Motion of a Particle on a Catenary Curve: A Lagrangian Mechanics Approach with Elliptic Integral Solutions

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1 Abstract.

This paper aims to find the motion of a particle that is restricted to move along a catenary curve. The method used to find the dynamics of the particle , we use Lagrangian mechanics along with elliptical integrals to find solve the obtained equation of motion for time. We transform the kinetic and potential energies one generalized coordinate, which results in a short Lagrangian formulation.After applying the Euler–Lagrange equation and using lagrange multiplier which shortens it down and helps us to find the constraint force needed helps us to reach to a differential equation i.e., the equation of motion is derived, analyzing the interaction between inertial and gravitational forces. A similar process is carried upon for the particle going on a catenary under influence of gravity and gravity too. The Euler Lagrange Equation is modified for non conservative forces and a equation of motion is derived. Using Lagrangian Mechanics yield a simple and elegant method.

2 Introduction

The study of mechanical systems constrained to move along specific paths is a classical problem in physics, often solved using Newtonian or Lagrangian mechanics. The motion of a particle along a catenary curve is one of such systems which presents both mathematical elegance and physical relevance. It appears in contexts starting from cable dynamics to architectural structures. A catenary is the shape assumed by a flexible uniform chain hanging under its own weight and offers a natural constraint for analyzing gravitational motion.

In this work, we use the Lagrangian framework to analyze the motion of a particle constrained to this curve. Unlike in the Newtonian approach, which requires resolving forces into components along the constraint surface, the Lagrangian mechanics simplifies the analysis by only using energy expressions and generalized coordinates for Euler-Lagrange Equation. First we derive expressions for the kinetic and potential energies of the particle. Then we apply the Euler–Lagrange equation to obtain the equation of motion.

Beyond the differential equation itself, we explore the conservation of mechanical energy and leverage it to derive an expression for the particle's velocity as a function of position. This leads to an integral expression for the time required for the particle to traverse a segment of the catenary, ultimately involving elliptic integrals. Our results underscore the effectiveness of analytical mechanics in addressing complex systems and provide insight into the behavior of particles in constrained geometries.

3 Lagrangian Formulation of the Problem

We consider a mass m constrained to move without friction along the catenary $y = a\cosh(\frac{x}{a})$, where a is a constant. We choose x as the generalized coordinate.

Kinetic energy:

$$T = \frac{1}{2}m\cosh^2(x/a)\dot{x}^2$$

Taking gravitational potential energy with zero level at :

$$V = mgy = mga\cosh\left(\frac{x}{a}\right)$$

The particle is constrained to move along a catenary:

$$y = a\cosh\left(\frac{x}{a}\right) \quad \Rightarrow \quad f(x,y) = y - a\cosh\left(\frac{x}{a}\right) = 0$$

The Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$f = y - a\cosh\frac{x}{a}$$

Euler–Lagrange Equation

The Euler–Lagrange equation for the generalized coordinate is given by

$$\frac{\partial L}{\partial \mathbf{r}_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{r}}_k} + \sum_{i=1}^{C}\lambda_i\frac{\partial f_i}{\partial \mathbf{r}_k} = 0,$$

Therefore the final Euler Lagrange Equation for x coordinate is given as

$$m\ddot{x} + \lambda \cdot \sinh(x/a) = 0 \tag{1}$$

The Euler–Lagrange equation for the coordinate y is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}'}{\partial y} = 0$$

$$\Rightarrow m\ddot{y} + mg - \lambda = 0$$

By using constraint $y = a\cosh(\frac{x}{a})$, we can differentiate y to find $\dot{y}$ and $\ddot{y}$:

$$\dot{y} = \sinh(x/a)\cdot\dot{x} \quad \Rightarrow \quad \ddot{y} = \frac{1}{a}\cosh(x/a)\dot{x}^2 + \sinh(x/a)\ddot{x}$$

Substituting $\ddot{y}$ in the Euler lagrange equation for y coordinate.

$$m\left(\frac{1}{a}\cosh(x/a)\dot{x}^2 + \sinh(x/a)\ddot{x}\right) + mg - \lambda = 0 \tag{2}$$

Solving (1) and (2) together gives:

$$\lambda = \frac{-m\ddot{x}}{\sinh(\frac{x}{a})} \tag{3}$$

From here, we get the constraint force as $\lambda\nabla f(x,y)$.

