 2X_t + (X_t)^2 = (1 - (X_t))^2 \tag{75}$$

Hence the equation becomes:

$$ds^2 = ((1 - (X_t))^2 - (X_r)^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{76}$$

We must analyze what happens in this Natario geometry if an observer in the center of the bubble starts to send photons to the front part of the bubble over the direction of motion.A photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$.Then applying the rule of the null-like geodesics $ds^2 = 0$ to the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime we have:

$$0 = ((1 - (X_t))^2 - (X_r)^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{77}$$

Dividing both sides by dt^2 we have:

$$0 = ((1 - (X_t))^2 - (X_r)^2) + 2(X_r \frac{dr}{dt}) - (\frac{dr}{dt})^2 \tag{78}$$

Making the following algebraic substitution:

$$U = \frac{dr}{dt} \tag{79}$$

We have:

$$0 = ((1 - (X_t))^2 - (X_r)^2) + 2(X_r U) - (U)^2 \tag{80}$$

Multiplying both sides of the equation above by -1 and rearranging the terms of the equation we get the result shown below:

$$(U)^2 - 2(X_r U) - ((1 - (X_t))^2 - (X_r)^2) = 0 \tag{81}$$

The solution of the quadratic equation is then given by:

$$U = \frac{2(X_r) \pm \sqrt{4(X_r)^2 + 4((1 - (X_t))^2 - (X_r)^2)}}{2} \tag{82}$$

$$U = \frac{2(X_r) \pm \sqrt{4(X_r)^2 + 4(1 - (X_t))^2 - 4(X_r)^2}}{2} \tag{83}$$

The simplified algebraic expression becomes:

$$U = \frac{2(X_r) \pm \sqrt{4(1 - (X_t))^2}}{2} \tag{84}$$

Which leads to:

$$U = \frac{2(X_r) \pm 2(1 - (X_t))}{2} \tag{85}$$

And the final result is then given by:

$$U = X_r \pm (1 - (X_t)) \tag{86}$$

The above equation have two possible solutions U respectively $U = X_r + (1 - (X_t))$ and $U = X_r - (1 - (X_t))$ being each solution U a root of the quadratic form.Remember that a photon according to General Relativity always moves in a null-like geodesics in which $ds^2 = 0$ and in our case a photon can be sent to the front or the rear parts of the bubble both parts being encompassed by $ds^2 = 0$ with each part being a root U and a solution of the quadratic form.The solutions U for the front and the rear parts of the bubble are then respectively given by:

$$U_{front} = X_r - (1 - (X_t)) = X_r + X_t - 1 \tag{87}$$

$$U_{rear} = X_r + (1 - (X_t)) = X_r - X_t + 1 \tag{88}$$

We are interested in the behavior of the photon sent to the front part of the bubble which means:

$$U_{front} = X_r - (1 - (X_t)) = X_r + X_t - 1 \tag{89}$$

$$X_t = 2f(r)ra \tag{90}$$

$$X_r = 2[2f(r)^2 + rf'(r)]at = 4f(r)^2at + 2rf'(r)at = 2f(r)2f(r)at + 2rf'(r)at \tag{91}$$

Note that unlike the section 4 in [36],[37] and [43] when we got only one contravariant shift vector component X^r and it was this component that dictated the Horizon behavior of the front solution for the quadratic form now we get two covariant shift vector components X_r and X_t for the front solution of the quadratic form.This would be algebraically more complicated to be manipulated but fortunately we can write the component X_r in function of the component X_t simplifying greatly the analysis.Using the following algebraic expressions both written in function of X_t

$$\frac{X_t}{r} = 2f(r)a \tag{92}$$

$$\frac{X_t}{f(r)} = 2ra \tag{93}$$

We can write X_r in function of X_t as follows:

$$X_r = 2f(r)2f(r)at + 2rf'(r)at = 2f(r)\frac{X_t}{r}t + f'(r)\frac{X_t}{f(r)}t \tag{94}$$

$$X_r = 2\frac{f(r)}{r}tX_t + \frac{f'(r)}{f(r)}tX_t \tag{95}$$

Simplifying we get:

$$X_r = tX_t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] \tag{96}$$

And the final solution of the quadratic form for the photon sent to the front part of the bubble is finally given by:

$$U_{front} = X_r + X_t - 1 = tX_t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + X_t - 1 \tag{97}$$

The expression simplified leads to:

$$U_{front} = (X_t)(t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + 1) - 1 \tag{98}$$

$$U_{front} = (2f(r)ra)(t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + 1) - 1 \tag{99}$$

The final solution of the quadratic form for the photon sent to the front part of the bubble is:

$$U_{front} = (X_t)(t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + 1) - 1 \quad (100)$$

$$U_{front} = (2f(r)ra)(t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + 1) - 1 \quad (101)$$

Again note that unlike the section 4 in [36],[37] and [43] where the dominant term was the component X^{rs} now the dominant term is the component X_t and $X_t = 2f(r)ra$.

Considering again a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]) and assuming a continuous behavior for $f(r)$ from 0 to $\frac{1}{2}$ and in consequence a continuous behavior for $2f(r)ra$ from 0 to ra we can clearly see that inside the bubble $2f(r)ra = 0$ because $f(r) = 0$ and outside the bubble $2f(r)ra = ra$ because $f(r) = \frac{1}{2}$ and assuming also continuous values from 0 to ra then somewhere in the Natario warped region where $0 < f(r) < \frac{1}{2}$ we have the situation in which $2f(r)ra = 1$ because 1 lies in the continuous interval from 0 to ra and in consequence $X_t = 1$.

The final solution of the quadratic form for the photon sent to the front part of the bubble when $X_t = 1$ is:

$$U_{front}(X_t = 1) = (X_t)(t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + 1) - 1 = t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] + 1 - 1 \quad (102)$$

Simplifying the result leads ourselves to:

$$U_{front}(X_t = 1) = t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] \neq 0!!! \quad (103)$$

Note that unlike the section 4 in [36],[37] and [43] the result is not zero !!!.The photon do not stops in the Natario warped region and the Horizon no longer exists!!!.

The place where $X_t = 2f(r)ra = 1$ is the place where the Natario shape function is $f(r) = \frac{1}{2ra}$ well inside the Natario warped region in which $0 < \frac{1}{2ra} < \frac{1}{2}$ with $a >= 1$ and of course $r > 0$.

Rewriting the solution of the quadratic form for the photon sent to the front part of the bubble when $X_t = 1$ using the value of the Natario shape function $f(r) = \frac{1}{2ra}$ we get:

$$U_{front}(X_t = 1) = t[2\frac{f(r)}{r} + \frac{f'(r)}{f(r)}] = t[2\frac{\frac{1}{2ra}}{r} + \frac{f'(r)}{\frac{1}{2ra}}] \neq 0!!! \quad (104)$$

$$U_{front}(X_t = 1) = t[\frac{1}{r^2 a} + 2f'(r)ra] \neq 0!!! \quad (105)$$

The solution of the quadratic form for the photon sent to the front part of the bubble when $X_t = 1$ using the value of the Natario shape function $f(r) = \frac{1}{2ra}$ is:

$$U_{front}(X_t = 1) = t[\frac{1}{r^2 a} + 2f'(r)ra] \neq 0!!! \quad (106)$$

Note that both the expressions $\frac{1}{r^2 a}$ and $2f'(r)ra$ have fractionary values close to zero but always greater than zero and above everything else not zero at all!!!.In the first expression we have both $r > 0$ and $a \geq 1$ and in the second expression the derivative of the shape function must have low values in order to reduce the needs of negative energy density to sustain a warp bubble.

Then we can easily see that $0 < \frac{1}{r^2 a} < 1$ due to $r > 0$ and $a \geq 1$ in the fraction and $0 < 2f'(r)ra < 1$ see section 3 in [29] for the low values of the square derivative of the Natario shape function able to reduce the negative energy density requirements implying in a low value for the derivative of the shape function.The expressions can be written as follows:

$$\frac{1}{r^2 a} \neq 0!!! \rightarrow \frac{1}{r^2 a} \simeq 0!!! \rightarrow \frac{1}{r^2 a} > 0!!! \rightarrow r > 0 \rightarrow a \geq 1 \quad (107)$$

$$2f'(r)ra \neq 0!!! \rightarrow 2f'(r)ra \simeq 0!!! \rightarrow 2f'(r)ra > 0!!! \quad (108)$$

Then the solution of the quadratic form for the photon sent to the front part of the bubble when $X_t = 1$ using the value of the Natario shape function $f(r) = \frac{1}{2ra}$ is better written as:

$$U_{front}(X_t = 1) = t[\frac{1}{r^2 a} + 2f'(r)ra] \cong 0!!! \quad (109)$$

$$U_{front}(X_t = 1) = t[\frac{1}{r^2 a} + 2f'(r)ra] > 0!!! \quad (110)$$

The result is close to zero but it is always greater than zero!!!.The photon do not stops but moves at a very low speed!!!.The Horizon do not exists!!!.(see section 3 in [38]),(see section 5 in [36]).

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for variable speed vs and constant acceleration a in a $1 + 1$ spacetime.We know that in the case of the Natario warp drive with constant speed the negative energy density covers the entire bubble.(see Appendices M,N,O,P and Q).Unfortunately we dont have the distribution of the negative energy density for the case of variable speeds(see Section 4).Then we dont know if the negative energy density covers the entire bubble in the case of variable speeds but if this happens and since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which would perhaps exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 83 in [20].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [20] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [34],pg 268 in [35] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

Back again to the solution of the photon sent to the front of the bubble:

$$U_{front} = X_r - (1 - (X_t)) = X_r + X_t - 1 \tag{111}$$

This is the solution for a Natario warp drive metric with variable velocities.Compare with the solution of the photon sent to the front of the bubble in a Natario warp drive metric with fixed velocity given in section 4 in [36],[37] and [43]:

$$U_{front} = X^{rs} - 1 \tag{112}$$

The term in X_t affects the whole structure of the spacetime geometry eliminating once for all the problem of the Horizon. When the velocity is constant the term in X_t vanishes leaving only the term $U_{front} = X_r - 1$ in $X_r = X^{rs}$ and in consequence $X^{rs} - 1$ and as already seen in section 4 in [36],[37] and [43] the Horizon appears.

In the Appendix L we presented the solutions for the generic quadratic forms in the $3+1$ spacetime with a lapse function.The dimensional reduction of this generic form to a $1+1$ spacetime gives the following result:

$$ds^2 = (\alpha^2 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{113}$$

$$\frac{dx^1}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{114}$$

$$\frac{dx^1}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{115}$$

Examining the Natario warp drive equation for variable speeds with a lapse function in a 1+1 spacetime:

$$ds^2 = (\alpha^2 - [X_r]^2)dt^2 + 2(X_r dr)dt - dr^2 \tag{116}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^t X^t) = (1 - 2X_t + X_t X^t) \tag{117}$$

But we know that $X^t = X_t$,$X^r = X_r = X^1 = X_1$ also $\gamma_{11} = 1$ and $(dx^1)^2 = dr^2$

$$\alpha^2 = (1 - 2X_t + X_t X^t) = (1 - 2X_t + X_t^2) = (1 - X_t)^2 \tag{118}$$

The solutions are:

$$\frac{dx^1}{dt} = \frac{dr}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} = X_1 + \alpha \Rightarrow \alpha = (1 - X_t) \tag{119}$$

$$\frac{dx^1}{dt} = \frac{dr}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} = X_1 - \alpha \Rightarrow \alpha = (1 - X_t) \tag{120}$$

And we recover the Horizons solutions with the lapse function presented in this section.

7 Horizons in the Natario warp drive with a variable speed vs in the original $3+1$ ADM formalism with a lapse function α in Polar Coordinates

In the previous section we used photons sent to the front of the bubble in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $2+1$ spacetime.($2+1$ dimensions).

The equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the original $3+1$ ADM formalism in polar coordinates is given by:(see Appendix J for details)

$$ds^2 = (1 - 2X_t + X_tX^t - X_rX^r - X_\theta X^\theta)dt^2 + 2(X_{rs}dr + X_\theta d\theta)dt - dr^2 - r^2d\theta^2 \tag{121}$$

$$ds^2 = (\alpha^2 - X_rX^r - X_\theta X^\theta)dt^2 + 2(X_rdr + X_\theta d\theta)dt - dr^2 - r^2d\theta^2 \tag{122}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^tX^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^tX^t) = (1 - 2X_t + X_tX^t) \tag{123}$$

The term $1 - 2X_t + (X_t)^2$ in the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime can be simplified as:

$$1 - 2X_t + (X_t)^2 = (1 - (X_t))^2 \tag{124}$$

$$\alpha^2 = (1 - (X_t))^2 \tag{125}$$

$$\alpha = (1 - (X_t)) \tag{126}$$

Because $X^t = X_t$ and $\gamma_{tt} = 1$(see Appendices I and J)[3]

Actually Polar Coordinates are given in the $2+1$ spacetime.The generic quadratic form and its solutions for the $2+1$ spacetime are given by:(see Appendix L)

$$ds^2 = (\alpha^2 - X_1X^1 - X_2X^2)dt^2 + 2(X_1dx^1 + X_2dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{127}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{128}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{129}$$

[3] geometrized units $c = G = 1$

The equation of the Natario warp drive vector nX in polar coordinates with a variable speed vs due to a constant acceleration a is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \quad (130)$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by(see Appendices A and B for pedagogical purposes and C for the final result):

$$X^t = 2f(r)rcos\theta a \quad (131)$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \quad (132)$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \quad (133)$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Remember also that $\gamma_{tt} = 1$.Then the covariant shift vector components X_t,X_r,X_θ,$X_1 = X_r$ and $X_2 = X_\theta$ are given by:

$$X_t = \gamma_{tt}X^t = X^t \quad (134)$$

$$X_i = \gamma_{ii}X^i \quad (135)$$

$$X_t = \gamma_{tt}X^t = 2f(r)rcos\theta a \quad (136)$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = X^r = X^r = 2[2f(r)^2 + rf'(r)]atcos\theta = X_1 \quad (137)$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = -2f(r)at[2f(r) + rf'(r)]r^2\sin\theta = X_2 \quad (138)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \quad (139)$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \quad (140)$$

$\gamma_{11} = \gamma_{rr} = 1$,$\gamma_{22} = \gamma_{\theta\theta} = r^2$,$\alpha = (1 - (X_t))$

$X_1 = X_r = 2[2f(r)^2 + rf'(r)]atcos\theta$,$X_2 = X_\theta = -2f(r)at[2f(r) + rf'(r)]r^2\sin\theta$

Note that now the photon moves in a $2+1$ spacetime and this means motion in r and θ.

