On the Equations $2m^2 + 2m = y^n$ and $m(m + 2) = y^n$: Connections to Quadratic Forms and Major Conjectures in Number Theory

Mar Detic

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Abstract

This paper provides a comprehensive analysis of the Diophantine equations $2m^2+2m = y^n$ and $m(m+2) = y^n$ for integers $m, y \geq 0$ and $n \geq 2$. We demonstrate that the first equation has infinitely many solutions for $n = 2$ (via a Pell equation) and only the trivial solution for $n \geq 3$ (by Erdős–Selfridge), while the second has no nontrivial solutions for any $n \geq 2$. We explore connections to Fermat's Last Theorem, the Beal Conjecture, and the ABC Conjecture. Additionally, we show that for odd $m = 2k + 1$, the equation $m(m + 2) = y^n$ becomes $4(k + 1)^2 - 1 = y^n$, connecting it to arithmetic progressions and Pell-type equations. We demonstrate that attempts to express these equations in Beal form fail, and we highlight the role of discriminants and factorization in determining the existence of solutions.

1 Introduction

Consider the Diophantine equations:

$$2m^2 + 2m = y^n, \tag{1}$$
$$m(m + 2) = y^n, \tag{2}$$

where $m, y \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 2}$. These equations can be rewritten respectively as:

$$2m(m + 1) = y^n,$$
$$m(m + 2) = y^n.$$

We aim to classify all solutions (m, y, n) to these equations and explore their connections to fundamental conjectures in number theory.

For equation (1), it is known that for $n \geq 3$, the only solution is $m = 0$ (and hence $y = 0$), a result implied by the Erdős–Selfridge theorem [3], which states that the product of consecutive integers is never a perfect power (except in trivial cases). For $n = 2$, however, there are infinitely many solutions. Equation (2) behaves differently: it has no nontrivial solutions for any $n \geq 2$.

This paper makes several contributions:

- We provide a unified analysis of both equations, highlighting their connections to quadratic forms and their discriminants
- We demonstrate the crucial role of Pell's equation in generating infinitely many solutions to (1) for $n = 2$
- We show why equation (2) admits no nontrivial solutions for any $n \geq 2$
- We explore connections to Fermat's Last Theorem, the Beal Conjecture, and the ABC Conjecture
- We demonstrate a connection between $m(m + 2) = y^n$ and arithmetic progressions through the transformation $m = 2k + 1$
- We attempt to express these equations in Beal form and show why such attempts necessarily fail
- We place these results in the broader context of quadratic forms and their representation properties

2 The Case $n = 2$: Quadratic Forms and Pell Equations

2.1 Equation (1): $2m^2 + 2m = y^2$

For $n = 2$, equation (1) becomes:

$$2m(m + 1) = y^2.$$

Completing the square yields:

$$(2m + 1)^2 - 2y^2 = 1.$$

This is a Pell equation, which has the fundamental solution $(x, y) = (3, 2)$, giving $m = 1$. All solutions are generated by:

$$x + y\sqrt{2} = (3 + 2\sqrt{2})^k, \quad k \geq 1.$$

Thus, there are infinitely many solutions, including:

$$\begin{aligned} m &= 1, \quad y = 2; \\ m &= 8, \quad y = 12; \\ m &= 49, \quad y = 70. \end{aligned}$$

2.2 Equation (2): $m(m + 2) = y^2$

For $n = 2$, equation (2) becomes:

$$m(m + 2) = y^2.$$

Rewriting gives:

$$(m + 1)^2 - y^2 = 1,$$

which factors as:

$$(m+1-y)(m+1+y) = 1.$$

The only integer solutions occur when:

$$m+1-y = 1 \quad \text{and} \quad m+1+y = 1,$$

or

$$m+1-y = -1 \quad \text{and} \quad m+1+y = -1.$$

Both cases yield $m = -1, y = 0$, which is trivial. Thus, no nontrivial solutions exist for equation (2) when $n = 2$.

2.3 Quadratic Form Perspective

Both equations can be viewed as quadratic forms:

$$Q_1(m) = 2m^2 + 2m,$$
$$Q_2(m) = m^2 + 2m.$$

The discriminants are:

$$\Delta_1 = 2^2 - 4 \cdot 2 \cdot 0 = 4,$$
$$\Delta_2 = 2^2 - 4 \cdot 1 \cdot 0 = 4.$$

Despite having the same discriminant, the equations behave differently due to their distinct factorization properties. Equation (1) connects to Pell's equation, while equation (2) factors as a difference of squares but yields only trivial solutions.

