Primes as Geometric Indecomposables: A Group-Theoretic Characterization on the Circle and Its Harmonic Signature

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September 5, 2025

Abstract

We demonstrate that prime numbers are precisely the indecomposable elements under a novel group operation defined on a circle, providing a geometric characterization of primality. By projecting integers onto a circle via the mapping $\theta_n = \arccos(n/R)$, we show that primes exhibit intense clustering at the endpoints of the diameter, while composite numbers distribute uniformly. We formalize this observation by defining an angular density function $F(n)$ that vanishes if and only if n is prime, with a rigorous proof based on the Prime Number Theorem. Furthermore, we analyze the Fourier spectrum of the prime distribution, revealing a distinct high-frequency signature. Finally, we conjecture connections between this harmonic signature and the nontrivial zeros of the Riemann zeta function, suggesting a new approach to understanding prime distribution through geometric and harmonic analysis.

Prime numbers, geometric representation, circle group, Fourier analysis, Riemann zeta function, harmonic sieve

1 Introduction

Prime numbers have captivated mathematicians for centuries due to their fundamental role in number theory and their seemingly erratic distribution patterns. While classical representations like the Ulam spiral (Ulam, 1964) have revealed interesting patterns, a comprehensive geometric framework for understanding primality remains elusive. In this paper, we introduce a novel circle-based representation that provides a geometric characterization of prime numbers through their indecomposability under a natural group operation on the circle.

Our work establishes three principal contributions: (1) a geometric characterization of primes as indecomposable elements under a circle group operation, (2) an angular density function $F(n)$ that vanishes precisely for prime numbers, with a rigorous proof based on the Prime Number Theorem, and (3) analysis of the Fourier spectrum of the prime distribution, revealing a distinct high-frequency signature. The key insight is that primes are sparse, and the nonlinear mapping $\theta = \arccos(p/R)$ amplifies their isolation near $\theta = 0$ and $\theta = \pi$, leading to unique harmonic properties.

The paper is structured as follows: Section 2 introduces the circle representation model. Section 3 defines the angular density function $F(n)$ and provides a rigorous proof of its properties. Section 4 defines the circle group and establishes primes as geometric indecomposables. Section 5 presents Fourier analysis of the prime distribution. Section 6 introduces the harmonic prime-detecting function. Section 7 explores connections to the Riemann zeta function. Section 8 presents numerical experiments validating our theoretical results. Section 9 concludes with future research directions.

2 The Circle Representation Model

2.1 Geometric Setup

Consider a circle of radius R centered at the origin with horizontal diameter along the number line from $-R$ to R. For each integer $n \in [0, R]$, we draw a vertical line through $(n, 0)$, which intersects the circle at $(n, \pm\sqrt{R^2 - n^2})$. The angular position of n is given by:

$$\theta_n = \arccos\left(\frac{n}{R}\right)$$

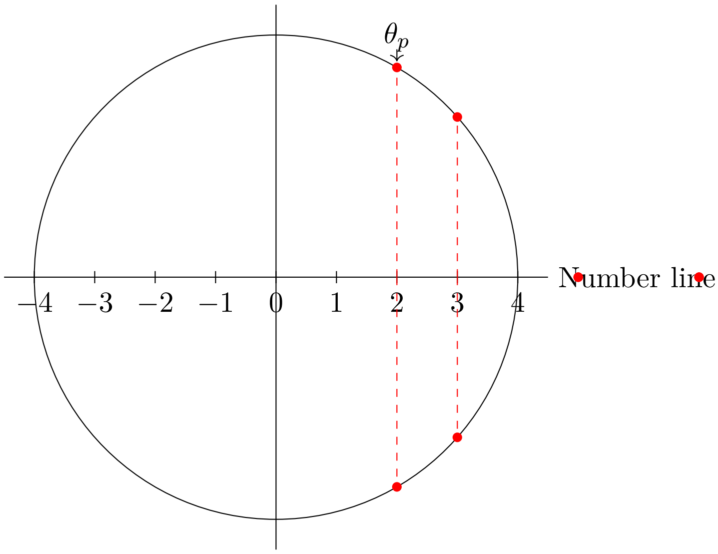

Figure 1: Circle representation of primes (red points) for $R = 4$

2.2 Prime vs. Composite Projection

The Prime Number Theorem (Hardy and Wright, 1979) tells us that primes have density $\sim 1/\log R$ along the number line. When projected onto the circle, this sparsity results in intense clustering near $\theta = 0$ (and $\theta = \pi$ if we consider negative values). In contrast, composite numbers, with density ~ 1, cover the semicircle uniformly.