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\sinh\left(\frac{x}{a}\right), 1\right)$$

From here we get the constraint force (Normal force by the catenary in our case) as

$$\vec{F} = \lambda\nabla f(x,y) = \lambda\left(\sinh\left(\frac{x}{a}\right), 1\right) = \frac{-m\ddot{x}}{\sinh(\frac{x}{a})}\left(\sinh\left(\frac{x}{a}\right), 1\right)$$

From here we get magnitude of constraint fore as

$$\|\vec{F}\| = \frac{-m\ddot{x}\cosh(\frac{x}{a})}{\sinh(\frac{x}{a})} \tag{4}$$

We get the equation of motion as :

$$\boxed{\cosh^2(x/a)\ddot{x} + \frac{\cosh(x/a)\sinh(x/a)}{a}\dot{x}^2 + g\sinh(x/a) = 0.} \tag{5}$$

Energy Conservation

Since the Lagrangian is explicitly independent of time, the energy is conserved. The total energy is:

$$E = T + V = \frac{1}{2}m\cosh^2(x/a)\dot{x}^2 + mga\cosh(x/a).$$

Solving for :

$$\dot{x} = \pm\sqrt{\frac{2}{m}\frac{E - mga\cosh(x/a)}{\cosh^2(x/a)}}$$

Finding the Time

To find the time for the mass to move along the catenary, we use energy conservation. We derived:

$$E = \frac{1}{2}m\cosh^2(x/a)\dot{x}^2 + mga\cosh(x/a).$$

Step 1: Express
Rearrange for :

$$\frac{1}{2}m\cosh^2(x/a)\dot{x}^2 = E - mga\cosh(x/a)$$

.

$$\dot{x}^2 = \frac{2}{m}\frac{E - mga\cosh(x/a)}{\cosh^2(x/a)}$$

.

Taking the square root:

$$\dot{x} = \pm\sqrt{\frac{2}{m}\frac{E - mga\cosh(x/a)}{\cosh^2(x/a)}}.$$

Step 2: Integrate for Time
Since , we separate variable

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E - mga\cosh(x/a))/\cosh^2(x/a)}}.$$

Rewriting:

$$t = \int \frac{\cosh(x/a)dx}{\sqrt{\frac{2}{m}(E - mga\cosh(x/a))}}.$$

Step 3: Solve the Integral
Let:

$$E = mga\cosh(x_0/a),$$

where x_0 is the initial position of the mass. Then:

$$E - mga\cosh(x/a) = mga(\cosh(x_0/a) - \cosh(x/a)).$$

Substituting:

$$t = \int_{-x_0}^{-x} \frac{\cosh(x/a)dx}{\sqrt{\frac{2}{m}mga(\cosh(x_0/a) - \cosh(x/a))}}.$$

Now to simplify the integration, we can break the integration into 2 parts each with lower limit zero. We can also substitute $\sqrt{\cosh(x_0/a)}$ as A and $\sqrt{\frac{2}{m}mga(\cosh(x_0/a)}$ as B. Also we can substitute $\frac{1}{a}$ as K.

The new time integral will be

$$I = \frac{1}{B}\int_0^x \frac{\cosh(K(x))}{\sqrt{A - \cosh(K(x))}}\, dx - \frac{1}{B}\int_0^L \frac{\cosh(K(x))}{\sqrt{A - \cosh(K(x))}}\, dx$$

Let I be $\frac{1}{B}\int_0^x \frac{\cosh(K(x))}{\sqrt{A-\cosh(K(x))}}\, dx$

For the anti-derivative, consider first

$$I = \int_0^x \frac{\cosh(K(x))}{\sqrt{A - \cosh(K(x))}}\, dx$$

$$K(x) = t \quad \Rightarrow \quad I = \frac{1}{K}\int_0^{Kx} \frac{\cosh(t)}{\sqrt{A - \cosh(t)}}\, dt$$

$$t = iu \quad \Rightarrow \quad I = \frac{i}{K}\int_0^{iK(x)} \frac{\cos(u)}{\sqrt{A - \cos(u)}}\, du$$

$$u = 2v \quad \Rightarrow \quad I = \frac{2i}{K}\int_0^{\frac{iK(x)}{2}} \frac{1 - 2\sin^2(v)}{\sqrt{(A-1) + 2\sin^2(v)}}\, dv$$

Now, we face two standard elliptic integrals

$$I = \frac{2i}{K\sqrt{A-1}}\left(A\, F\left(\frac{iKx}{2}\middle|\frac{2}{1-A}\right) + (1-A)\, E\left(\frac{iKx}{2}\middle|-\frac{2}{A-1}\right)\right)$$

Back to the original definite integral

$$I = -\frac{2ai}{\sqrt{\cosh(\frac{x_0}{a}) - 1}}\times$$

$$\left(\cosh(\frac{x_0}{a})\left[F\left(\frac{ix_0}{2a}| - \operatorname{csch}^2\left(\frac{x_0}{2a}\right)\right) - F\left(\frac{ix}{2a}| - \operatorname{csch}^2\left(\frac{x}{2a}\right)\right]\right)\right.$$

$$\left.-(\cosh(\frac{x_0}{a}) - 1)\left[E\left(\frac{ix_0}{2a}| - \operatorname{csch}^2\left(\frac{x_0}{2a}\right)\right) - E\left(\frac{ix}{2a}| - \operatorname{csch}^2\left(\frac{x}{2a}\right)\right)\right]\right.$$

4 Conclusion

This study presents a modern analytical treatment of a classical mechanics problem using Lagrangian formalism. The derivation of motion equations, conservation laws, and time integration not only reaffirm classical results but also open avenues for extensions into elastic, relativistic, or computational formulations. The approach demonstrates how variational principles provide a unifying language across physics and engineering disciplines