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1+r^2}}{1+r^2} = \frac{[2[2f(r)^2 + rf'(r)]atcos\theta] + [-2f(r)at[2f(r) + rf'(r)]r^2\sin\theta] + \alpha\sqrt{1+r^2}}{1+r^2} \quad (141)$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} = \frac{X_r + X_\theta + \alpha\sqrt{1+r^2}}{1+r^2} = \frac{[2[2f(r)^2 + rf'(r)]atcos\theta] + [-2f(r)at[2f(r) + rf'(r)]r^2\sin\theta] - \alpha\sqrt{1+r^2}}{1+r^2} \quad (142)$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta+\alpha\sqrt{1+r^2}}{1+r^2}=\frac{[2[2f(r)^2+rf'(r)]atcos\theta]+[-2f(r)at[2f(r)+rf'(r)]r^2\sin\theta]+\alpha\sqrt{1+r^2}}{1+r^2} \quad (143)$$

$$\frac{dr}{dt}+\frac{d\theta}{dt}=\frac{X_r+X_\theta+\alpha\sqrt{1+r^2}}{1+r^2}=\frac{[2[2f(r)^2+rf'(r)]atcos\theta]+[-2f(r)at[2f(r)+rf'(r)]r^2\sin\theta]-\alpha\sqrt{1+r^2}}{1+r^2} \quad (144)$$

In two dimensions the photon moves in a $2+1$ spacetime and this means motion in r and θ the Horizon do not occurs even if $\theta=0$,$\cos\theta=1$,$\sin\theta=0$ and $r^2d\theta^2=0$ and we recover in this case the problem of the Horizon with the lapse function in the $1+1$ spacetime.

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for variable speed vs in a $3+1$ spacetime in Polar Cooordinates with a lapse function and we know that in the Natario warp drive with constant speeds the negative energy density covers the entire bubble.(see Appendices M,N,O,P and Q).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

Unfortunately we dont have the distribution of the negative energy density for the case of variable speeds(see Section 4).Then we dont know if the negative energy density covers the entire bubble in the case of variable speeds but if this happens and since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which would perhaps exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 83 in [20].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [20] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [34],pg 268 in [35] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

8 Horizons in the Natario warp drive with a variable speed vs in the original $3+1$ ADM formalism with a lapse function α in $3D$ Spherical Coordinates

In some of the previous sections we used photons sent to the front of the bubble in the $1+1$ spacetime.($1+1$ dimensions).

Now we use photons sent to the front of the bubble in the $3+1$ spacetime.($3+1$ dimensions).

The equation of the Natario warp drive spacetime in $3D$ Spherical Coordinates with a variable speed vs in the original $3+1$ ADM formalism with a lapse function is given by:(see Appendix J)

$$ds^2 = ((1-2X_t+X_tX^t)-X_rX^r-X_\theta X^\theta-X_\phi X^\phi)dt^2+2(X_rdr+X_\theta d\theta+X_\phi d\phi)dt-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2 \tag{145}$$

$$ds^2 = (\alpha^2 - X_rX^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_rdr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{146}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^tX^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^tX^t) = (1 - 2X_t + X_tX^t) \tag{147}$$

The term $1 - 2X_t + (X_t)^2$ in the Natario warp drive equation for variable speed vs and constant acceleration a in a $1+1$ spacetime can be simplified as:

$$1 - 2X_t + (X_t)^2 = (1 - (X_t))^2 \tag{148}$$

$$\alpha^2 = (1 - (X_t))^2 \tag{149}$$

$$\alpha = (1 - (X_t)) \tag{150}$$

Because $X^t = X_t$ and $\gamma_{tt} = 1$(see Appendices I and J)[4]

The generic quadratic form and its solutions for the $3+1$ spacetime are given by:(see Appendix L)

$$ds^2 = (\alpha^2 - X_1X^1 - X_2X^2 - X_3X^3)dt^2 + 2(X_1dx^1 + X_2dx^2 + X_3dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{151}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{152}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{153}$$

[4] geometrized units $c = G = 1$

Remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2 \sin^2\theta$.Remember also that $\gamma_{tt} = 1$. Then the covariant shift vector components X_r,X_θ,X_ϕ,$X_1 = X_r$,$X_2 = X_\theta$ and $X_3 = X_\phi$ are given by:(see Appendix J)

$$X_i = \gamma_{ii} X^i \tag{154}$$

$$X_t = \gamma_{tt} X^t \tag{155}$$

$$X_t = \gamma_{tt} X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) = X^t \tag{156}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X^r \tag{157}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{158}$$

$$X_\phi = \gamma_{\phi\phi} X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{159}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \tag{160}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \tag{161}$$

Note that now the photon moves in a $3 + 1$ spacetime and this means motion in r,θ and ϕ.

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi + \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \tag{162}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{X_r + X_\theta + X_\phi - \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \tag{163}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{U + \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \tag{164}$$

$$\frac{dr}{dt} + \frac{d\theta}{dt} + \frac{d\phi}{dt} = \frac{U - \alpha\sqrt{1 + r^2 + r^2 sin^2\theta}}{1 + r^2 + r^2 sin^2\theta} \tag{165}$$

$$U = V + X_\phi \tag{166}$$

$$V = X_r + X_\theta \tag{167}$$

$$V = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) + -r^2(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{168}$$

$$V = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) - r^2(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{169}$$

$$U = V + r^2 sin^2\theta(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \quad (170)$$

In three dimensions the photon moves in a $3+1$ spacetime and this means motion in r,θ and ϕ the Horizon do not occurs even with $\theta = 0$,$\cos\theta = 1$,$\sin\theta = 0$,$\phi = 90$,$\sin\phi = 1$,$\cos\phi = 0$,$r^2 d\theta^2 = 0$ and $r^2 \sin^2 d\phi^2 = 0$ and we recover in this case the problem of the Horizon with the lapse function in the $1+1$ spacetime.

Of course this point of view about the Horizons reflects only the geometrical point of view of the Natario warp drive equation for variable speed vs in a $3+1$ spacetime in Spherical Coordinates with a lapse function and we know that in the Natario warp drive with constant speeds the negative energy density covers the entire bubble.(see Appendices M,N,O,P and Q).Since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which exists in the front of the bubble never reaching the bubble walls.

Unfortunately we dont have the distribution of the negative energy density for the case of variable speeds(see Section 4).Then we dont know if the negative energy density covers the entire bubble in the case of variable speeds but if this happens and since the negative energy density have repulsive gravitational behavior(see pg 116 in [19]) the photon of light would then be deflected by the repulsive behavior of the negative energy density which would perhaps exists in the front of the bubble never reaching the bubble walls.

The solution that allows contact with the bubble walls was presented in pg 83 in [20].Although the light cone of the external part of the large warp bubble is causally disconnected from the astronaut who lies inside the center of the large warp bubble he(or she) can somehow generate micro warp bubbles and since the astronaut is external to the micro warp bubble he(or she) contains the entire light cone of the micro warp bubble so these bubbles can be "created" at sublight speed by the astronaut and then perhaps these micro warp bubbles can be "post-programmed" to achieve superluminal speed using perhaps an idea similar to the idea outlined in fig 7 pg 83 in [20] to be sent to the large warp bubble keeping it in causal contact.Remember that one source of negative energy repels a source of positive energy but attracts another source of negative energy.This idea seems to be endorsed by pg 34 in [34],pg 268 in [35] where it is mentioned that warp drives can only be created or controlled by an observer that contains the entire forward light cone of the bubble.

9 Conclusion

In this work we introduced a new tridimensional $3D$ spherical coordinates warp drive vector using the Natario mathematical techniques.We focused ourselves in the application of the Hodge Star in $3D$ spherical coordinates for variable speeds.

Our focus was concentrated in the Natario methods to obtain a warp drive vector.We know that we used a language and a presentation method or style that may be regarded as exhaustive tedious and monotonous for experienced or seasoned readers but we are concerned about beginners,newcomers,novices or intermediate students not familiarized with the techniques Natario used to develop warp drive vectors so our extensive mathematical demonstrations QED Quod Erad Demonstratum will benefit this audience at least we hope.We gave our best efforts trying to accomplish this goal but only this audience will tell in the future if we succeeded (or not).

Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The final form of the Hodge Star for this warp drive vector was calculated no longer over $*d(r\cos\theta)$ as Natario did but instead over $*d(r\sin\phi\cos\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.

Remember also that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model so it must possesses variable velocities.

One the major drawbacks concerning warp drives is the problem of the Horizons(causally disconnected portions of spacetime) in which an observer in the center of the bubble cannot signal nor control the front part of the bubble. The behavior of a photon sent to the front of the warp bubble in the case of a Natario warp drive with variable velocity and a lapse function was also one of the main purposes of this work.We presented the behavior of a photon sent to the front of the bubble in the Natario warp drive in the $1+1$ and $3+1$ spacetimes in polar and spherical coordinates with the lapse function using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity and we provided here the step by step mathematical calculations in order to outline the final results found in our work which are the following ones:

For the case of the lapse function the Horizon do not exists at all.Due to the extra terms in the lapse function that affects the whole spacetime geometry this solution allows to circumvent the problem of the Horizon.

In this work we developed Horizons for variable velocities in a tridinemsional spacetime..

The application of the Horizons in the tridimensional $3D$ spherical coordinates warp drive vector using variable speeds and the ADM(Arnowitt-Dresner-Misner) formalism equations in General Relativity with the approach of MTW(Misner-Thorne-Wheeler) and Alcubierre complements the works in [8],[10],[11][12], [13],[36],[37][38] and [43].

The Natario warp drive is possibly the best candidate for a realistic interstellar space travel.(see Appendices M,N,O,P and Q).See also the works in [24].[25].[21],[22],[26],[27],[28],[29] and [23].

The warp drive as an artificial superluminal geometric tool that allows to travel faster than light may well have an equivalent in the Nature.According to the modern Astronomy the Universe is expanding and as farther a galaxy is from us as faster the same galaxy recedes from us.The expansion of the Universe is accelerating and if the distance between us and a galaxy far and far away is extremely large the speed of the recession may well exceed the light speed limit.(see pg 98 in [39] and pg 377 in [40]).

For the experimental verification of the acceleration of the Universe see for example the bottom of pg 355 and top of pg 356 eq 8.155 in [42].