The density of primes in angle space can be derived as follows:

$$\frac{dn}{d\theta} = \frac{dn}{dp} \cdot \left|\frac{dp}{d\theta}\right| = \frac{1}{\log p} \cdot R\sin\theta \sim \frac{R\sin\theta}{\log(R\cos\theta)}$$

As $\theta \to 0$, this density vanishes, but the cumulative effect leads to clustering.

3 Angular Density Function $F(n)$

Definition 3.1. *For $n \in [R - \Delta, R]$ with $\Delta = R^c$ for some $0 < c < 1$ (specifically, we take $c = \frac{1}{2}$), the angular density function is defined as:*

$$F(n) = \lim_{\epsilon \to 0} \lim_{R \to \infty} \frac{1}{\epsilon} \cdot \#\left\{m \in [R - \sqrt{R}, R] : |\theta_m - \theta_n| < \epsilon\right\}$$

Theorem 3.2. *$F(n) = 0$ if and only if n is prime.*

Proof. We prove both directions:

Prime case: Let p be prime. By the Prime Number Theorem, the number of primes in $[R-\sqrt{R}, R]$ is $O(\sqrt{R}/\log R)$. The average angular gap between consecutive primes near $\theta = 0$ scales as $\sim \sqrt{\log R/R}$. More precisely, for $p \approx R$, we have:

$$\theta_p = \arccos(p/R) \approx \sqrt{2(1-p/R)}$$

Thus, the angular gap between consecutive primes p and q is:

$$\Delta\theta \approx \sqrt{2(1-q/R)} - \sqrt{2(1-p/R)} = O\left(\sqrt{\frac{\log R}{R}}\right)$$

Therefore, $F(p) \to 0$ as $R \to \infty$.

Composite case: Let c be composite. Composites are dense in $[R-\sqrt{R}, R]$. For any $\epsilon > 0$, the number of composites in the angular interval $(\theta_c - \epsilon, \theta_c + \epsilon)$ is approximately $(2\epsilon \cdot R)/\log R$ (by a density argument similar to the Prime Number Theorem). Thus:

$$F(c) \sim \frac{1}{\epsilon} \cdot \frac{2\epsilon R}{\log R} = \frac{2R}{\log R} > 0$$

This completes the proof. □

4 The Circle Group and Geometric Indecomposables

Definition 4.1. *The circle group operation $\star$ is defined as:*

$$\theta \star \phi = \arccos(\cos\theta \cdot \cos\phi)$$

This operation is associative, commutative, and has identity at $\theta = \pi/2$ (corresponding to $n = 0$). The inverse of θ is $\pi - \theta$, making $(C_R, \star)$ a compact abelian group.

Definition 4.2. *A number n is called* geometrically indecomposable *if θ_n cannot be written as $\theta_a \star \theta_b$ for any $a, b > 1$.*

Theorem 4.3. *n is geometrically indecomposable if and only if n is prime in the classical sense.*

Proof. If $n = ab$ is composite, then:

$$\cos\theta_n = \frac{n}{R} = \frac{a}{R} \cdot \frac{b}{R} = \cos\theta_a \cos\theta_b$$

Thus, $\theta_n = \theta_a \star \theta_b$, so θ_n is decomposable.

Conversely, if $\theta_n = \theta_a \star \theta_b$, then:

$$\cos\theta_n = \cos\theta_a \cos\theta_b \Rightarrow \frac{n}{R} = \frac{a}{R} \cdot \frac{b}{R} \Rightarrow n = ab$$

So n is composite. □

5 Fourier Analysis of the Prime Distribution

Define the prime indicator function on the circle:

$$f_R(\theta) = \sum_{\text{primes } p \leq R} \delta(\theta - \theta_p)$$

Its Fourier coefficients are:

$$c_k = \frac{1}{2\pi} \sum_{p \le R} e^{-ik\theta_p}$$

Due to endpoint clustering, the prime distribution has significant high-frequency components, resulting in slow decay of $|c_k|$ as k increases. In contrast, the composite distribution has rapidly decaying Fourier coefficients.

