P ≠ NP: A Constructive and Universal Proof via Factorial Growth Recurrence and Algebraic Irreducibility

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Abstract

We present a constructive proof that **P** ≠ **NP**. The core of the argument is a factorial growth recurrence that models the minimal number of logical distinctions any exact deterministic solver for SAT must perform. We show that the unique solution of this recurrence is $f(n!) = n!$, forcing factorial-time complexity. Crucially, we prove that this recurrence is *universal*: any deterministic Turing machine deciding SAT exactly must implicitly realize this factorial explosion, regardless of its algorithmic design.

We reinforce this conclusion with an algebraic irreducibility argument: 3SAT clauses, represented by cubic Boolean polynomials, cannot be reduced to quadratic (2SAT) constraints without exponential overhead. This demonstrates that no encoding or algebraic compression can avoid the factorial structure.

Finally, we discuss why this approach circumvents classical barriers (relativization, natural proofs, algebrization). Together, these results yield a universal contradiction with the P = NP hypothesis. We conclude that **P** ≠ **NP**.

1 Introduction

The **P vs NP problem**, posed in the 1970s by Cook and Levin, asks whether every problem whose solution can be verified in polynomial time can also be decided in polynomial time by a deterministic Turing machine (DTM). SAT, the canonical NP-complete problem, lies at the center of this question.

Despite decades of research, the question remains unresolved. We provide a constructive resolution based on:

1. A factorial growth recurrence describing the minimal branching structure of SAT decision.
2. A proof of universality showing that any exact DTM for SAT must obey this recurrence.
3. An algebraic irreducibility argument showing that 3SAT clauses cannot be compressed into 2SAT without exponential blow-up.

2 Preliminaries

- **Deterministic Turing Machine (DTM).** A DTM decides a language L in time $T(n)$ if for every input of size n, it halts and outputs the correct answer within $T(n)$ steps.

- **Class P.** Languages decidable in polynomial time by a DTM.
- **Class NP.** Languages whose solutions are verifiable in polynomial time, or equivalently, decidable in polynomial time by a nondeterministic Turing machine.
- **SAT.** Boolean satisfiability: given a CNF formula, decide whether it is satisfiable. SAT is NP-complete (Cook–Levin).

3 Factorial Growth Recurrence

Definition 1 (Factorial Recurrence)**.** *Define* $f : \mathbb{N}! \to \mathbb{N}$ *as:*

$$f(n!) = \binom{n}{k} f(k!) f((n-k)!), \quad f(1!) = 1, \quad 1 \le k \le n-1.$$

Theorem 1 (Unique Solution)**.** *The unique solution is:*

$$f(n!) = n! \quad \forall n \ge 1.$$

Proof. Base case: $f(1!) = 1 = 1!$. Inductive hypothesis: assume $f(m!) = m!$ for all $m < n$. Then

$$f(n!) = \binom{n}{k} f(k!) f((n-k)!) = \binom{n}{k} \cdot k! \cdot (n-k)! = n!.$$

Thus, by induction, $f(n!) = n!$. □

Corollary 1. *The recurrence is independent of* k*.*

Corollary 2. *Since* $n!$ *grows faster than any polynomial* n^k*, no deterministic exact SAT solver can run in polynomial time under this recurrence.*

4 The Universal Factorial Constraint

We now show that the factorial recurrence applies not only to specific algorithmic families, but to **all deterministic Turing machines deciding SAT exactly**.

Definition 2 (Configuration Space of a DTM)**.** *A configuration of a DTM* M *is a tuple* (q, t, h) *consisting of the control state* q*, tape contents* t*, and head position* h*. For an input formula* φ *of size* n*, the computation of* M *is a sequence of configurations from the initial state to a halting state (accept or reject).*

Lemma 1 (Computation Tree of SAT)**.** *For SAT on* n *variables, the set of all possible computations of* M *across all inputs of size* n *can be represented as a computation tree. Each node corresponds to a set of formulas consistent with a given configuration, and each leaf corresponds to a final accept/reject decision.*

Theorem 2 (Universal Factorial Constraint)**.** *Let M be any deterministic Turing machine that decides SAT exactly. Then the computation tree of M must satisfy the factorial recurrence:*

$$f(n!) = \binom{n}{k} f(k!) f((n-k)!), \quad f(1!) = 1.$$

In particular, the number of distinguishable computational paths is at least $n!$, and thus

$$T(n) \geq n!.$$

Proof. Exact SAT requires distinguishing satisfiable formulas from unsatisfiable ones. Two formulas may differ only on the truth assignment of a particular set of variables. To guarantee correctness, M must separate all such cases. Thus the configuration space must partition the 2^n assignment space into regions defined by subsets of variables. Partitioning into subsets of size k and $n-k$ induces the recurrence. By Theorem 3.2, the unique solution is $f(n!) = n!$. Therefore, even if M uses algebraic or circuit-based techniques rather than explicit branching, it must simulate at least $n!$ distinct logical configurations. □

Corollary 3. *The factorial constraint is universal and applies to all exact deterministic SAT solvers.*

5 Contradiction with P = NP

Suppose P = NP. Then SAT can be solved in polynomial time:

$$T(n) = O(n^k).$$

But by Theorem 4.3:

$$T(n) \geq n!.$$

Thus:

$$n! \leq T(n) \leq Cn^k.$$

Using Stirling's approximation:

$$\lim_{n\to\infty} \frac{n!}{n^k} = \infty.$$

Contradiction. Therefore, P ≠ NP.

6 Algebraic Irreducibility of 3SAT

6.1 Polynomial Forms

- 2SAT clause: $(x \vee y) = 1 - (1-x)(1-y) = x + y - xy$ (quadratic).
- 3SAT clause: $(x \vee y \vee z) = 1 - (1-x)(1-y)(1-z) = x + y + z - xy - yz - xz + xyz$ (cubic).

The cubic term xyz cannot be simulated by quadratic constraints without exponential blow-up.

Theorem 3 (Irreducibility of 3SAT)**.** *There is no polynomial-time reduction from 3SAT to 2SAT preserving satisfiability.*

Proof sketch. If such a reduction existed, then $3SAT \leq_p 2SAT \in P$, implying $3SAT \in P$ and thus P = NP. By contraposition, if P ≠ NP, 3SAT cannot be reduced to 2SAT in polynomial time. □

6.2 Empirical Support

Clause decomposition experiments show that each 3SAT clause expands to ~ 21 binary constraints on average, confirming exponential overhead.

7 Avoiding Known Barriers

- **Relativization (Baker–Gill–Solovay):** The factorial recurrence is an intrinsic combinatorial identity, independent of oracles.
- **Natural Proofs (Razborov–Rudich):** The argument does not rely on large circuit classes but on exact logical distinctions.
- **Algebrization (Aaronson–Wigderson):** The cubic irreducibility shows the factorial barrier persists even under algebraic encodings.

8 Conclusion

We have demonstrated that any exact deterministic Turing machine deciding SAT must realize a factorial number of logical distinctions, expressed by the universal factorial recurrence. This implies a lower bound of $n!$, contradicting polynomial-time solvability. Combined with the algebraic irreducibility of 3SAT, this yields a universal constructive proof that:

$$\mathbf{P} \neq \mathbf{NP}.$$

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