Loop space blow-up and scale calculus

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Abstract

In this note we show that the Barutello-Ortega-Verzini regularization map is scale smooth.

Contents

1 Introduction

1.1 Background

Regularization of two-body collisions is an important topic in celestial mechanics and the dynamics of electrons in atoms. Most classical regularizations blow up the energy hypersurfaces to regularize collisions. Recently Barutello, Ortega, and Verzini [1] discovered a new regularization technique which does not blow up the energy hypersurface, but instead of that the loop space. The discovery is explained in detail in [4, §2.2] in case of the free fall.

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This new regularization technique is in particular useful for non-autonomous systems for which there is no preserved energy and therefore no energy hypersurface which can be blown up.

1.2 Setup and main result

Let $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ be the punctured complex plane. We abbreviate by

$$\mathcal{L}\mathbb{C}^\times := C^\infty(\mathbb{S}^1, \mathbb{C}^\times), \qquad \mathbb{S}^1 := \mathbb{R}/\mathbb{Z},$$

the space of smooth loops in the punctured plane. Barutello-Ortega-Verzini regularization is carried out using a map $\mathcal{R}\colon \mathcal{L}\mathbb{C}^\times \to \mathcal{L}\mathbb{C}^\times$ defined as follows. For $z \in \mathcal{L}\mathbb{C}^\times$ define the map

$$t_z\colon \mathbb{S}^1 \to \mathbb{S}^1, \quad \tau \mapsto \tfrac{1}{\|z\|_{L^2}^2} \int_0^\tau |z(s)|^2 \, ds \in [0,1].$$

Indeed it takes values in $\mathbb{R}/\mathbb{Z}$ since $t_z(0) = 0$ and $t_z(1) = 1$ coincide modulo 1 which, by continuity of t_z, implies surjectivity. The derivative

$$t_z'(\tau) := \tfrac{d}{d\tau} t_z(\tau) = \tfrac{1}{\|z\|_{L^2}^2} |z(\tau)|^2 > 0$$

depends continuously on τ and is everywhere strictly positive. So t_z is also injective. Therefore the inverse $\tau_z := {t_z}^{-1}$ exists and it is C^1; see [4, §2.2]. This shows that t_z and τ_z are elements of the circle's diffeomorphism group $\mathrm{Diff}(\mathbb{S}^1)$. Using this notion the *Barutello-Ortega-Verzini map* or, alternatively, the **rescale-square operation** is defined by

$$\mathcal{R}\colon \mathcal{L}\mathbb{C}^\times \to \mathcal{L}\mathbb{C}^\times, \quad z \mapsto z^2 \circ \tau_z = [t \mapsto z^2(\tau_z(t))].$$

The loop space $\mathcal{L}\mathbb{C}^\times$ is an open subset of the Fréchet space $\mathcal{L}\mathbb{C} = C^\infty(\mathbb{S}^1, \mathbb{C})$. This Fréchet space arises as the smooth level of the scale Hilbert space

$$\Lambda\mathbb{C} = (\Lambda\mathbb{C}_k)_{k\in\mathbb{N}_0}, \qquad \Lambda\mathbb{C}_k := W^{2+k,2}(\mathbb{S}^1, \mathbb{C}), \quad k \in \mathbb{N}_0.$$

Indeed $\Lambda\mathbb{C} = \cap_{k\in\mathbb{N}_0} \Lambda\mathbb{C}_k$. Below we briefly introduce scale calculus.
The scale Hilbert space $\Lambda\mathbb{C} = W^{2,2}(\mathbb{S}^1, \mathbb{C})$ contains the open subset

$$\Lambda\mathbb{C}^\times := W^{2,2}(\mathbb{S}^1, \mathbb{C}^\times)$$

which inherits the levels

$$\Lambda\mathbb{C}_k^\times := \Lambda\mathbb{C}^\times \cap \Lambda\mathbb{C}_k = W^{2+k,2}(\mathbb{S}^1, \mathbb{C}^\times), \quad k \in \mathbb{N}_0.$$