3 The Case $n \geq 3$: Erdős–Selfridge and Beyond

3.1 Equation (1): $2m^2 + 2m = y^n$

For $n \geq 3$, we claim that the only solution is $m = 0$.

Proof sketch. Since $\gcd(m, m+1) = 1$, the equation $2m(m+1) = y^n$ implies that m and $m+1$ must be n-th powers up to factors of 2. Thus, either:

$$m = 2a^n, \quad m+1 = b^n, \quad \text{so} \quad b^n - 2a^n = 1, \quad \text{or}$$
$$m = a^n, \quad m+1 = 2b^n, \quad \text{so} \quad 2b^n - a^n = 1.$$

For $n \geq 3$, these exponential Diophantine equations have no solutions by Mihăilescu's theorem [5] and bounds on exponential Diophantine equations. The trivial solution $m = 0$ gives $y = 0$. □

This is a consequence of the Erdős–Selfridge theorem [3].

3.2 Equation (2): $m(m+2) = y^n$

For $n \geq 3$, we again apply the Erdős–Selfridge theorem. If m is odd, $\gcd(m, m+2) = 1$, so m and $m+2$ must be perfect n-th powers:

$$m = a^n, \quad m+2 = b^n \implies b^n - a^n = 2.$$

By Catalan's conjecture (Mihăilescu's theorem [5]), no solutions exist for $n \geq 3$. If m is even, let $m = 2k$:

$$4k(k+1) = y^n.$$

Then k and $k+1$ are consecutive and coprime, so by Erdős–Selfridge, the only solution is $k = 0$ ($m = 0$).

4 Connection to Fermat's Last Theorem

Fermat's Last Theorem states that $a^n + b^n = c^n$ has no positive integer solutions for $n > 2$. While our equations are not sums of powers, they share the theme that for $n \geq 3$, nontrivial solutions are absent. Both results reflect the deep number-theoretic principle that perfect powers are rare in arithmetic sequences and products.

5 Connection to the Beal Conjecture

The Beal Conjecture [2] states that if $a^x + b^y = c^z$ with $x, y, z > 2$, then a, b, c must share a common prime factor.

5.1 Attempt to Express in Beal Form

For equation (1), suppose we assume $2m = a^x$ (a perfect power). Then:

$$a^x(m+1) = y^n \implies m+1 = \left(\frac{y}{a}\right)^n.$$

Let $b = y/a$, so $m = b^n - 1$ and $2(b^n - 1) = a^x$. This leads to:

$$a^x - 2b^n = -2.$$

For $n \geq 3$, this has no solutions by modular constraints.

Alternatively, suppose both $2m^2$ and $2m$ are perfect powers:

$$2m^2 = A^p, \quad 2m = B^q,$$

with $p, q, n \geq 3$. Then:

$$A^p + B^q = y^n. \tag{3}$$

This is in Beal form. However, from $2m = B^q$, we have $m = B^q/2$. Substituting into $2m^2 = A^p$:

$$2\left(\frac{B^q}{2}\right)^2 = A^p \implies \frac{B^{2q}}{2} = A^p.$$

Thus:

$$B^{2q} = 2A^p. \tag{4}$$

For integer solutions, B^{2q} must be twice a perfect power. For $p = q = 3$:

$$B^6 = 2A^3.$$

This requires that A^3 has a factor of 2, so $A = 2^{1/3}C$, which is impossible. More generally, equation (4) has no solutions for $p, q \geq 3$ by unique factorization.

Similar arguments show that equation (2) cannot be expressed in Beal form with all exponents ≥ 3.

6 Connection to the ABC Conjecture

The ABC Conjecture [4] states that for any $\epsilon > 0$, there exists C_ϵ such that for coprime a, b, c with $a + b = c$:

$$\max(|a|, |b|, |c|) \leq C_\epsilon \cdot \left(\prod_{p|abc} p \right)^{1+\epsilon}.$$

For equation (1), apply ABC to $m + (m+1) = 2m + 1$ with $a = m$, $b = m + 1$, $c = 2m + 1$. The radical is $O(m)$. Then:

$$2m + 1 \leq C_\epsilon \cdot m^{1+\epsilon}.$$

If $2m(m+1) = y^n$, then $y^n \approx 2m^2$. The ABC Conjecture would imply:

$$y^n \lesssim C_\epsilon \cdot m^{1+\epsilon} \approx C_\epsilon \cdot (y^{n/2})^{1+\epsilon},$$

so $y^n \lesssim y^{n/2(1+\epsilon)}$, forcing $n \leq 2$ for small ϵ. Thus, ABC implies that (1) has only finitely many solutions for $n \geq 3$.