10 Appendix A:mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^3 space basis-Polar Coordinates

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{171}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{172}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{173}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{174}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{175}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{176}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{177}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{178}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{179}$$

Back again to the Natario equivalence between polar and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{180}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{181}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{182}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{183}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{184}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{185}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{186}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{187}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{188}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{189}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{190}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{191}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{192}$$

Because and according to eqs 3.90 and 3.91 pg $74(a)(b)$ in [2],tb 3.2 pg $68(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{193}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{194}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{195}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{196}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{197}$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{198}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{199}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{200}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{201}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{202}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{203}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{204}$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(d\varphi\wedge dr)\sim -r\sin\theta(dr\wedge d\varphi) \tag{205}$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dr\wedge(rd\theta)\sim r(dr\wedge d\theta) \tag{206}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{207}$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \tag{208}$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \tag{209}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{210}$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{211}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{212}$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{213}$$

We prefer the first expression above for the Natario warp drive vector nX in Polar Coordinates with constant speeds(pg 2 and 5 in [1]):

$$nX=X^r e_r+X^\theta e_\theta \tag{214}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])

$$X^r=2v_s f(r)\cos\theta \tag{215}$$

$$X^\theta=-v_s(2f(r)+(r)f'(r))\sin\theta \tag{216}$$

11 Appendix B:mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs or for the first term $vs * dx$ from the Natario vector $nX = vs * dx + x * dvs$(a variable speed) in a R^4 space basis-Polar Coordinates

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{217}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{218}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{219}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{220}$$

$$rd\theta \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{221}$$

$$r\sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{222}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{223}$$

$$*rd\theta = r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{224}$$

$$*r\sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{225}$$

Back again to the Natario equivalence between polar and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta dt \wedge d\theta \wedge d\varphi + r\sin^2\theta dt \wedge dr \wedge d\varphi = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{226}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{227}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{228}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{229}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(dt\wedge d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(dt\wedge d\varphi\wedge dr)] \tag{230}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] - [r\sin^2\theta(dt\wedge d\varphi\wedge dr)] \tag{231}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{232}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] + [r\sin^2\theta(dt\wedge dr\wedge d\varphi)] \tag{233}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{234}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{235}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{236}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \tag{237}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \quad (238)$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]::

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (239)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p=3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \quad (240)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (241)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt\wedge d(\sin^2\theta)\wedge d\varphi - dt\wedge\sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) \quad (242)$$

$$*[d(r^2)d\varphi] = 2rdt\wedge dr\wedge d\varphi - dt\wedge r^2\wedge dd\varphi = 2r(dt\wedge dr\wedge d\varphi) \quad (243)$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + r\sin^2\theta(dt\wedge dr\wedge d\varphi) \quad (244)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (245)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (246)$$

$$f(r)r^2 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (247)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (248)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (249)$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim dt\wedge(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(dt\wedge d\theta\wedge d\varphi) \quad (250)$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim dt\wedge(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(dt\wedge d\varphi\wedge dr)\sim -r\sin\theta(dt\wedge dr\wedge d\varphi) \quad (251)$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dt\wedge dr\wedge(rd\theta)\sim r(dt\wedge dr\wedge d\theta) \quad (252)$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(dt\wedge d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)] \quad (253)$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \quad (254)$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \quad (255)$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (256)$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \quad (257)$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \quad (258)$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \quad (259)$$

We prefer the first expression above for the Natario warp drive vector nX in Polar Coordinates (pg 2 and 5 in [1]):

$$nX=X^r e_r+X^\theta e_\theta \quad (260)$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])

$$X^r=2v_s f(r)\cos\theta \quad (261)$$

$$X^\theta=-v_s(2f(r)+(r)f'(r))\sin\theta \quad (262)$$

12 Appendix C:mathematical demonstration of the Natario vector $nX = *(vsx) = vs * dx + x * dvs$ for a variable speed vs and a constant acceleration a in Polar Coordinates

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G in [8],[43] for an explanation about this statement)

In the Appendices A and B we gave the mathematical demonstration of the Natario vector $nX = vs * dx$ in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the Natario vector nX for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{263}$$

$$*dx = *d(rcos\theta) = *d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) = *d[f(r)r^2 \sin^2\theta d\varphi] \tag{264}$$

The equation of the Natario vector nX(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{265}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{266}$$

With the contravariant shift vector components explicitly given by:

$$X^r = 2v_s f(r)\cos\theta \tag{267}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{268}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{269}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1^5$,eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0^6$ pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{270}$$

$$dt \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{271}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dt = r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{272}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{273}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{274}$$

[5]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[6]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{275}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{276}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\varphi)] \tag{277}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{278}$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \tag{279}$$

The term $*dx$ was obtained in the Appendices A and B as follows:(see pg 5 in [1])

$$*dx = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta \tag{280}$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + x(2[atf'(r)e_r + f(r)ae_t]) \tag{281}$$

But remember that we are in polar coordinates(pg 4 in [1]) in which $x = rcos\theta$(see pg 5 in [1]) and this leaves us with:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{282}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{283}$$

Then we can start with a warp bubble initially at the rest using the Natario vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the Natario vector for constant speeds shown below:

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \tag{284}$$

Working some algebra with the Natario vector for variable velocities we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (285)$$

$$nX = 4f(r)^2 at\ cos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta + 2atf'(r)rcos\theta e_r + 2f(r)rcos\theta ae_t \quad (286)$$

$$nX = 2f(r)rcos\theta ae_t + 4f(r)^2 at\ cos\theta e_r + 2atf'(r)rcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (287)$$

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (288)$$

Then the Natario vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \quad (289)$$

Or being:

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (290)$$

The contravariant shift vector components are respectively given by the following expressions:

$$X^t = 2f(r)rcos\theta a \quad (291)$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \quad (292)$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \quad (293)$$

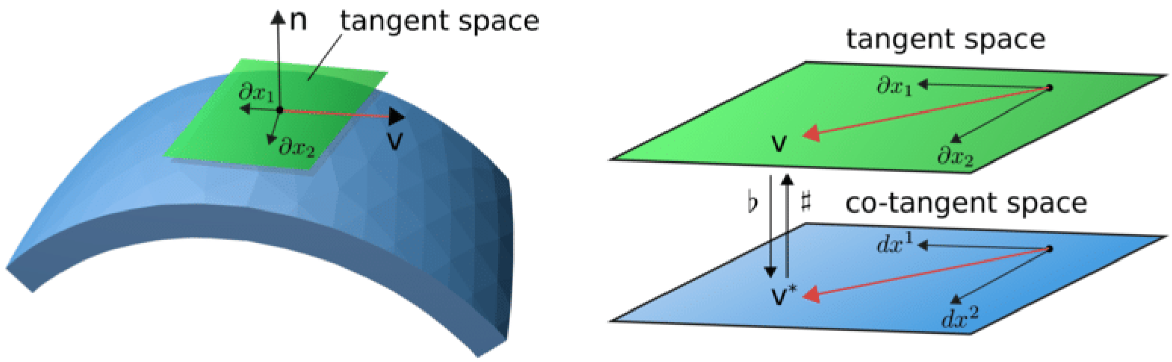

Figure 1: Artistic Presentation of Tangent and Cotangent Spaces I.(Source:Internet)

13 Appendix D:Tangent and Cotangent Spaces I

The Canonical Basis of the Hodge Star * in spherical coordinates in R^3 can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{294}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{295}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{296}$$

The Canonical Basis of the Hodge Star * in spherical coordinates in R^4 can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{297}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{298}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{299}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[7],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[8] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{300}$$

As a matter of fact we have for the Canonical Basis and the Hodge Star * in R^4 the following equations (see pg 47 eqs 2.67 to 2.70 in [3]):

$$*e_0 = e_1 \wedge e_2 \wedge e_3 \tag{301}$$

$$*e_1 = e_0 \wedge e_2 \wedge e_3 \tag{302}$$

$$*e_2 = e_0 \wedge e_3 \wedge e_1 \tag{303}$$

$$*e_3 = e_0 \wedge e_1 \wedge e_2 \tag{304}$$

In R^3 the corresponding equations are:(see pg 55 in [5])(see also pg 54 fig 4.2 in [5] for a graphical presentation of the Hodge Star * in R^3)(see pg 18 eq 1.55 in [6]):

$$*e_1 = e_2 \wedge e_3 \tag{305}$$

$$*e_2 = e_3 \wedge e_1 = -e_1 \wedge e_3 \tag{306}$$

$$*e_3 = e_1 \wedge e_2 \tag{307}$$

The Canonical Basis e_i are related to the partial derivatives $\frac{\partial}{\partial x_i}$ or simplifying related to ∂x_i wether in R^3 or R^4 and are graphically represented by the partial derivatives ∂x_i included in the tangent space of the picture given in the beginning of this section.

[7]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[8]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

On the other hand in R^4 we also have the following relations for the Hodge Star *:(see pg 92 in [3])

$$*dt = dx \wedge dy \wedge dz \tag{308}$$

$$*dx = dt \wedge dy \wedge dz \tag{309}$$

$$*dy = dt \wedge dz \wedge dx \tag{310}$$

$$*dz = dt \wedge dx \wedge dy \tag{311}$$

Also for R^4 considering the $((w,v)(\epsilon\Lambda_p^3)(R^{1,3}))$ formalism we may have the following relations:(see pg 382 in [4])($x^1 = x$,$x^2 = y$,$x^3 = z$)

$$*dt = dx^1 \wedge dx^2 \wedge dx^3 \tag{312}$$

$$*dx^1 = dt \wedge dx^2 \wedge dx^3 \tag{313}$$

$$*dx^2 = dt \wedge dx^3 \wedge dx^1 \tag{314}$$

$$*dx^3 = dt \wedge dx^1 \wedge dx^2 \tag{315}$$

In R^3 we would have the following relations:(see pg 117 eqs 4.6 and 4.7 in [7])(see pg 298 in [4])

$$*dx = dy \wedge dz \tag{316}$$

$$*dy = dz \wedge dx \tag{317}$$

$$*dz = dx \wedge dy \tag{318}$$

The differentials dx,dy,dz or dx^1, dx^2 and dx^3 are related to the cotangent space differentials included in the picture given in the beginning of this section.

See the graphical presentations of the relations between tangent and cotangent spaces in pg 55 fig 2.28 and pg 70 fig 3.1 in [4].See pg 168 fig 5.19 for a graphical presentation of $dx \wedge dy$,pg 169 fig 5.20 for a graphical presentation of $dy \wedge dz$ and pg 170 fig 5.21 for a graphical presentation of $dz \wedge dx$ all in [4].

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{319}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{320}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{321}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

See also Appendix E.

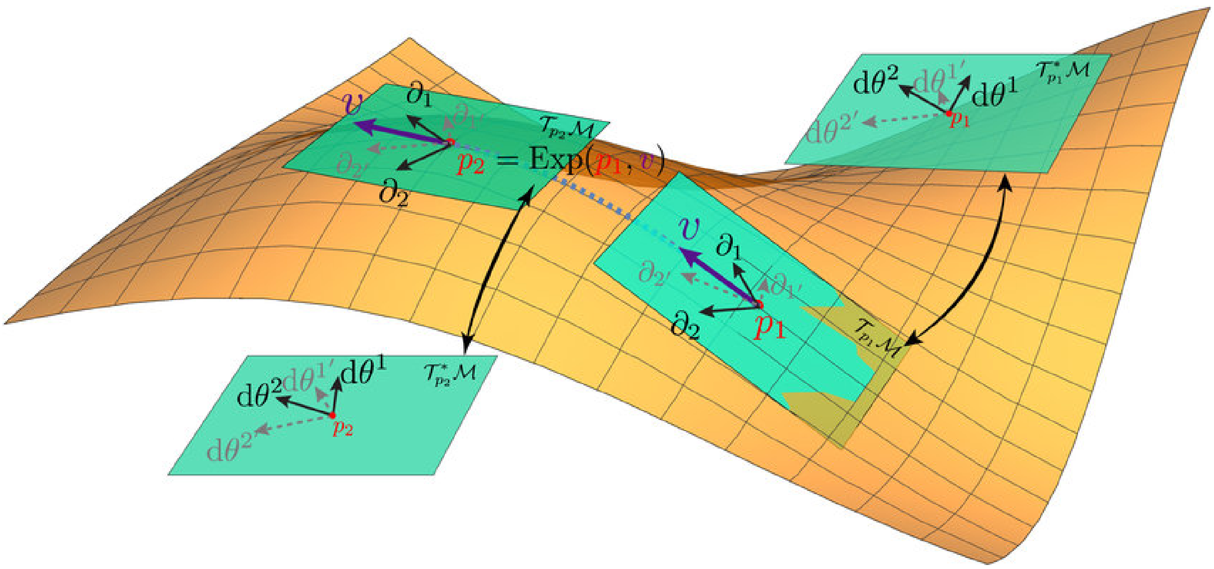

Figure 2: Artistic Presentation of Tangent and Cotangent Spaces II.(Source:Internet)

14 Appendix E:Tangent and Cotangent Spaces II

Consider a curve R in R^4 defined in function of a given set of coordinates u^0,u^1,u^2 and u^3 as being $R = R(u^0, u^1, u^2, u^3)$.

A total derivative of R is given by:

$$dR = \frac{\partial R}{\partial u^0}du^0 + \frac{\partial R}{\partial u^1}du^1 + \frac{\partial R}{\partial u^2}du^2 + \frac{\partial R}{\partial u^3}du^3 \tag{322}$$

Applying the Einstein summing convention:

$$dR = \frac{\partial R}{\partial u^i}du^i = e_i du^i \tag{323}$$

or

$$dR = \frac{\partial R}{\partial u^j}du^j = e_j du^j \tag{324}$$

With $i, j = 0, 1, 2, 3$ as the coordinates, $\frac{\partial R}{\partial u^i}$ and $\frac{\partial R}{\partial u^j}$ as the directional partial derivatives of R with respect to each coordinate and e_i and e_j are the respective Canonical Basis.

Defining $ds^2 = dR \bigotimes dR$ we have:

$$ds^2 = dR \bigotimes dR = \frac{\partial R}{\partial u^i}du^i \bigotimes \frac{\partial R}{\partial u^j}du^j = e_i du^i \bigotimes e_j du^j \tag{325}$$

$$ds^2 = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j}du^i du^j = e_i e_j du^i du^j = g_{ij} du^i du^j \tag{326}$$

$$g_{ij} = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j} = e_i e_j \tag{327}$$

The directional partial derivatives of R and their respective Canonical Basis are related to the ∂_i and ∂_j tangent spaces of the picture depicted in the beginning of this section while the differentials du^i and du^j are related to the respective cotangent spaces.See pg 148 problem 17 in [14],pg 132 eq 10.12 pg 133 eqs 10.14a,10.14b and 10.15 in [15].

$g_{ij} = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j} = e_i e_j$ is the spacetime metric tensor of General Relativity.