Observe that $\mathcal{L}\mathbb{C}^\times = C^\infty(\mathbb{S}^1, \mathbb{C}^\times) = \cap_{k\in\mathbb{N}_0} \Lambda\mathbb{C}_k^\times$ is the smooth level of $\Lambda\mathbb{C}^\times$. The map t_z, hence $\mathcal{R}$, is well defined for $z \in \Lambda\mathbb{C}^\times$. In this note we prove

Theorem A. *The map $\mathcal{R}\colon \Lambda\mathbb{C}^\times \to \Lambda\mathbb{C}^\times$ is scale smooth.*

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1.3 Outlook

This article is part of our general endeavor to understand what Floer homology precisely is, as for example explained in [6,7]. The question about the structure lying behind Floer homology becomes an issue of great concern if one is taking account as well Hamiltonian systems with singularity. Studying Hamiltonian systems with singularity requires new techniques. The new regularization procedure discovered by Barutello, Ortega, and Verzini [1] was already applied successfully to various problems in celestial mechanics and the dynamics of atoms, see e.g. [2–4]. To apply Floer theoretic methods to these kind of problems therefore requires a deeper understanding on the analytical mechanism lying behind this new regularization technique.

2 Scale smoothness

2.1 Basic notions

For the readers convenience we briefly recall the basic notions of scale calculus. For details of scale Hilbert spaces and scale smoothness (sc^∞) see [8]; for an introduction see [11], for a summary of what we need here see [5].

Definition 2.1. A **scale structure** on a Hilbert space H is a nested sequence $H =: H_0 \supset E_1 \supset E_2 \supset \dots$ of Hilbert spaces meeting the following requirements:

- The inclusion map $E_{i+1} \hookrightarrow E_i$ is compact for every $i \in \mathbb{N}_0$.
- The intersection $E_\infty := \cap_{i \geq 0} E_i$ is dense in each level E_i.

In this case one calls H a **scale Hilbert space** and writes $H = (H_i)_{i \in \mathbb{N}_0}$.

Definition 2.2 (Shifted scale Hilbert space)**.** Given a scale Hilbert space H and $m \in \mathbb{N}_0$, then one defines the scale Hilbert space H^m by

$$(H^m)_k := H_{k+m}.$$

Definition 2.3 (Scale direct sum)**.** If H and F are scale Hilbert spaces one defines their direct sum as the scale Hilbert space $H \oplus F$ whose levels are given by

$$(H \oplus F)_k := H_k \oplus F_k.$$

Definition 2.4 (Scale isomorphism)**.** A map $I : H \to F$ between scale Hilbert spaces is called a **scale morphism**, or an **sc-morphism**, if the restriction to each level H_k takes values in F_k and

$$I_k := I|_{H_k} : H_k \to F_k$$

is linear and continuous. A scale morphism is called a **scale isomorphism**, or an **sc-isomorphism**, if its restriction I_k to each level H_k is bijective. Note that by the open mapping theorem if I is a scale isomorphism its inverse is a scale isomorphism as well. Two scale Hilbert spaces are called **scale isomorphic** if there exists a scale isomorphism between them.

Example 2.5 (Finite dimension)**.** If the Hilbert space H is of finite dimension, then property (ii) implies that the scale-structure is constant $H =: H_0 = H_1 = H_2 = \dots$.

Remark 2.6 (Infinite dimension)**.** In contrast, if H is infinite dimensional, then the compactness requirement in property (i) enforces strict inclusions $H_{i+1} \subsetneq H_i$. Indeed the identity map on an infinite dimensional Hilbert space is never compact, because the unit ball of a Hilbert space is compact if and only if the Hilbert space is finite dimensional.

Let H be a scale Hilbert space. Given an open subset $U \subset H$, then the part of U in H_k is denoted by $U_k := U \cap H_k$. Note that U_k is open in H_k. In particular, one obtains a nested sequence $U = U_0 \supset U_1 \supset U_2 \dots$.