A similar argument applies to equation (2).

7 Quadratic Forms and Representation of Perfect Powers

The equations studied here are special cases of the general problem of when quadratic forms represent perfect powers. Consider:

$$Q(m) = am^2 + bm + c = y^n.$$

For $n = 2$, this is a quadratic in m, and solutions exist when the discriminant is a perfect square. For $n \geq 3$, the equation defines a curve of genus ≥ 1, and by Faltings' theorem, there are finitely many solutions. However, the Erdős–Selfridge theorem gives stronger results for specific forms.

The different behavior of equations (1) and (2) stems from their transformation properties:

- Equation (1) connects to Pell's equation through completion of the square
- Equation (2) factors as a difference of squares but yields only trivial solutions

8 Connection to Arithmetic Progressions and Perfect Squares

A significant connection emerges when we consider the quadratic form $m(m+2)$ in the context of arithmetic progressions. Let $m = 2k+1$, where k is an integer, ensuring m is odd. Then:

$$m(m+2) = (2k+1)(2k+3) = 4k^2 + 8k + 3.$$

Alternatively, we can express:

$$m(m+2) = (m+1)^2 - 1 = (2k+2)^2 - 1 = 4(k+1)^2 - 1.$$

Thus, the equation $m(m+2) = y^n$ becomes:

$$4(k+1)^2 - 1 = y^n. \tag{5}$$

For $n = 2$, this is:

$$4(k+1)^2 - 1 = y^2 \quad \Rightarrow \quad 4(k+1)^2 - y^2 = 1.$$

Factoring:

$$(2(k+1) - y)(2(k+1) + y) = 1.$$

The only integer solutions occur when:

$$2(k+1) - y = 1 \quad \text{and} \quad 2(k+1) + y = 1,$$

or

$$2(k+1) - y = -1 \quad \text{and} \quad 2(k+1) + y = -1.$$

Both yield $k = -1, y = 0$, giving $m = -1$, a trivial solution. This confirms that $m(m+2) = y^2$ has no nontrivial solutions.

For $n \geq 3$, equation (5) becomes:

$$4(k+1)^2 - y^n = 1.$$

This is an exponential Diophantine equation. By known results [1], the only solution is $k = -1, y = 0$, giving $m = -1$.

If m is even, let $m = 2k$. Then:

$$m(m+2) = 4k(k+1) = y^n.$$

Since k and $k+1$ are consecutive, by the Erdős–Selfridge theorem, the only solution is $k = 0$, giving $m = 0$.

This analysis shows that $m(m+2) = y^n$ is equivalent to $4(k+1)^2 - 1 = y^n$ for odd m, connecting it to arithmetic progressions and explaining the absence of nontrivial solutions.

9 Conclusion

We have analyzed the Diophantine equations $2m^2 + 2m = y^n$ and $m(m+2) = y^n$ and highlighted their distinct behaviors. The first equation has infinitely many solutions for $n = 2$ via a Pell equation, while for $n \geq 3$ only the trivial solution exists, consistent with the Erdős–Selfridge theorem. The second equation has no nontrivial solutions for any $n \geq 2$.

The proper transformation $m = 2k + 1$ for odd m reveals that $m(m+2) = 4(k+1)^2 - 1$, connecting the problem to arithmetic progressions and Pell-type equations. This transformation clarifies why perfect powers rarely occur in such quadratic forms.

Additionally, we have explored the connections of these equations to major number-theoretic conjectures:

- The rarity of nontrivial solutions for $n \geq 3$ reflects the philosophy of Fermat's Last Theorem.
- Attempts to express these equations in Beal form fail, highlighting the constraints on perfect powers in sums and products.
- The ABC Conjecture provides a general framework explaining the finiteness of solutions for higher powers.
- Discriminants and factorization properties play a crucial role in determining whether quadratic forms represent perfect powers.

This study synthesizes elementary and advanced number theory, demonstrating how specific Diophantine equations connect to broader conjectures and principles, and clarifies the precise mechanisms behind the existence (or absence) of solutions.

References

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