15 Appendix F:mathematical demonstration of the warp drive vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^3 space basis-$3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{328}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{329}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{330}$$

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho\sin\phi\cos\theta)$:

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\cos\theta) = sin\phi[d(r\cos\theta)] + (r\cos\theta)d(\sin\phi) \tag{331}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta dr + r(d\cos\theta)] + (r\cos\theta)(cos\phi d\phi) \tag{332}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - r sin\theta(d\theta)] + (r\cos\theta)[cos\phi(d\phi)] \tag{333}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - sin\theta(rd\theta)] + cos\phi[(r\cos\theta)(d\phi)] \tag{334}$$

Applying the Hodge Star * to the term $[cos\theta(dr) - sin\theta(rd\theta)]$ we will get the same results already shown in the Appendix A and the first part of the $3D$ spherical warp drive vector is the one of the Appendix A multiplied by $\sin\phi$ .Then we must concern ourselves with the term $cos\phi[(r\cos\theta)(d\phi)]$ and the following Canonical Basis for the Hodge Star * since the other two were covered in the Appendix A.

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{335}$$

The term $cos\phi[(r\cos\theta)(d\phi)]$ must become compatible with the Canonical Basis for the Hodge Star above and this can be achieved by the following substitution:

$$cos\phi[(r\cos\theta)(d\phi)] = cos\phi[(r\sin\theta\cot\theta)(d\phi)] = cos\phi[\cot\theta(r\sin\theta)(d\phi)] \tag{336}$$

$$cos\phi[\cot\theta * ((r\sin\theta)(d\phi))] = cos\phi[\cot\theta(r(dr \wedge d\theta))] = cos\phi[\cot\theta(e_\phi)] \tag{337}$$

In the Appendix A we used the term $d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ and its respective Hodge Star $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ also used by Natario in pg 5 in [1] because this term corresponds to the term $[cos\theta(*dr) - sin\theta(*rd\theta)]$ now being multiplied by $sin\phi$.In the $3D$ spherical warp drive this term also appears multiplied by $sin\phi$ but we must look for a corresponding expression concerning the term $cos\phi[\cot\theta * ((r\sin\theta)(d\phi))] = cos\phi[\cot\theta(r(dr \wedge d\theta))]$.

The desired expression is the following one:

$$cos\phi[d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{338}$$

Its respective Hodge Star is:

$$cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{339}$$

Using the relations in the expression above to deal with the Hodge Star * given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{340}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{341}$$

$p = 2$ stands for the R^3.Then we have:

$$*d[(\frac{1}{2})(r^2)\cot\theta d\theta] = (\frac{1}{2})(\cot\theta) * d(r^2 d\theta) + (\frac{1}{2})(r^2) * d(\cot\theta d\theta) + (\frac{1}{2})(r^2)\cot\theta * d(d\theta) \tag{342}$$

$$*d(r^2 d\theta) = d(r^2) \wedge d\theta + r^2 \wedge d(d\theta) = d(r^2) \wedge d\theta = 2rdr \wedge d\theta \tag{343}$$

$$*d(\cot\theta d\theta) = d\cot\theta \wedge d\theta + \cot\theta \wedge d(d\theta) = d\cot\theta \wedge d\theta = -\csc^2\theta d\theta \wedge d\theta = 0 \tag{344}$$

$$*d(d\theta) = 0 \tag{345}$$

$$*d[(\frac{1}{2})(r^2)\cot\theta d\theta] = (\frac{1}{2})(\cot\theta) * d(r^2 d\theta) = (\frac{1}{2})(\cot\theta)(2rdr \wedge d\theta) = (\cot\theta)(rdr \wedge d\theta) \tag{346}$$

And

$$cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] = cos\phi[(\cot\theta)(rdr \wedge d\theta)] = cos\phi[\cot\theta(e_\phi)] \tag{347}$$

Because due to the Canonical Basis of the Hodge Star:

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{348}$$

Then in the $3D$ spherical coordinates we have the following Hodge Star:

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{349}$$

Also in Appendix A we used the term $*d[f(r)r^2\sin^2\theta d\phi]$ corresponding to the term $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ because Natario also used it in pg 5 in [1].Now this term must be multiplied by $\sin\phi$.

From the Appendix A we have:

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{350}$$

Defining the Natario Vector as in pg 5 in [1] in polar coordinates with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) \tag{351}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) \tag{352}$$

We can get finally the latest expressions for the Natario Vector in polar coordinates nX also shown in pg 5 in [1]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{353}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{354}$$

We choose the polar coordinates Natario vectors $nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right)$ and

$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta$

But in $3D$ spherical coordinates we have:

$$\sin\phi(*d[f(r)r^2\sin^2\theta d\phi]) = \sin\phi(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) \tag{355}$$

Like the term $*d[f(r)r^2\sin^2\theta d\phi]$ is associated to the term $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ and now these terms must be multiplied by $\sin\phi$ we must find the corresponding term for $cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]]$.

The term we are looking for is the following one:

$$cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{356}$$

Solving the Hodge Star we have:

$$*d[(f(r))(r^2)\cot\theta d\theta] \tag{357}$$

$$(f(r))\cot\theta * d(r^2 d\theta) + (f(r))(r^2) * d(\cot\theta d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta)) + ((f(r))(r^2)\cot\theta) * d(d\theta) \tag{358}$$

As already seen before the terms $*d(\cot\theta d\theta) = 0$ and $*d(d\theta) = 0$.Then the Hodge Star becomes:

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta * d(r^2 d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta) \tag{359}$$

$$*d(r^2 d\theta) = d(r^2)\wedge d\theta + r^2\wedge d(d\theta) = d(r^2)\wedge d\theta = 2rdr\wedge d\theta \tag{360}$$

$$*d(f(r)d\theta) = d(f(r)\wedge d\theta + f(r)\wedge d(d\theta) = d(f(r)\wedge d\theta = f'(r)dr\wedge d\theta \tag{361}$$

Still with the Hodge Star:

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta * d(r^2 d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta) \tag{362}$$

$$*d(r^2 d\theta) = 2rdr \wedge d\theta \tag{363}$$

$$*d(f(r)d\theta) = f'(r)dr \wedge d\theta \tag{364}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta(2rdr \wedge d\theta) + (r^2)(\cot\theta)f'(r))(dr \wedge d\theta) \tag{365}$$

The Canonical Basis for the Hodge Star is:

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{366}$$

Then the Hodge Star now becomes:

$$*d[(f(r))(r^2)\cot\theta d\theta] = 2(f(r))\cot\theta(rdr \wedge d\theta) + (\cot\theta)rf'(r)(rdr \wedge d\theta) \tag{367}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))](rdr \wedge d\theta) \tag{368}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{369}$$

At last we are ready to present the new tridimensional $3D$ spherical warp drive vector.We already know that in the $3D$ spherical coordinates $d(r\sin\phi\cos\theta)$ we have the following Hodge Star:

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{370}$$

But as we already demonstrated in this section the Hodge Star above can be associated to the following one:

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{371}$$

With:

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{372}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{373}$$

Then our tridimensional $3D$ spherical Hodge Star can be given by:

$$\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi] \tag{374}$$

Natario defined two warp drive vectors in pg 5 in [1] as being:(see Appendix A)

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{375}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{376}$$

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{377}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{378}$$

We choose this one:$nX = vs(t)*d\left(f(r)r^2\sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta$.Then we have the original Natario warp drive vector in polar coordinates:

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = vs(t)[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] \tag{379}$$

Now and finally[9] we can present the final form of our new warp drive vector in tridimensional $3D$ spherical coordinates as being:

$$nX = vs(t)[\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]]] \tag{380}$$

$$nX = vs(t)\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + vs(t)cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{381}$$

$$nX = (\sin\phi)vs(t)[*d[f(r)r^2\sin^2\theta d\phi]] + (cos\phi)vs(t)[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{382}$$

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{383}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{384}$$

$$nX = vs(t)[\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{385}$$

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{386}$$

This is the final form of our new $3D$ spherical warp drive vector.Note that Natario in pg 4 in [1] defined the x-axis as the polar axis.if the motion occurs only in the x-axis in polar coordinates then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector in $3D$ spherical coordinates reduces to the original Natario warp drive vector in polar coordinates.

Only in a real $3D$ spherical coordinates motion our new warp drive vector accounts for a significant difference

[9] at last!!!we know that this section is being written in a tedious and monotonous style but we are writing this for beginners or introductory students eagerly needing these mathematical demonstrations QED Quod Erad Demonstratum in order to allow these students to more easily understand the whole process of the obtention of warp drive vectors

For our new $3D$ spherical coordinates warp drive vector

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{387}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{388}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{389}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{390}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{391}$$

16 Appendix G:mathematical demonstration of the warp drive vector $nX = vs * dx$ for a constant speed vs or for the first term $vs * dx$ from the warp drive vector $nX = vs * dx + x * dvs$(a variable speed) in a R^4 space basis-$3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{392}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim dt \wedge (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (dt \wedge d\varphi \wedge dr) \tag{393}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{394}$$

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{395}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{396}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{397}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho \sin\phi \cos\theta)$:

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\cos\theta) = sin\phi[d(r\cos\theta)] + (r\cos\theta)d(\sin\phi) \tag{398}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - sin\theta(rd\theta)] + cos\phi[(r\cos\theta)(d\phi)] \tag{399}$$

Applying the Hodge Star * to the terms above we will get the same results already shown in the Appendix F.As a matter of fact comparing the Appendices A and B the given final result is the same in both Appendices except for the fact that in Appendix A the Hodge Star is taken over R^3 and in Appendix B the Hodge Star is taken over R^4.

So the expressions for the Hodge Star of the term $d(r\sin\phi\cos\theta)$ covered in the last(and gigantic or enormous) Appendix F taken over R^3 that uses the terms

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{400}$$

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{401}$$

Will appear in identical form if we compute the Hodge Star for the same term

$$d(r\sin\phi\cos\theta)$$

in R^4.The only difference is the term in R^4

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \tag{402}$$

Different than its counterpart in R^3

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{403}$$

But since the term $f\wedge d\alpha = 0$ wether in R^4 or R^3 the final result of the Hodge Star is the same wether in R^4 or R^3 and we do not need to repeat here the tedious and monotonous piles of calculations shown in the (monster)Appendix F since the results are the same ones.

Our new $3D$ spherical coordinates warp drive vector in R^4 with constant speed $vs\ nX = vs * dx$ or for the first term $vs * dx$ of the new $3D$ spherical coordinates warp drive vector in R^4 with variable speed $vs\ nX = vs * dx + x * dvs$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{404}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{405}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{406}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{407}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{408}$$

17 Appendix H:mathematical demonstration of the new warp drive vector $nX = *(vsx) = vs * dx + x * dvs$ for a variable speed vs and a constant acceleration a in $3D$ Spherical Coordinates

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G in [8],[43] for an explanation about this statement)

In the Appendices F and G we gave the mathematical demonstration of the new warp drive vector nX in the R^3 and R^4 space basis in $3D$ spherical coordinates where the velocity vs is constant.Hence the complete expression of the Hodge star that generates the warp drive vector $nX = vs * dx$ for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{409}$$

$$*dx = *d(r \sin\phi \cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2 \sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{410}$$

$$\sin\phi[*d[f(r)r^2 \sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{411}$$

Our new $3D$ spherical coordinates warp drive vector in R^4 with constant speed vs $nX = vs * dx$ or for the first term $vs * dx$ of the new $3D$ spherical coordinates warp drive vector in R^4 with variable speed vs $nX = vs * dx + x * dvs$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{412}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{413}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{414}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{415}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{416}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{417}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[10],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[11] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\phi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\phi) \quad (418)$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dt = r^2 \sin\theta (dr \wedge d\theta \wedge d\phi) \quad (419)$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \quad (420)$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \quad (421)$$

[10]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[11]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{422}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{423}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\phi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\phi)] \tag{424}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{425}$$

The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \tag{426}$$

The term $*dx$ was obtained in the Appendices F and G as follows:

$$*dx = *d(r \sin\phi \cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2 \sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2) \cot\theta d\theta]] \tag{427}$$

$$\sin\phi[*d[f(r)r^2 \sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2) \cot\theta d\theta]] \tag{428}$$

$$\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi] \tag{429}$$

The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is now given by:

$$nX = vs(\sin\phi[2f(r)\ cos\theta e_r - [2f(r)+rf'(r)] \sin\theta e_\theta] + cos\phi[cot\theta[2(f(r))+(rf'(r))]e_\phi]) + x(2[atf'(r)e_r + f(r)ae_t]) \tag{430}$$

But remember that we are in $3D$ spherical coordinates in which $x = r \sin\phi \cos\theta$ and this leaves us with:

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{431}$$

$$A = vs(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{432}$$

$$B = (r \sin\phi \cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{433}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{434}$$

$$A = (2f(r)at)(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{435}$$

$$B = (r\sin\phi\cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{436}$$

Then we can start with a warp bubble initially at the rest using the warp drive vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed vs constant.The term B due to the acceleration $x * (dvs)$ now disappears the speed vs is no longer $vs = 2f(r)at$ and we are left again with the warp drive vector for constant speeds shown below:

$$nX = A \rightarrow A = vs * dx \tag{437}$$

$$A = vs(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{438}$$

Working some algebra with the new warp drive vector for variable velocities we get:[12]