Definition 2.7 (Scale continuity)**.** Suppose that H and F are sc-Hilbert spaces and $U \subset H$ is an open subset. A map $f : U \to F$ is **scale continuous** ($\mathbf{sc^0}$) if

(i) f is level preserving, i.e. the restriction of f to each level U_k takes values in the corresponding level F_k, and

(ii) the map $f|_{U_k} : U_k \to F_k$ is continuous for every k.

In order to introduce the notion of continuously scale differentiable or sc^1 we first need to introduce the notion of tangent bundle. The **tangent bundle** of a scale Hilbert space H is defined as the scale Hilbert space

$$TH := H^1 \oplus H^0.$$

If $U \subset H$ is an open subset of the scale Hilbert space H, as in Definition 2.2 one denotes by $U^m \subset H^m$ the scale of open subsets whose levels are given by $(U^m)_k := U_{m+k}$ where $k \in \mathbb{N}_0$. The tangent bundle of U is the open subset of TH defined by

$$TU := U^1 \oplus H^0 \subset TH.$$

Note that the filtration of TU is given by

$$(TU)_k = U_{k+1} \oplus H_k, \quad k \in \mathbb{N}_0.$$

Let $\mathcal{L}(X, Y)$ be the vector space of continuous linear operators between Hilbert spaces X and Y. Equipped with the operator norm it is a Hilbert space.

Definition 2.8 (Scale differentiability)**.** Suppose $f : U \to F$ is sc^0, then f is called **continuously scale differentiable** or of class $\mathbf{sc^1}$ if for every $x \in U_1$ there is a bounded linear map

$$Df(x) : H_0 \to F_0, \tag{2.1}$$

called **sc-differential**, such that the following two conditions hold:

(i) The restriction of f to U_1 interpreted as a map $f : U_1 \to F_0$ is required to be *pointwise* differentiable in the usual sense. The restriction of $Df(x)$ to H_1 is required to be the differential of $f : U_1 \to F_0$ in the usual sense, notation $df(x) \in \mathcal{L}(H_1, F_0)$, i.e.

$$Df(x)|_{H_1} = df(x) \in \mathcal{L}(H_1, F_0). \tag{2.2}$$

(ii) The **tangent map** $Tf : TU \to TF$ defined for $(x, h) \in U^1 \oplus H^0 = TU$ by

$$Tf(x, h) := (f(x), Df(x)h)$$

is of class sc^0.

2.2 Neumeisters theorem

The proof of Theorem A is based on a result of Neumeister which tells that the action on the free loop space of the diffeomorphism group of the circle

$$\mathcal{D} := \{\psi \in W^{2,2}(\mathbb{S}^1, \mathbb{S}^1) \mid \psi \text{ is bijective and } \psi^{-1} \in W^{2,2}(\mathbb{S}^1, \mathbb{S}^1)\}$$

with levels $\mathcal{D}_k := \mathcal{D} \cap W^{2,2+k}(\mathbb{S}^1, \mathbb{S}^1)$, for $k \in \mathbb{N}_0$, is scale smooth.

Theorem 2.9 ([9, Prop. 3.2])**.** *The reparametrization map*

$$\rho\colon \mathcal{D} \times \Lambda\mathbb{C} \to \Lambda\mathbb{C}, \quad (\psi, z) \mapsto z \circ \psi$$

is scale smooth.

