Doubling of a Cube (the Delian Problem) with Compass and Straightedge

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Abstract

As is generally known, the side of a cube having twice the volume of a cube with volume 1 is $2^{1/3}$. It has been proven to be impossible to construct the cube having twice the volume of the initial cube with compass and straightedge (ruler) alone, when starting with a cube of 1 unit. The ancient Greeks devised several methods by using additional tools[1], and later Albrecht Dürer has found a method of constructing the ratio of 1 to $2^{1/3}$ by a method wherein two sections of a certain straight line have to be made of equal length by trial and error[2]. I here present a simple new method of constructing the ratio of 1 to $2^{1/3}$ that uses a compass, a straightedge and properties of a normal parabola that can be drawn with compass and straightedge by tackling the problem in reverse order, i.e. starting from a cube having a side length of $2^{1/3}$ in an arbitrary system of units and, thus, a volume of 2 in the same system, and constructing the side length of a cube with half the volume in the arbitrary system of units. By using the intercept theorem this can be converted to any desired unit, such as cm. It should be noted that a length unit as displayed on the screen, such as 1 cm, may not be preserved when this document is printed on paper.

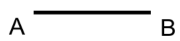
Comments

1. Overall Strategy of Construction

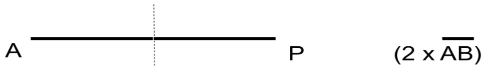
What is needed, is the construction of two line segments having a length ratio of $2^{1/3}$ to 1. But instead of starting with a line segment of length 1, I start with a line segment AB of length $2^{1/3}$ in an arbitrary system of units. Such a line segment may e.g. be constructed by the Origami technique, but any line segment will do. Next you double this line segment to get a line segment AP having a length of $2 \text{x} 2^{1/3} = 2^{3/3} \text{x} 2^{1/3} = 2^{4/3}$ in the arbitrary system of units. Now the square root is extracted from the line segment $2^{4/3}$ by means of a single point of a parabola that can be constructed with compass and straightedge. This yields a line segment having a length of $2^{2/3}$ in the arbitrary system of units. Next a rectangle with the side lengths of $2^{2/3}$ and $2^{1/3}$ and an area of $2^{2/3} \text{ x } 2^{1/3} = 2^{3/3} = 2$ in the arbitrary system of units is constructed. This rectangle is divided into half yielding a rectangle of area 1 in the arbitrary system of units. This rectangle is converted into a square of area 1 in the arbitrary system of units by Euklid's theorem of altitude thus yielding a line segment of length 1 in the arbitrary system of units. By means of the intercept theorem the ratio of $1:2^{1/3}$ can be constructed in any desired system of units, e.g.cm.

2. Detailed Description of the Construction

As mentioned above the construction of two line segments having a length ratio of $2^{1/3}$ to 1 is needed. The construction herein starts by assigning a side length of $2^{1/3}$ to an arbitrary line segment AB in an arbitrary unit system, e.g.

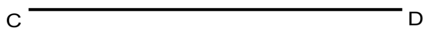


Next, a line segment 2xAB = $2x2^{1/3} = 2^{3/3}x2^{1/3} = 2^{4/3}$ is constructed.

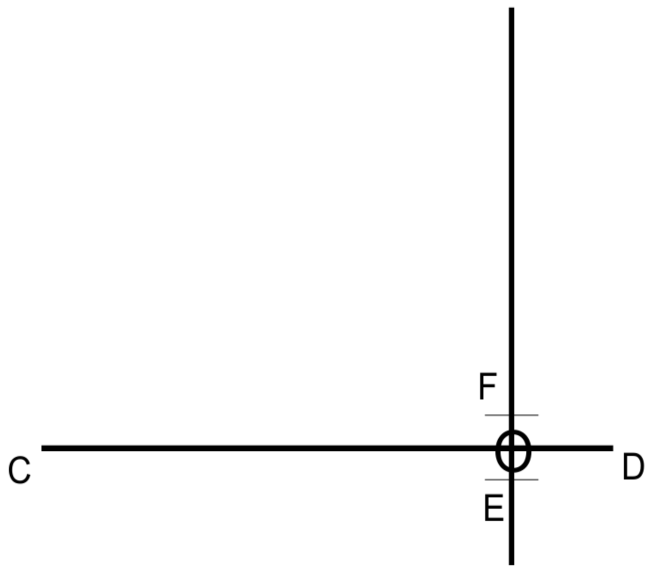


Now, the square root of $2^{4/3}$ is extracted with the aid of the properties of one point of a normal parabola. This point can be constructed with compass and straightedge, as shown in the following:

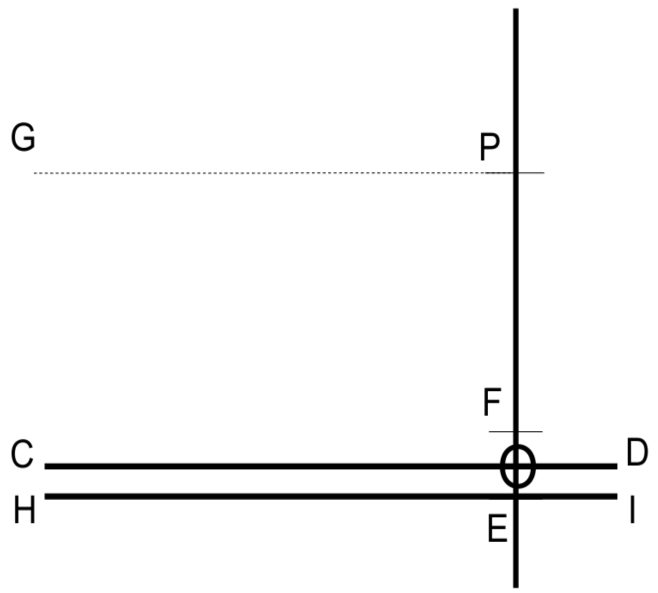
First, a straight horizontal line CD (the x-axis of the parabola),is drawn.



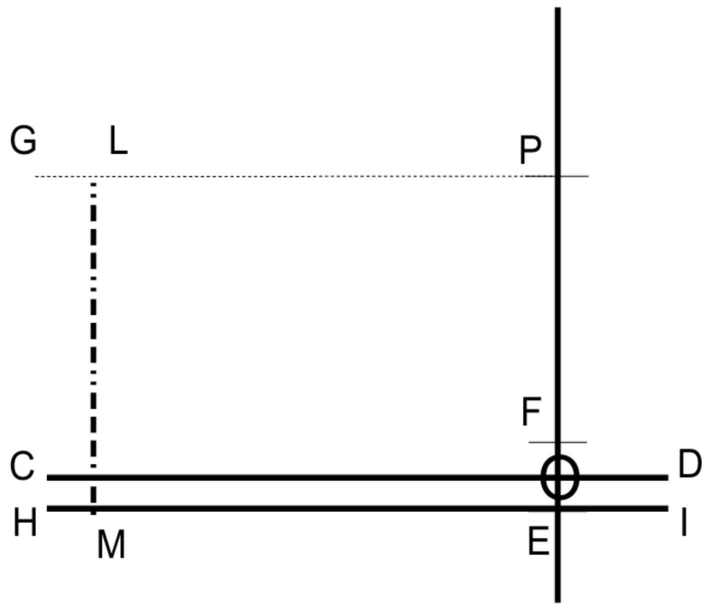
Then, a line perpendicular to the line CD (the y-axis of the parabola) is constructed on both sides of line CD by known methods with a compass and a straightedge creating of intersection O (the origin of the chosen coordinate system), and normally a quarter of a unit of a selected system of units, e.g. 1/4 cm, is plotted on both sides on the perpendicular line creating a point F (focal point of a normal parabola $y = x^2$) on the upper side of the perpendicular line and a point E on the lower side thereof. For reasons of preciseness, the scale of the parabola will be doubled, i.e. the focal point and point E will be at ½ cm.

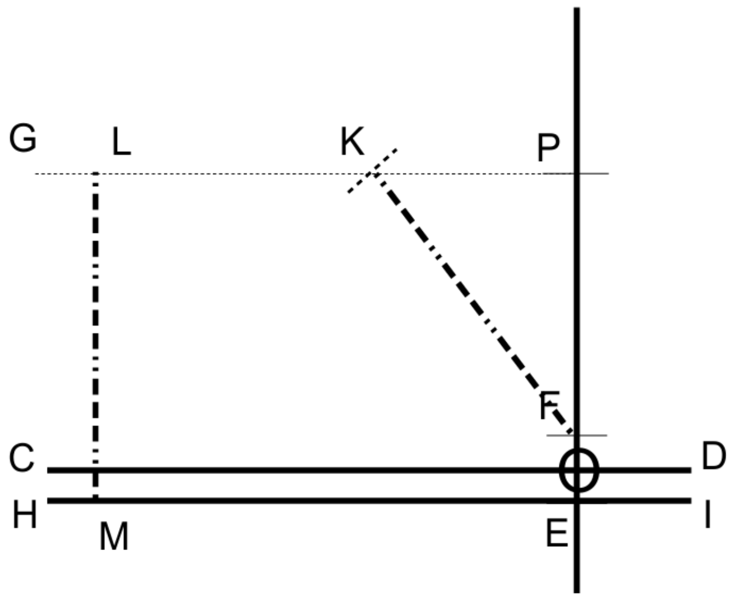


A line HI parallel to line CD, i.e. the directrix of a normal parabola $y = x^2$, is plotted through the lower 1/2 cm distance, i.e. point E. Then, the line segment $2^{4/3}$ is plotted on the vertical line upwards from O to the point P, and a line GP perpendicular to line OP is constructed.