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{439}$$

$$A = (2f(r)at)(\sin\phi[2f(r)\cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{440}$$

$$B = (r\sin\phi\cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{441}$$

$$A = (2f(r)at)\sin\phi[2f(r)\cos\theta e_r] - (2f(r)at)\sin\phi[2f(r)+rf'(r)]\sin\theta e_\theta + (2f(r)at)cos\phi[cot\theta[2(f(r))+(rf'(r))]e_\phi]] \tag{442}$$

$$B = 2(r\sin\phi\cos\theta)atf'(r)e_r + 2(r\sin\phi\cos\theta)f(r)ae_t \tag{443}$$

$$A = 4(f(r)^2at)(\sin\phi)(\cos\theta)e_r - (2f(r)at)[2f(r)+rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2(f(r))+(rf'(r))](cos\phi)(cot\theta)e_\phi \tag{444}$$

$$B = 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r + 2(rf(r)a))(\sin\phi)(\cos\theta)e_t \tag{445}$$

[12] again:we know that we are being tedious monotonous and repetitive but we are writing this mainly for beginners or introductory students

Rearranging the terms we have:

$$A = 4(f(r)^2 at)(\sin\phi)(\cos\theta)e_r - (2f(r)at)[2f(r)+rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2(f(r))+(rf'(r))](cos\phi)(cot\theta)e_\phi \quad (446)$$

$$A = (2f(r)at)\sin\phi[2f(r)\cos\theta e_r] - (2f(r)at)\sin\phi[2f(r)+rf'(r)]\sin\theta e_\theta + (2f(r)at)cos\phi[cot\theta[2(f(r))+(rf'(r))]e_\phi]] \quad (447)$$

$$(2f(r)at)[2f(r)](\sin\phi)(\cos\theta)e_r - (2f(r)at)[2f(r)+rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2f(r)+(rf'(r))](cos\phi)(cot\theta)e_\phi \quad (448)$$

$$B = 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r + 2(rf(r)a))(\sin\phi)(\cos\theta)e_t \quad (449)$$

Working the terms with e_r

$$(2f(r)at)\sin\phi[2f(r)\cos\theta e_r] + 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r \quad (450)$$

$$(2f(r)at)[2f(r)](\sin\phi)(\cos\theta)e_r + 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r \quad (451)$$

$$(2at)[2f(r)^2](\sin\phi)(\cos\theta)e_r + 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r \quad (452)$$

$$(2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta)e_r \quad (453)$$

At last we can give now the new warp drive vector for variable velocities in real $3D$ spherical coordinates using its respective contravariant shift vector components:[13]

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \quad (454)$$

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \quad (455)$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \quad (456)$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \quad (457)$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \quad (458)$$

[13]again:the section is extensive but a beginner needs all these QED Quod Erad Demonstratum mathematical demonstrations

Comparing the new warp drive vector for variable velocities in real $3D$ spherical coordinates with the Natario polar coordinates warp drive vector counterpart:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{459}$$

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{460}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{461}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{462}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{463}$$

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{464}$$

$$X^t = 2f(r)r(cos\theta)a \tag{465}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at(cos\theta) \tag{466}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)](\sin\theta) \tag{467}$$

Natario defined a motion in the $x - axis$ of polar coordinates (pgs 4 and 5 in [1]) then the polar plane $x - y$ makes an angle of 90 degrees with the $z - axis$ and since $\sin\phi = 1$ and $\cos\phi = 0$ it is easy to see that in this case the new warp drive vector for variable velocities in real $3D$ spherical coordinates reduces itself to the Natario polar coordinates warp drive vector counterpart:

The difference occurs only in a real tridimensional motion.

18 Appendix I:mathematical demonstration of the Natario warp drive equation for a constant speed vs in the original $3+1$ ADM Formalism according to MTW and Alcubierre

General Relativity describes the gravitational field in a fully covariant way using the geometrical line element of a given generic spacetime metric $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ where do not exists a clear difference between space and time.This generical form of the equations using tensor algebra is useful for differential geometry where we can handle the spacetime metric tensor $g_{\mu\nu}$ in a way that keeps both space and time integrated in the same mathematical entity (the metric tensor) and all the mathematical operations do not distinguish space from time under the context of tensor algebra handling mathematically space and time exactly in the same way.

However there are situations in which we need to recover the difference between space and time as for example the evolution in time of an astrophysical system given its initial conditions.

The $3+1$ ADM formalism allows ourselves to separate from the generic equation $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ of a given spacetime the 3 dimensions of space and the time dimension.(see pg 64 in [18])

Consider a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg 65) in [18].

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg 65 in [18])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg 66 in [18])
(see also fig 21.2 pg 506 in [17] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg 507 in [17])[14]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij}dx^i dx^j$ with $i,j = 1,2,3$ that measures the proper distance between two points inside each hypersurface
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

[14] we adopt the Alcubierre notation here

Combining the eqs (21.40),(21.42) and (21.44) pgs 507 and 508 in [17]
with the eqs (2.2.5) and (2.2.6) pg 67 in [18] using the signature $(-,+,+,+)$ we get the original equations of the $3+1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{468}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{469}$$

The components of the inverse metric are given by the matrix inverse :

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix} \tag{470}$$

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{471}$$

But since $dl^2 = \gamma_{ij} dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii} dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{472}$$

$$(dx^i + \beta^i dt)^2 = (dx^i)^2 + 2\beta^i dx^i dt + (\beta^i dt)^2 \tag{473}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii}(dx^i)^2 + 2\gamma_{ii}\beta^i dx^i dt + \gamma_{ii}(\beta^i dt)^2 \tag{474}$$

$$\beta_i = \gamma_{ii}\beta^i \tag{475}$$

$$\gamma_{ii}(\beta^i dt)^2 = \gamma_{ii}\beta^i\beta^i dt^2 = \beta_i\beta^i dt^2 \tag{476}$$

$$(dx^i)^2 = dx^i dx^i \tag{477}$$

$$\gamma_{ii}(dx^i + \beta^i dt)^2 = \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{478}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii} dx^i dx^i + 2\beta_i dx^i dt + \beta_i\beta^i dt^2 \tag{479}$$

$$ds^2 = (-\alpha^2 + \beta_i\beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{480}$$

Note that the expression above is exactly the eq (2.2.4) pg 67 in [18].It also appears as eq 1 pg 3 in [16].

With the original equations of the $3+1$ ADM formalism given below:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{481}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{482}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ii} - \frac{\beta^i\beta^i}{\alpha^2} \end{pmatrix} \tag{483}$$

and suppressing the lapse function making $\alpha = 1$ we have:

$$ds^2 = (-1 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii} dx^i dx^i \tag{484}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} -1 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ii} \end{pmatrix} \tag{485}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} -1 & \beta^i \\ \beta^i & \gamma^{ii} - \beta^i\beta^i \end{pmatrix} \tag{486}$$

changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-1 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{487}$$

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{488}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{489}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{490}$$

Remember that the equations given above corresponds to the generic warp drive metric given below:

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{491}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$(pg 2 in [1])

$$ds^2 = dt^2 - \sum_{i=1}^{3}(dx^i - X^i dt)^2 \tag{492}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = dt^2 - \sum_{i=1}^{3}\gamma_{ii}(dx^i - X^i dt)^2 \tag{493}$$

Comparing all these equations

$$ds^2 = (1 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{494}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{495}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & -\beta^i \\ -\beta^i & -\gamma^{ii} + \beta^i\beta^i \end{pmatrix} \tag{496}$$

$$ds^2 = dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{497}$$

With

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{498}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_iX^i$ with X^i as being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector.Hence we have:

$$ds^2 = (1 - X_iX^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{499}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_iX^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{500}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^iX^i \end{pmatrix} \tag{501}$$

Looking to the equation of the Natario warp drive vector nX with constant speed in polar coordinates(pg 2 and 5 in [1]):

$$nX = X^r e_r + X^\theta e_\theta \tag{502}$$

With the contravariant shift vector components X^r and X^θ given by:(see pg 5 in [1])(see also Appendices A and B for details)

$$X^r = 2v_s f(r)\cos\theta \tag{503}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{504}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Then the covariant shift vector components X_r and X_θ are given by:

$$X_i = \gamma_{ii}X^i \tag{505}$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = 2v_s f(r)\cos\theta = X^r \tag{506}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = -r^2 v_s(2f(r) + (r)f'(r))\sin\theta \tag{507}$$

The equations of the Natario warp drive in the $3+1$ ADM formalism are given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{508}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{509}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{i0} & g^{ii} \end{pmatrix} = \begin{pmatrix} 1 & X^i \\ X^i & -\gamma^{ii} + X^i X^i \end{pmatrix} \tag{510}$$

Then the equation of the Natario warp drive spacetime in polar coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{511}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{512}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{513}$$

Considering now the new Natario warp drive vector in $3D$ spherical coordinates with a constant speed vs nX given by::

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{514}$$

With the contravariant shift vector components X^r, X^θ and X^ϕ given by:
(see Appendices F and G for details)

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{515}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{516}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{517}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2\sin^2\theta$. Then the covariant shift vector components X_r,X_θ and X_ϕ are given by:

$$X_i = \gamma_{ii}X^i \tag{518}$$

$$X_r = \gamma_{rr}X^r = X_r = \gamma_{rr}X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = X^r \tag{519}$$

$$X_\theta = \gamma_{\theta\theta}X^\theta = r^2 X^\theta = -r^2 vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{520}$$

$$X_\phi = \gamma_{\phi\phi}X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta[vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{521}$$

Then the equation of the Natario warp drive spacetime in $3D$ spherical coordinates with a constant speed vs in the original $3+1$ ADM formalism is given by:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{522}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{523}$$

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \tag{524}$$

19 Appendix J:mathematical demonstration of the Natario warp drive equation for a variable speed vs and a constant acceleration a in the original $3+1$ ADM Formalism according to MTW and Alcubierre

In the Appendix C we defined a variable bubble velocity vs due to a constant acceleration a as follows:

$$vs = 2f(r)at \tag{525}$$

And we obtained the Natario vector nX for a Natario warp drive in polar coordinates with variable velocities defined as follows:

$$nX = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{526}$$

$$nX = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{527}$$

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{528}$$

Remember that $x = rscos\theta$(see pg 5 in [1]). Considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small r(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]) we can see that the Natario vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * d(vs)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1]).Working with some algebra we got:

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{529}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector in polar coordinates with variable velocities are defined by:(see Appendices A,B and C)

$$X^t = 2f(r)rcos\theta a \tag{530}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{531}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{532}$$

Consider again a 3 dimensional hypersurface Σ_1 in an initial time $t1$ that evolves to a hypersurface Σ_2 in a later time $t2$ and hence evolves again to a hypersurface Σ_3 in an even later time $t3$ according to fig 2.1 pg 65 in [18].Considering now an accelerating warp drive then the amount of time needed for the evolution of the hypersurface from Σ_2 to Σ_3 occurring in the lapse of time $t3$ is smaller than the amount of time needed for the evolution of the hypersurface from Σ_1 to Σ_2 occurring in the lapse of time $t2$ because due to the constant acceleration the speed of the warp bubble is growing from $t2$ to $t3$ and in the lapse of time $t3$ the warp drive is faster than in the lapse of time $t2$.

The hypersurface Σ_2 is considered and adjacent hypersurface with respect to the hypersurface Σ_1 that evolved in a differential amount of time dt from the hypersurface Σ_1 with respect to the initial time $t1$. Then both hypersurfeces Σ_1 and Σ_2 are the same hypersurface Σ in two different moments of time Σ_t and Σ_{t+dt}.(see bottom of pg 65 in [18])

The geometry of the spacetime region contained between these hypersurfaces Σ_t and Σ_{t+dt} can be determined from 3 basic ingredients:(see fig 2.2 pg 66 in [18])
(see also fig 21.2 pg 506 in [17] where $dx^i + \beta^i dt$ appears to illustrate the equation 21.40 $g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ at pg 507 in [17])[15]

- 1)-the 3 dimensional metric $dl^2 = \gamma_{ij} dx^i dx^j$ with $i, j = 1, 2, 3$ that measures the proper distance between two points inside each hypersurface
- 2)-the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} measured by observers moving in a trajectory normal to the hypersurfaces(Eulerian obsxervers) $d\tau = \alpha dt$ where α is known as the lapse function.Note that in a warp drive of constant velocity the elapsed times $t2$ and $t3$ are equal because the velocity do not varies between $t2$ and $t3$. Hence the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} is always the same as time goes by but for an accelerating warp drive the elapsed time $t3$ is smaller than the elapsed time $t2$ so the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity generated by a constant acceleration.
- 3)-the relative velocity β^i between Eulerian observers and the lines of constant spatial coordinates $(dx^i + \beta^i dt)$.β^i is known as the shift vector.