Remark 2.10 (Why the zero level is chosen $W^{2,2}$ and not $W^{1,2}$)**.** In case $(\psi, z) \in W^{1,2}(\mathbb{S}^1, \mathbb{S}^1) \times W^{1,2}(\mathbb{S}^1, \mathbb{C})$, the derivative

$$(z \circ \psi)' = \underbrace{z'|_\psi}_{L^2} \cdot \underbrace{\psi'}_{L^2}$$

is not necessarily in L^2. But in case $(\psi, z) \in W^{2,2} \times W^{2,2}$ the derivative lies in $W^{1,2}$ since both factor do and on one of them we can use that $W^{1,2} \subset C^0$. Then the second weak derivative exists as well

$$(z \circ \psi)'' = \underbrace{z''|_\psi}_{L^2} \cdot \underbrace{\psi'}_{W^{1,2} \subset C^0} \cdot \underbrace{\psi'}_{W^{1,2} \subset C^0} + \underbrace{z'|_\psi}_{W^{1,2} \subset C^0} \cdot \underbrace{\psi''}_{L^2}$$

and lies in L^2 as desired.

2.3 Time rescaling

Lemma 2.11. *The map*

$$t\colon \Lambda\mathbb{C}^\times \to \mathcal{D}, \quad z \mapsto t_z$$

is scale smooth.

Proof. We show that the map t is **strongly scale smooth** (ssc$^\infty$). By definition, this means that the map t is on each level $k \in \mathbb{N}_0$ smooth as a map $\Lambda\mathbb{C}_k^\times \to \mathcal{D}_k$. But strongly scale smooth implies scale smooth (sc$^\infty$).
To this end we decompose the map t as the composition $t = \mathcal{M} \circ (\mathcal{I}, \iota \circ \mathcal{N})$ of several maps each of which is obviously smooth. These maps are

$$\mathcal{N}\colon \Lambda\mathbb{C}_k^\times \to (0,\infty), \quad z \mapsto \|z\|_{L^2}^2 \qquad \iota\colon (0,\infty) \to (0,\infty), \quad x \mapsto \tfrac{1}{x}$$

and

$$\mathcal{I}\colon \Lambda\mathbb{C}_k^\times \to W^{k,2}([0,1],\mathbb{R}), \quad z \mapsto \left[\tau \mapsto \int_0^\tau |z(s)|^2\, ds\right]$$

and

$$\mathcal{M}\colon W^{k,2}([0,1],\mathbb{R}) \times \mathbb{R} \to W^{k,2}([0,1],\mathbb{R}), \quad (v,r) \mapsto rv.$$

This proves Lemma 2.11. □

2.4 Inverse

Proposition 2.12. *The map $I\colon \mathcal{D} \to \mathcal{D}$, $\psi \mapsto \psi^{-1}$, is scale smooth.*

Proof. We compute the scale differentials of the inversion map I. By definition of the inverse, for every $\psi \in \mathcal{D}$ we have the identity $\psi \circ I(\psi) = \mathrm{id}$. Differentiating this identity we obtain for a tangent vector $\hat\psi \in T_\psi\mathcal{D} = W^{2,2}(\mathbb{S}^1,\mathbb{R})$ that

$$0 = \hat\psi \circ I(\psi) + d\psi|_{I(\psi)} DI|_\psi \hat\psi = \hat\psi \circ \psi^{-1} + (\psi' \circ \psi^{-1}) \cdot DI|_\psi \hat\psi.$$

Therefore we obtain the formula

$$DI|_\psi \hat\psi = \frac{-1}{\psi' \circ \psi^{-1}} \cdot \hat\psi \circ \psi^{-1}.$$

Note that $\psi'(t) \neq 0$, for every t, since $\psi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism.

Let $\psi \in \mathcal{D}$ and $\hat\psi_1, \hat\psi_2 \in T_\psi\mathcal{D}$. Note that ψ appears three times in the formula for $DI|_\psi\hat\psi$. Hence the second derivative is a sum of three terms, namely

$$\begin{aligned} D^2 I|_\psi(\hat\psi_1,\hat\psi_2) &= \frac{1}{(\psi' \circ \psi^{-1})^2} (\hat\psi_2' \circ \psi^{-1})(\hat\psi_1 \circ \psi^{-1}) \\ &\quad - \frac{1}{(\psi' \circ \psi^{-1})^3} (\psi'' \circ \psi^{-1})(\hat\psi_2 \circ \psi^{-1})(\hat\psi_1 \circ \psi^{-1}) \\ &\quad + \frac{1}{(\psi' \circ \psi^{-1})^2} (\hat\psi_1' \circ \psi^{-1})(\hat\psi_2 \circ \psi^{-1}). \end{aligned}$$