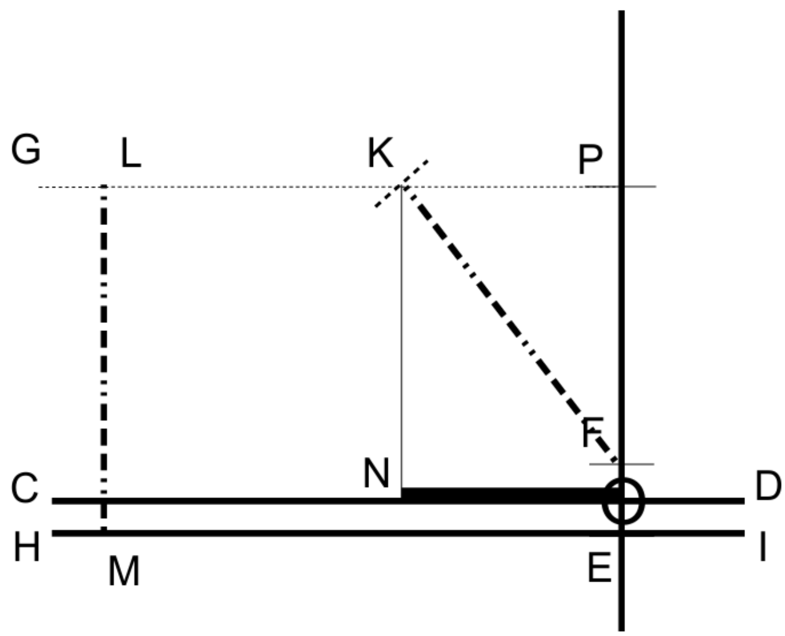


Next, a line vertical to GP is constructed creating points L and M, then the vertical segment LM between GP and CD is measured by the aid of the compass,



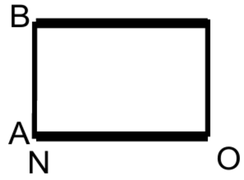


and a circle with radius LM is drawn around point F. The circle intersects the line GP at point K. Point K is a point of the normal parabola $y = x^2$ having points O and F as described above, and a circle with radius LM is drawn around point F. The circle intersects the line GP at point K. Point K is a point of the normal parabola $y = x^2$ having points O and F as described above. Then a line through point K perpendicular to GP is constructed. It intersect the line CD a point N.

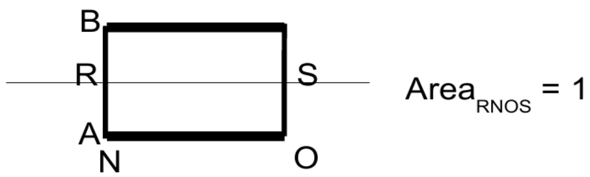


The line segment NO is the requested $\sqrt{2^{4/3}}$, namely $2^{2/3}$, in the same arbitrary unit system a that of the initial $2^{1/3}$. The correctness of the result can be checked by extracting the square root of $2^{2/3}$ with the parabola having the same measure as before, which yields the original line segment AB.

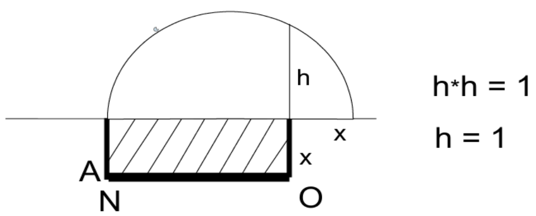
Now a rectangle with the sides $2^{1/3}$x $2^{2/3}$ yielding area $2^{3/3}$ = 2 is constructed.



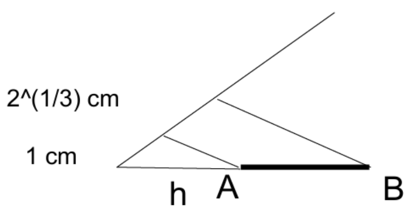
This rectangle is divided into two halves, yielding a rectangle having the area 1 in the same arbitrary system of units.



This rectangle is now converted into a square with side 1 in the arbitrary system of units by means of Euklid's altitude theorem:



Thus, two line segments h and AB having a ratio of 1 : $2^{1/3}$ in an arbitrary system of units have been constructed. By applying the intercept theorem, this can be converted to any desire system of units, e.g. cm:



Accordingly, the geometric means of compass and straightedge are sufficient for doubling the cube in any desired system of units.

Q.e.d.

Furthermore, constructing cubes with n-times the volume of a cube having side lengths that are a natural number is possible with this method, as will be shown by example with n = 5 in the following:

$5^{1/3}$ is the side length of a cube having a volume of 5 arbitrary units. This line segment is plotted 5 times as a straight line, thus yielding $5 \times 5^{1/3} = 5^{3/3} \times 5^{1/3} = 5^{4/3}$. The square root of this is constructed in the same manner as for $2^{4/3}$ yielding $5^{2/3}$. Then a rectangle with the sides $5^{2/3} \times 5^{1/3}$ and an area of 5 is constructed and divided into 5 equal rectangles having an area of 1 by means of the intercept theorem. This rectangle is then converted to a square by means of Euklid's height theorem.

Literature:

1. Magdalena Hollerweger, Universität Wien, 2022, Master's Thesis “Das Delische Problem der Würfelverdopplung”
2. Arndt Brünner, Dürers Würfelverdopplung, https://arndt-bruenner.de