Combining the eqs (21.40),(21.42) and (21.44) pgs 507 and 508 in [17]
with the eqs (2.2.5) and (2.2.6) pg 67 in [18] using the signature $(-, +, +, +)$ we get the original equations of the $3 + 1$ ADM formalism given by the following expressions:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{533}$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{534}$$

[15] we adopt the Alcubierre notation here

The spacetime metric in $3+1$ is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{535}$$

Remember that in an accelerating warp drive the lapse of proper time $d\tau$ between both hypersurfaces Σ_t and Σ_{t+dt} becomes shorter and shorter as times goes by due to an ever growing velocity that makes the warp drive moves faster and faster being this velocity generated by the extra terms in the Natario vector.These extra terms must be inserted inside the spacetime metric in $3+1$ using a mathematical structure similar to the one of the lapse function as follows:

$$\alpha^2 = \gamma_{tt}(1+\beta^t)^2 = \gamma_{tt}(1+2\beta^t+\beta^t\beta^t) = (\gamma_{tt}+2\gamma_{tt}\beta^t+\gamma_{tt}\beta^t\beta^t) \tag{536}$$

$$\beta_t = \gamma_{tt}\beta^t \tag{537}$$

Remember that here we are working with geometrized units in which $c=1$ so $\gamma_{tt}=1$

$$\alpha^2 = (1+2\beta_t+\beta_t\beta^t) \tag{538}$$

The spacetime metric in $3+1$ is then given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{539}$$

Since $dl^2 = \gamma_{ij}dx^i dx^j$ must be a diagonalized metric then $dl^2 = \gamma_{ii}dx^i dx^i$ and we have:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{540}$$

$$ds^2 = -\gamma_{tt}(1+\beta^t)^2 dt^2 + \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{541}$$

From the Appendix I we can write the $3+1$ metric as:

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ii}dx^i dx^i \tag{542}$$

Note that the expression above is exactly the eq (2.2.4) pg 67 in [18].It also appears as eq 1 pg 3 in [16].Changing the signature from $(-,+,+,+)$ to signature $(+,-,-,-)$ we have:

$$ds^2 = -(-\alpha^2 + \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{543}$$

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{544}$$

$$ds^2 = (1+2\beta_t+\beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii}dx^i dx^i \tag{545}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{546}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1+2\beta_t+\beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{547}$$

The warp drive spacetime according to Natario is defined by the following equation but we changed the metric signature from $(-,+,+,+)$ to $(+,-,-,-)$ and we modified the equation to insert the terms due to the lapse function α^2.(pg 2 in [1])

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} (dx^i - X^i dt)^2 \tag{548}$$

The Natario equation given above is valid only in cartezian coordinates.For a generic coordinates system we must employ the equation that obeys the $3+1$ ADM formalism:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{549}$$

Comparing all these equations

$$ds^2 = (\alpha^2 - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{550}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{551}$$

$$ds^2 = \alpha^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{552}$$

$$\alpha^2 = \gamma_{tt}(1 + \beta^t)^2 \tag{553}$$

$$\alpha^2 = (1 + 2\beta_t + \beta_t\beta^t) \tag{554}$$

$$ds^2 = \gamma_{tt}(1 + \beta^t)^2 dt^2 - \gamma_{ii}(dx^i + \beta^i dt)^2 \tag{555}$$

$$ds^2 = (1 + 2\beta_t + \beta_t\beta^t - \beta_i\beta^i)dt^2 - 2\beta_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{556}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 + 2\beta_t + \beta_t\beta^t - \beta_i\beta^i & -\beta_i \\ -\beta_i & -\gamma_{ii} \end{pmatrix} \tag{557}$$

With these

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{558}$$

$$ds^2 = \gamma_{tt}(1 - X^t)^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{559}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^tX^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^tX^t) = (1 - 2X_t + X_tX^t) \tag{560}$$

The generic equations for the Natario warp drive that obeys the $3+1$ ADM formalism with variable velocities are given below:

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{561}$$

$$ds^2 = \gamma_{tt}(1 - X^t)^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{562}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt}X^t + \gamma_{tt}X^t X^t) = (1 - 2X_t + X_t X^t) \tag{563}$$

We can see that $\beta^i = -X^i$,$\beta_i = -X_i$ and $\beta_i\beta^i = X_i X^i$ with X^i being the contravariant form of the Natario shift vector and X_i being the covariant form of the Natario shift vector both for the spatial components of the Natario vector.In the same way we can see that $\beta^t = -X^t$,$\beta_t = -X_t$ and $\beta_t\beta^t = X_t X^t$ with X^t being the contravariant form of the Natario shift vector and X_t being the covariant form of the Natario shift vector for the time component of the Natario vector.Hence we have the equations of the generic Natario warp drive in the $3+1$ ADM formalism with variable velocities:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{564}$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{565}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{566}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{567}$$

Looking to the equation of the Natario vector in polar coordinates nX with variable velocities:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{568}$$

The contravariant shift vector components X^t,X^r and X^θ of the Natario vector are defined by:(see Appendices A,B and C)

$$X^t = 2f(r) r cos\theta a \tag{569}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at cos\theta \tag{570}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(rs)] \sin\theta \tag{571}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2$ with $\gamma_{rr} = 1$ and $\gamma_{\theta\theta} = r^2$. Remember also that $\gamma_{tt} = 1$.Then the covariant shift vector components X_t,X_r and X_θ are given by:

$$X_t = \gamma_{tt} X^t \tag{572}$$

$$X_i = \gamma_{ii} X^i \tag{573}$$

$$X_t = \gamma_{tt} X^t = 2f(r) r cos\theta a \tag{574}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = X^r = X^r = 2[2f(r)^2 + rf'(r)] atcos\theta \tag{575}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -2f(r)at[2f(r) + rf'(r)]r^2 \sin\theta \tag{576}$$

The equations of the generic Natario warp drive in the $3+1$ ADM formalism with variable velocities are given by:

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{577}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{578}$$

Then the equation of the Natario warp drive spacetime for a variable velocity and a constant acceleration in the original $3+1$ ADM formalism in polar coordinates is given by:

$$ds^2 = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{579}$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_{rs} dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{580}$$

With

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt} X^t + \gamma_{tt} X^t X^t) = (1 - 2X_t + X_t X^t) \tag{581}$$

having the behavior of a lapse function.

We have:

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr dt + X_\theta d\theta dt) - dr^2 - r^2 d\theta^2 \tag{582}$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{583}$$

Back again to the equations of the generic Natario warp drive in the 3+1 ADM formalism with variable velocities:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{584}$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{585}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} \alpha^2 - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{586}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{i0} & g_{ii} \end{pmatrix} = \begin{pmatrix} 1 - 2X_t + X_t X^t - X_i X^i & X_i \\ X_i & -\gamma_{ii} \end{pmatrix} \tag{587}$$

With

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt} X^t + \gamma_{tt} X^t X^t) = (1 - 2X_t + X_t X^t) \tag{588}$$

having the behavior of a lapse function.

The Natario warp drive vector for variable velocities in real $3D$ spherical coordinates and its respective contravariant shift vector components are given by:(see Appendices F,G and H)

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{589}$$

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{590}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{591}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{592}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{593}$$

But remember that $dl^2 = \gamma_{ii}dx^i dx^i = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ with $\gamma_{rr} = 1$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2 \sin^2\theta$.Remember also that $\gamma_{tt} = 1$. Then the covariant shift vector components X_r,X_θ and X_ϕ are given by:

$$X_i = \gamma_{ii} X^i \tag{594}$$

$$X_t = \gamma_{tt} X^t \tag{595}$$

$$X_t = \gamma_{tt} X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) = X^t \tag{596}$$

$$X_r = \gamma_{rr} X^r = X_r = \gamma_{rr} X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = X^r \tag{597}$$

$$X_\theta = \gamma_{\theta\theta} X^\theta = r^2 X^\theta = -r^2(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{598}$$

$$X_\phi = \gamma_{\phi\phi} X^\phi = r^2 sin^2\theta X^\phi = r^2 sin^2\theta(2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{599}$$

From the equations of the generic Natario warp drive in the $3+1$ ADM formalism with variable velocities:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{600}$$

$$ds^2 = (1 - 2X_t + X_t X^t - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{601}$$

$$\alpha^2 = \gamma_{tt}(1 - X^t)^2 = \gamma_{tt}(1 - 2X^t + X^t X^t) = (\gamma_{tt} - 2\gamma_{tt} X^t + \gamma_{tt} X^t X^t) = (1 - 2X_t + X_t X^t) \tag{602}$$

We have the Natario warp drive equation for variable velocities in real $3D$ spherical coordinates

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{603}$$

$$ds^2 = (\alpha^2 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{604}$$

$$ds^2 = ((1-2X_t+X_t X^t) - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr dt + X_\theta d\theta dt + X_\phi d\phi dt) - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{605}$$

$$ds^2 = ((1-2X_t+X_t X^t) - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{606}$$

20 Appendix K:Generic quadratic forms in the $3+1$ ADM spacetime without the lapse function.

The Natario warp drive equations with signature $(+,-,-,-)$ that obeys the original $3+1$ ADM formalism are given below:

in Polar Coordinates:(see Appendix I).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta)dt^2 + 2(X_r dr + X_\theta d\theta)dt - dr^2 - r^2 d\theta^2 \tag{607}$$

in $3D$ Spherical Coordinates:(see also Appendix I).

$$ds^2 = (1 - X_r X^r - X_\theta X^\theta - X_\phi X^\phi)dt^2 + 2(X_r dr + X_\theta d\theta + X_\phi d\phi)dt - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{608}$$

Using quadratic forms and the null-like geodesics $ds^2 = 0$ of General Relativity,Horizons can be easily computed for the dimensionally reduced $1+1$ spacetime versions of these equations because only the quadratic form dr^2 exists but in the $3+1$ spacetime we have the presence of 3 quadratic forms respectively dr^2,$r^2 d\theta^2$ and $r^2\sin^2\theta d\phi^2$.Algebraic solutions for the null-like geodesics $ds^2 = 0$ of General Relativity of the $3+1$ equations above are extremely difficult due to the presence of these 3 quadratic forms considering solutions for each quadratic form dr^2 or $r^2 d\theta^2$ or $r^2\sin^2\theta d\phi^2$ isolated.

The best effort to solve the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above is to find out a solution that encompasses all the 3 quadratic forms dr^2 and $r^2 d\theta^2$ and $r^2\sin^2\theta d\phi^2$ grouped together.

We will demonstrate all the required mathematics step by step.

Back to the $3+1$ ADM formalism compact generic equation given below:(see Appendix I)

$$ds^2 = dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{609}$$

Expanding the equation above we have:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{610}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{611}$$

Dividing by dt^2 we have:

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{dx^i dx^i}{dt^2} \tag{612}$$

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{613}$$

$$0 = (1 - X_i X^i) + 2X_i \frac{dx^i}{dt} - \gamma_{ii}(\frac{dx^i}{dt})^2 \tag{614}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{615}$$

We have now a generic quadratic form in the term U^i:

$$0 = (1 - X_i X^i) + 2X_i U^i - \gamma_{ii}(U^i)^2 \tag{616}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2X_i - (1 - X_i X^i) = 0 \tag{617}$$

$$\gamma_{ii}(U^i)^2 - 2X_i + (X_i X^i - 1) = 0 \tag{618}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2X_i \pm \sqrt{[-2X_i]^2 - 4[\gamma_{ii}(X_i X^i - 1)]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[\gamma_{ii}(X_i X^i] + 4[\gamma_{ii}]}}{2\gamma_{ii}} \tag{619}$$

But since:

$$X_i = \gamma_{ii} X^i \tag{620}$$

We have:

$$U^i = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[X_i]^2 + 4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{+4[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm 2\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} \tag{621}$$

$$U^i = \frac{2X_i \pm 2\sqrt{\gamma_{ii}}}{2\gamma_{ii}} = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{622}$$

At last we have the final solution of this generic quadratic form in the term U^i given by:

$$U^i = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{623}$$

But this expression actually means:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{\sum_{i=1}^{3} X_i \pm \sum_{i=1}^{3} \sqrt{\gamma_{ii}}}{\sum_{i=1}^{3} \gamma_{ii}} = \frac{\sum_{i=1}^{3} X_i \pm \sqrt{\sum_{i=1}^{3} \gamma_{ii}}}{\sum_{i=1}^{3} \gamma_{ii}} \tag{624}$$

The subscript γ_{ii} is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [41]).Then $\sum_{i=1}^{3} \sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3} \gamma_{ii}}$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 \pm \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{625}$$

The generic quadratic form in the term U^i for the null-like geodesics $ds^2 = 0$ is given by:

$$U^i = \frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{626}$$

Expanding the terms in the expression above we have:

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i \pm \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 \pm \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{627}$$

The line element in the $3+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{628}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1 - X_2 X^2 - X_3 X^3)dt^2 + 2(X_1 dx^1 + X_2 dx^2 + X_3 dx^3)dt - \gamma_{11} dx^1 dx^1 - \gamma_{22} dx^2 dx^2 - \gamma_{33} dx^3 dx^3 \tag{629}$$

$$ds^2 = (1 - X_1 X^1 - X_2 X^2 - X_3 X^3)dt^2 + 2(X_1 dx^1 + X_2 dx^2 + X_3 dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{630}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{631}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{632}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{633}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 + U^3 = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{634}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{635}$$

$$\sum_{i=1}^{3}[U^i] = \sum_{i=1}^{3}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{636}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

The line element in the $2+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{637}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11} dx^1 dx^1 - \gamma_{22} dx^2 dx^2 \tag{638}$$

$$ds^2 = (1 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11} (dx^1)^2 - \gamma_{22} (dx^2)^2 \tag{639}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{640}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{641}$$

$$\sum_{i=1}^{2} [U^i] = \sum_{i=1}^{2} [\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{642}$$

$$\sum_{i=1}^{2} [U^i] = \sum_{i=1}^{2} [\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 + U^2 = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{643}$$

$$\sum_{i=1}^{2} [U^i] = \sum_{i=1}^{2} [\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{644}$$

$$\sum_{i=1}^{2} [U^i] = \sum_{i=1}^{2} [\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{645}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $2+1$ spacetime equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

The line element in the $1+1$ ADM spacetime without the lapse function is:

$$ds^2 = (1 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{646}$$

Expanding the terms in the expression above we have:

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11} dx^1 dx^1 \tag{647}$$

$$ds^2 = (1 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{648}$$

The generic quadratic form in the term $U^i = \frac{dx^i}{dt}$ for the null-like geodesics $ds^2 = 0$ have two roots given by:

$$U^i = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{649}$$

$$U^i = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} = \frac{dx^i}{dt} = \frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{650}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{651}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = U^1 = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{652}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i + \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 + \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{653}$$

$$\sum_{i=1}^{1}[U^i] = \sum_{i=1}^{1}[\frac{X_i - \sqrt{\gamma_{ii}}}{\gamma_{ii}}] = \frac{dx^1}{dt} = \frac{X_1 - \sqrt{\gamma_{11}}}{\gamma_{11}} \tag{654}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $1+1$ spacetime equations given above with the solution that encompasses the single quadratic forms $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

21 Appendix L:Generic quadratic forms in the $3+1$ ADM spacetime with the lapse function.

This Appendix is a continuation of the Appendix K but this time we consider the lapse function.We provide all the step by step mathematical calculations.