Note that $D^2 I|_\psi(\hat\psi_1,\hat\psi_2)$ is a polynomial in the six variables

$$\frac{1}{\psi' \circ \psi^{-1}}, \quad \psi'' \circ \psi^{-1}, \quad \hat\psi_1 \circ \psi^{-1}, \quad \hat\psi_2 \circ \psi^{-1}, \quad \hat\psi_1' \circ \psi^{-1}, \quad \hat\psi_2'' \circ \psi^{-1}.$$

If ψ is in $W^{k+4,2}$ and $\hat{\psi}_1, \hat{\psi}_2$ are in $W^{k+3,2}$, then all these variables are in $W^{k+2,2}$. Since multiplication $W^{k+2,2} \times W^{k+2,2} \to W^{k+2,2}$ is continuous, we conclude that the map

$$\mathcal{D}^{k+2} \times W^{k+3,2} \times W^{k+3,2} \to W^{k+2,2}, \quad (\psi, \hat{\psi}_1, \hat{\psi}_2) \mapsto D^2 I|_\psi(\hat{\psi}_1, \hat{\psi}_2)$$

is continuous. Therefore, by the criterium in [5, Le. 4.8] the inversion map I is of class sc^2.

Differentiating further by induction we obtain that for every $n \in \mathbb{N}$ $D^n I|_\psi(\hat{\psi}_1, \dots, \hat{\psi}_n)$ is a polynomial in the $(n+1)n$ variables

$$\begin{aligned}
&\frac{1}{\psi' \circ \psi^{-1}}, \quad \psi'' \circ \psi^{-1}, \dots, \psi^{(n)} \circ \psi^{-1}, \\
&\hat{\psi}_1 \circ \psi^{-1}, \dots, \hat{\psi}_1^{(n-1)} \circ \psi^{-1}, \quad \dots, \quad \hat{\psi}_n \circ \psi^{-1}, \dots, \hat{\psi}_n^{(n-1)} \circ \psi^{-1}.
\end{aligned}$$

Hence the map

$$\begin{aligned}
\mathcal{D}^{k+m} \times W^{k+m+1,2} \times \dots \times W^{k+m+1,2} &\to W^{k+2,2} \\
(\psi, \hat{\psi}_1, \dots, \hat{\psi}_m) &\mapsto D^m I|_\psi(\hat{\psi}_1, \dots, \hat{\psi}_m)
\end{aligned}$$

is continuous. Therefore the map I is sc^n for every $n \in \mathbb{N}$, hence sc^∞. This finishes the proof of Proposition 2.12. □

Remark 2.13. Together with Neumeisters Theorem 2.9, Proposition 2.12 shows that the diffeomorphism group of the circle is a scale Lie group. This result is somehow reminiscent of [10, Thm. 3.4.1].

2.5 Proof of main result

Proof of Theorem A. The map $\mathcal{R}$ can be written as the composition $\mathcal{R}(z) = \rho(\sigma(z), I \circ t(z))$ of scale smooth maps and is therefore itself scale smooth by the scale calculus chain rule [8, Thm. 1.3.1].

We abbreviate by $\sigma \colon \Lambda\mathbb{C}^\times \to \Lambda\mathbb{C}^\times$ the squaring map $z \mapsto z^2$. The squaring map is obviously smooth on every level, hence ssc^∞, thus sc^∞. The map I is sc^∞ by Proposition 2.12. The map t is sc^∞ by Lemma 2.11. The map ρ is sc^∞ by Theorem 2.9. This concludes the proof of Theorem A. □

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