Back to the $3+1$ ADM formalism compact generic equation with the lapse function given below:(see Appendix J)

$$ds^2 = \alpha^2 dt^2 - \sum_{i=1}^{3} \gamma_{ii}(dx^i - X^i dt)^2 \tag{655}$$

Expanding the equation above we have:

$$ds^2 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{656}$$

The null-like geodesics $ds^2 = 0$ is:

$$0 = (\alpha^2 - X_i X^i)dt^2 + 2X_i dx^i dt - \gamma_{ii} dx^i dx^i \tag{657}$$

Dividing by dt^2 we have:

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{dx^i dx^i}{dt^2} \tag{658}$$

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i dt}{dt^2} - \gamma_{ii} \frac{(dx^i)^2}{dt^2} \tag{659}$$

$$0 = (\alpha^2 - X_i X^i) + 2X_i \frac{dx^i}{dt} - \gamma_{ii} (\frac{dx^i}{dt})^2 \tag{660}$$

Introducing the term U^i as being:

$$U^i = \frac{dx^i}{dt} \tag{661}$$

We have now a generic quadratic form in the term U^i:

$$0 = (\alpha^2 - X_i X^i) + 2X_i U^i - \gamma_{ii}(U^i)^2 \tag{662}$$

Rearranging the terms in this quadratic form we have:

$$\gamma_{ii}(U^i)^2 - 2X_i - (\alpha^2 - X_i X^i) = 0 \tag{663}$$

$$\gamma_{ii}(U^i)^2 - 2X_i + (X_i X^i - \alpha^2) = 0 \tag{664}$$

The solution of this generic quadratic form in the term U^i is given by:

$$U^i = \frac{2X_i \pm \sqrt{[-2X_i]^2 - 4[\gamma_{ii}(X_i X^i - \alpha^2)]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[\gamma_{ii}(X_i X^i] + 4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} \tag{665}$$

But since:

$$X_i = \gamma_{ii} X^i \tag{666}$$

We have:

$$U^i = \frac{2X_i \pm \sqrt{4[X_i]^2 - 4[X_i]^2 + 4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm \sqrt{+4\alpha^2[\gamma_{ii}]}}{2\gamma_{ii}} = \frac{2X_i \pm 2\alpha\sqrt{[\gamma_{ii}]}}{2\gamma_{ii}} \tag{667}$$

$$U^i = \frac{2X_i \pm 2\alpha\sqrt{\gamma_{ii}}}{2\gamma_{ii}} = \frac{X_i \pm \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{668}$$

At last we have the final solution of this generic quadratic form for the null-like geodesics $ds^2 = 0$ in the term U^i given by:

$$U^i = \frac{X_i \pm \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{669}$$

The solution have two roots:

$$U^i = \frac{X_i + \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{670}$$

$$U^i = \frac{X_i - \alpha\sqrt{\gamma_{ii}}}{\gamma_{ii}} \tag{671}$$

The subscript γ_{ii} is inside the root $\sqrt{\gamma_{ii}}$ so the sum must be taken also inside the root.(see pg 5,pg 227 section 7.3 and pg 241 section 7.10 in [41]).Then $\sum_{i=1}^{3} \sqrt{\gamma_{ii}}$ actually must be $\sqrt{\sum_{i=1}^{3} \gamma_{ii}}$

Adapting the results from the previous section we have for the equation of the $3+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1X^1 - X_2X^2 - X_3X^3)dt^2 + 2(X_1dx^1 + X_2dx^2 + X_3dx^3)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 - \gamma_{33}(dx^3)^2 \tag{672}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 + \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{673}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt} = \frac{X_1 + X_2 + X_3 - \alpha\sqrt{\gamma_{11} + \gamma_{22} + \gamma_{33}}}{\gamma_{11} + \gamma_{22} + \gamma_{33}} \tag{674}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $3+1$ spacetime equations given above with the solution that encompasses all the 3 quadratic forms $(dx^1)^2$,$(dx^2)^2$ and $(dx^3)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt} + \frac{dx^3}{dt}$.

Adapting the results from the previous section we have for the equation of the $2+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1 X^1 - X_2 X^2)dt^2 + 2(X_1 dx^1 + X_2 dx^2)dt - \gamma_{11}(dx^1)^2 - \gamma_{22}(dx^2)^2 \tag{675}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 + \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{676}$$

$$\frac{dx^1}{dt} + \frac{dx^2}{dt} = \frac{X_1 + X_2 - \alpha\sqrt{\gamma_{11} + \gamma_{22}}}{\gamma_{11} + \gamma_{22}} \tag{677}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $2+1$ spacetime equations given above with the solution that encompasses all the 2 quadratic forms $(dx^1)^2$ and $(dx^2)^2$ grouped together.The solution is given in function of $\frac{dx^1}{dt} + \frac{dx^2}{dt}$.

Adapting the results from the previous section we have for the equation of the $1+1$ spacetime in the ADM formalism:

$$ds^2 = (\alpha^2 - X_1 X^1)dt^2 + 2(X_1 dx^1)dt - \gamma_{11}(dx^1)^2 \tag{678}$$

$$\frac{dx^1}{dt} = \frac{X_1 + \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{679}$$

$$\frac{dx^1}{dt} = \frac{X_1 - \alpha\sqrt{\gamma_{11}}}{\gamma_{11}} \tag{680}$$

We solved the null-like geodesics $ds^2 = 0$ in the case of the $1+1$ spacetime equations given above with the solution that encompasses the single quadratic form $(dx^1)^2$.The solution is given in function of $\frac{dx^1}{dt}$.

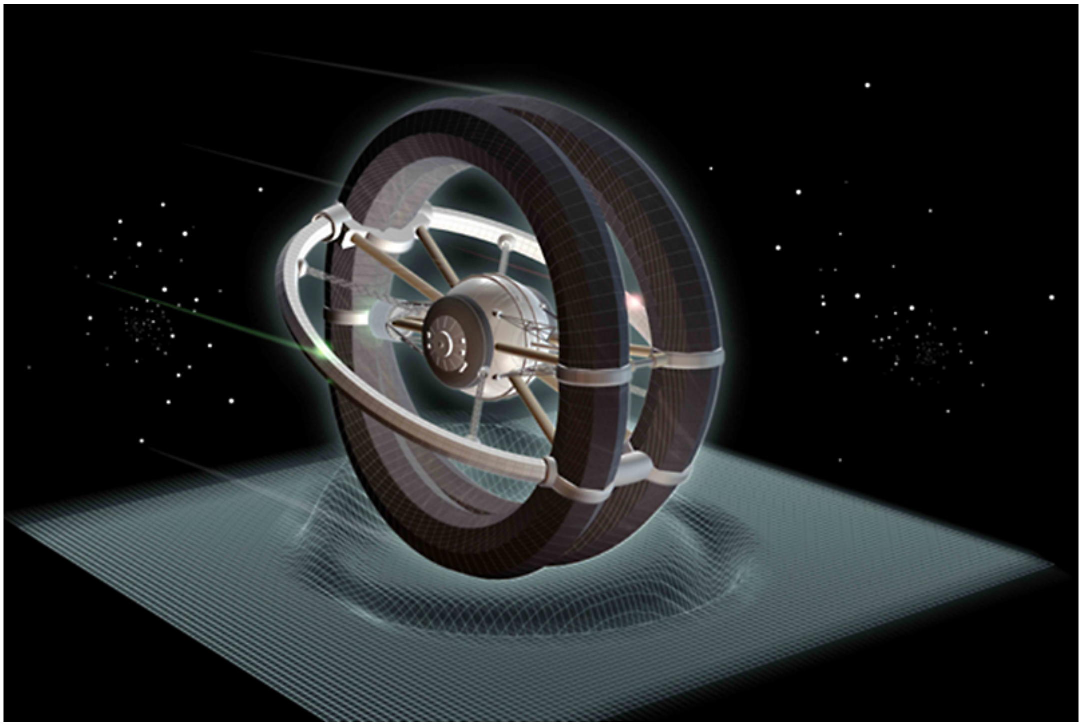

Figure 3: Artistic representation of the Natario warp drive .Note in the bottom of the figure the Alcubierre expansion of the normal volume elements .(Source:Internet)

22 Appendix M:Artistic Presentation of the Natario warp drive-Polar Coordinates

According to the geometry of the Natario warp drive the spacetime contraction in one direction(radial) is balanced by the spacetime expansion in the remaining direction(perpendicular).(pg 5 in [1]).

The expansion of the normal volume elements in the Natario warp drive is given by the following expressions(pg 5 in [1]).

$$K_{rr} = \frac{\partial X^r}{\partial r} = -2v_s f'(r)\cos\theta \tag{681}$$

$$K_{\theta\theta} = \frac{1}{r}\frac{\partial X^\theta}{\partial\theta} + \frac{X^r}{r} = v_s f'(r)\cos\theta; \tag{682}$$

$$K_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial X^\varphi}{\partial\varphi} + \frac{X^r}{r} + \frac{X^\theta\cot\theta}{r} = v_s f'(r)\cos\theta \tag{683}$$

$$\theta = K_{rr} + K_{\theta\theta} + K_{\varphi\varphi} = 0 \tag{684}$$

If we expand the radial direction the perpendicular direction contracts to keep the expansion of the normal volume elements equal to zero.This figure is a pedagogical example of the graphical presentarion of the Natario warp drive.

The "bars" in the figure were included to illustrate how the expansion in one direction can be counterbalanced by the contraction in the other directions.These "bars" keeps the expansion of the normal volume elements in the Natario warp drive equal to zero.

Note also that the graphical presentation of the Alcubierre warp drive expansion of the normal volume elements according to fig 1 pg 10 in [16] is also included

Note also that the energy density in the Natario warp drive $3+1$ spacetime being given by the following expressions(pg 5 in [1]):

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(f'(r))^2 \cos^2\theta + \left(f'(r) + \frac{r}{2} f''(r) \right)^2 \sin^2\theta \right]. \quad (685)$$

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{df(r)}{dr})^2 \cos^2\theta + \left(\frac{df(r)}{dr} + \frac{r}{2} \frac{d^2 f(r)}{dr^2} \right)^2 \sin^2\theta \right]. \quad (686)$$

Is being distributed around all the space involving the ship(above the ship $\sin\theta = 1$ and $\cos\theta = 0$ while in front of the ship $\sin\theta = 0$ and $\cos\theta = 1$).The negative energy in front of the ship "deflect" photons or other particles so these will not reach the ship inside the bubble.The illustrated "bars" are the obstacles that deflects photons or incoming particles from outside the bubble never allowing these to reach the interior of the bubble.[16]

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [19])

-)-Energy directly above the ship($y - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[\left(\frac{df(r)}{dr} + \frac{r}{2} \frac{d^2 f(r)}{dr^2} \right)^2 \sin^2\theta \right]. \quad (687)$$

-)-Energy directly in front of the ship($x - axis$)

$$\rho = -\frac{1}{16\pi} K_{ij} K^{ij} = -\frac{v_s^2}{8\pi} \left[3(\frac{df(r)}{dr})^2 \cos^2\theta \right]. \quad (688)$$

[16] See also Appendix N

The distribution of energy presented in this Appendix is valid only for the Natario warp drive vector in Polar Coordinates without the lapse function.For the case of the lapse function see Section 3 and Appendix I in [10].

Also the Zero-Expansion behavior is valid only in Polar Coordinates and do not occurs in $3D$ Spherical Coordinates.See [9].

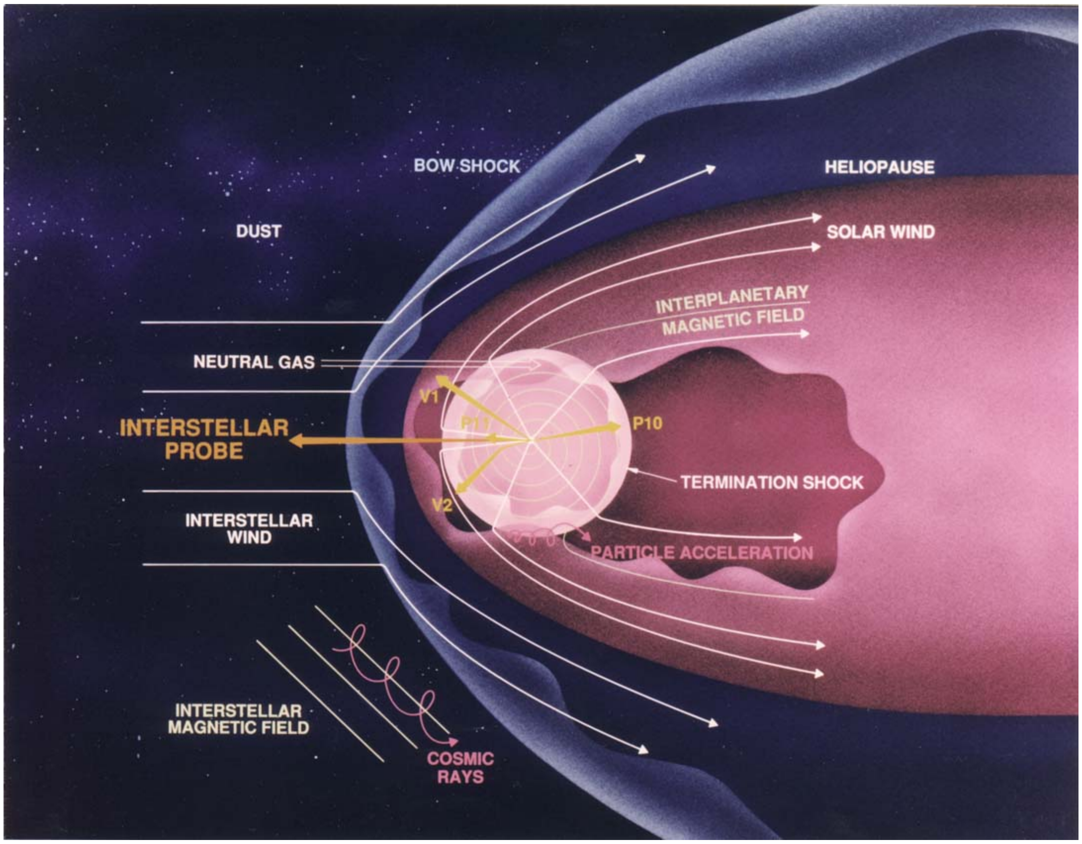

Figure 4: Artistic representation of a Natario warp wrive in a real superluminal space travel .Note the negative energy in front of the ship deflecting incoming hazardous interstellar matter(brown arrows).(Source:Internet)

23 Appendix N:Artistic Presentation of a Natario warp drive in a real faster than light interstellar spaceflight

Above is being presented the artistic presentation of a Natario warp drive in a real interstellar superluminal travel.The "ball" or the spherical shape is the Natario warp bubble with the negative energy surrounding the ship in all directions and mainly protecting the front of the bubble.[17]

The brown arrows in the front of the Natario bubble are a graphical presentation of the negative energy in front of the ship deflecting interstellar dust,neutral gases,hydrogen atoms,interstellar wind photons etc.[18]

The spaceship is at the rest and in complete safety inside the Natario bubble.

[17] See Appendix M

[18] see Appendices P and Q for the composition of the Interstellar Medium IM)

In order to allow to the negative energy density of the Natario warp drive the deflection of incoming hazardous particles from the Interstellar Medium(IM) the Natario warp drive energy density must be heavier or denser when compared to the IM density.

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [19])

24 Appendix O:The Natario warp drive negative energy density in Cartezian coordinates

The negative energy density according to Natario in Polar Coordinates is given by(see pg 5 in [1])[19]:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2\cos^2\theta + \left(f'(rs) + \frac{r}{2}f''(rs)\right)^2\sin^2\theta\right] \tag{689}$$

In the bottom of pg 4 in [1] Natario defined the x-axis as the polar axis.In the top of page 5 we can see that $x = r\cos(\theta)$ implying in $\cos(\theta) = \frac{x}{r}$ and in $\sin(\theta) = \frac{y}{r}$

Rewriting the Natario negative energy density in cartezian coordinates we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2(\frac{x}{rs})^2 + \left(f'(rs) + \frac{r}{2}f''(rs)\right)^2(\frac{y}{rs})^2\right] \tag{690}$$

Considering motion in the equatorial plane of the Natario warp bubble (x-axis only) then $[y^2 + z^2] = 0$ and $r^2 = [(x - xs)^2]$ and making $xs = 0$ the center of the bubble as the origin of the coordinate frame for the motion of the Eulerian observer then $r^2 = x^2$ because in the equatorial plane $y = z = 0$.

Rewriting the Natario negative energy density in cartezian coordinates in the equatorial plane we should expect for:

$$\rho = T_{\mu\nu}u^{\mu}u^{\nu} = -\frac{1}{16\pi}K_{ij}K^{ij} = -\frac{v_s^2}{8\pi}\left[3(f'(rs))^2\right] \tag{691}$$

The negative energy density have repulsive gravitational behavior and is distributed along all the bubble volume even in the equatorial plane so any hazardous incoming objects in front of the bubble (Doppler blueshifted photons or space dust or debris) would then be deflected by the repulsive behavior of the negative energy in front of the bubble never reaching the bubble walls(see pg 116 in [19])

The distribution of energy presented in this Appendix is valid only for the Natario warp drive vector in Polar Coordinates without the lapse function.For the case of the lapse function see Section 3 and Appendix I in [10].

But for the case of the warp drive vector in $3D$ Spherical Coordinates equations we can say nothing about the negative energy density at first sight and we need to compute "all-the-way-round" the Christoffel symbols Riemann and Ricci tensors and the Ricci scalar in order to obtain the Einstein tensor and hence the stress-energy-momentum tensor in a long and tedious process of tensor analysis liable of occurrence of calculation errors.

Or we can use computers with programs like $Maple$ or $Mathematica$ (see pg 342 in [17], pg 276 in [30],pgs $454, 457, 560$ in [31] pg 98 in [32],pg 178 in [33]).

Appendix C pgs $551 - 555$ in [31] shows how to calculate everything until the Einstein tensor from the basic input of the covariant components of the $3 + 1$ spacetime metric using $Mathematica$.[20]

[19] $f(r)$ is the Natario shape function.Equation written in the Geometrized System of Units $c = G = 1$

[20] Unfortunately we dont have access to anyone of these programs so we have our hands "tied up"

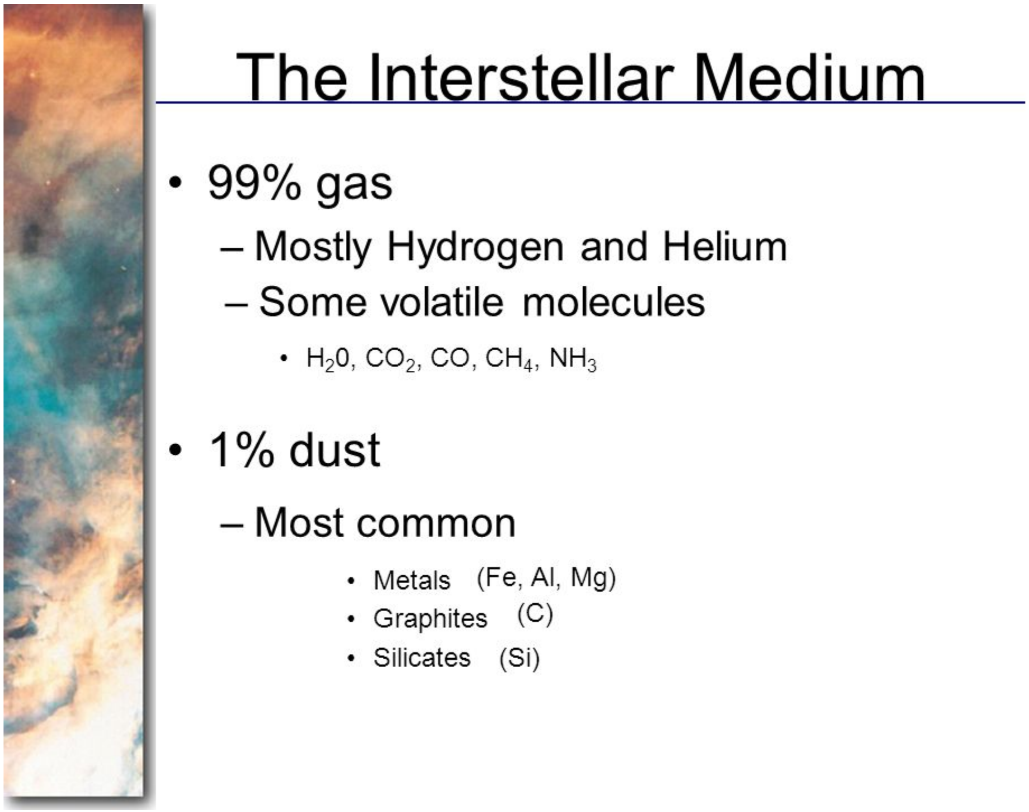

Figure 5: Composition of the Interstellar Medium IM(Source:Internet)

25 Appendix P:Composition of the Interstellar Medium IM

The problem of collisions between a warp drive spaceship moving at superluminal velocity and the potentially dangerous particles from the Interstellar Medium IM is not new.

It was first noticed in 1999 in the work of Chad Clark,Will Hiscock and Shane Larson(see [24]).Later on in 2010 it appeared again in the work of Carlos Barcelo,Stefano Finazzi and Stefano Liberatti(see [25]).In 2012 the same problem of collisions against hazardous IM particles appeared in the work of Brendan McMonigal,Geraint Lewis and Philip O'Byrne(see [21]).

The last work addressing interstellar collisions was the work in ([22]) in 2022.It covers the analysis of Siyu Bian ,Yi Wang,Zun Wang and Mian Zhu.

All these works uses the geometry of the original Alcubierre warp drive 1994 paper in [16] and the results outlined in these works are completely correct.

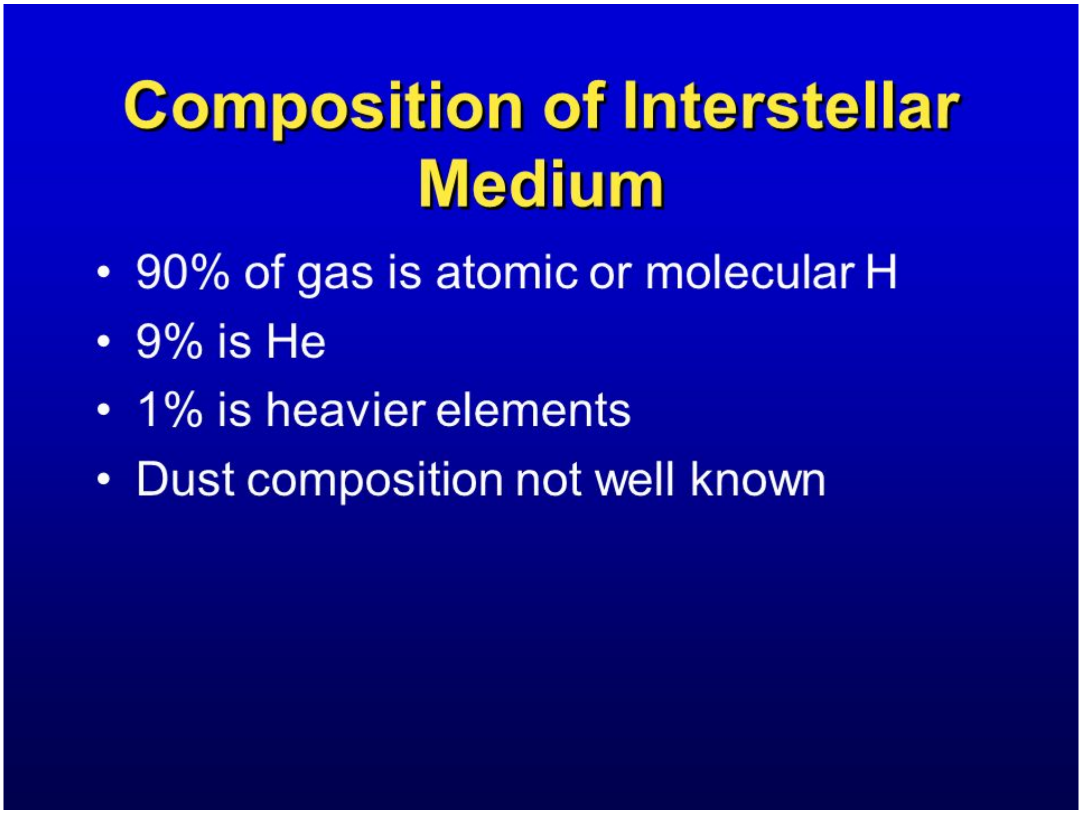

Figure 6: Composition of the Interstellar Medium IM(Source:Internet)

26 Appendix Q:Composition of the Interstellar Medium IM

The Natario warp drive is probably the best candidate(known until now) for an interstellar space travel considering the fact that a spaceship in a real superluminal interstellar spaceflight will encounter(or collide against) hazardous objects(asteroids,comets,interstellar dust and debris etc) and due to a different distribution of the negative energy in front of the ship with repulsive gravitational behavior(see pg 116 in [19]) deflecting all the incoming hazardous particles of the Interstellar Medium(see Appendices M,N and O) the Natario spacetime offers an excellent protection to the crew members as depicted in the works [26],[27] and specially [28],[29] and [23].

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