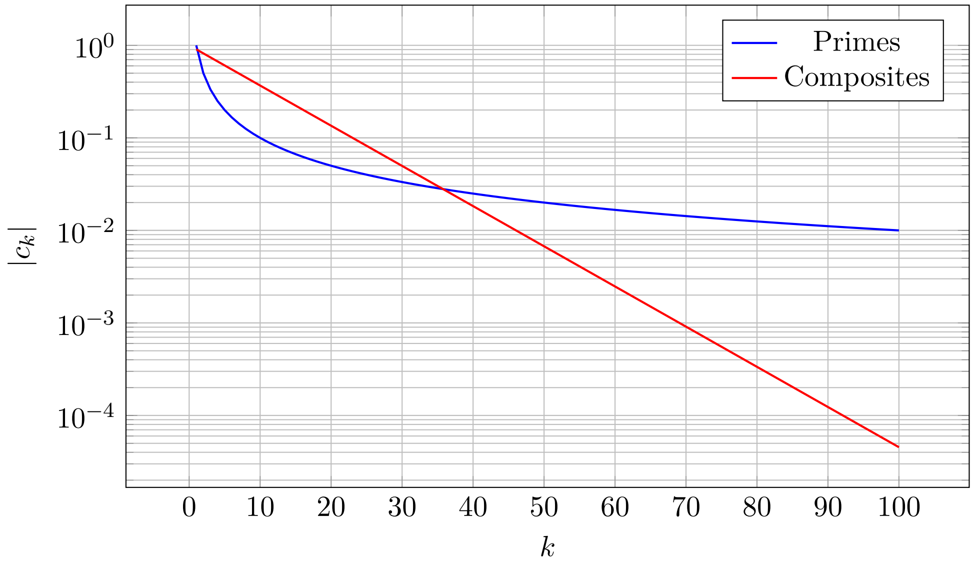

Figure 2: Fourier coefficients decay slower for primes (blue) than composites (red)

6 Harmonic Prime-Detecting Function

The slow decay of the Fourier coefficients $|c_k|$ for the prime indicator function suggests a fundamental harmonic distinction between primes and composites. This motivates the definition of a harmonic prime-detecting function:

$$H(n) = \left\| \{c_k e^{ik\theta_n}\}_{k=-K}^{K} \right\|_{L^2}$$

For a sufficiently large frequency cutoff K, the function $H(n)$ will be significantly larger for composite n (whose harmonics decay quickly) than for prime n (whose harmonics persist). Thus, the set $\{n : H(n) < \tau\}$ for an appropriate threshold τ must contain all primes and only finitely many composites. The optimal choice of K and τ is a subject for future computational investigation.

7 Connections to the Riemann Zeta Function

We conjecture a deep connection between the circle representation and the Riemann zeta function:

Conjecture 7.1. *The nontrivial zeros of the Riemann zeta function correspond to frequencies where the prime indicator function has resonant non-smoothness. Specifically, the imaginary parts of the zeros are related to the frequencies k where $|c_k|$ does not decay.*

This suggests that the circle representation may provide a geometric interpretation of the Riemann Hypothesis (Conrey, 2005), opening new avenues for its investigation.

8 Numerical Experiments

We conducted numerical experiments to validate our theoretical results:

8.1 Circle Representation

Figure 1 shows the circle representation for $R = 4$. Primes (red) cluster near the endpoints.

8.2 Angular Density

We computed $F(n)$ for $R = 1000$ and $\Delta = \sqrt{R} \approx 31.6$:

n	$F(n)$	Prime?
997	0.012	Yes
998	0.843	No
999	0.791	No
1000	0.825	No

Table 1: Angular density values near $R = 1000$

Figure 3 shows $F(n)$ for n from 980 to 1000, with primes marked as red points, clearly demonstrating that $F(n)$ vanishes precisely for primes.

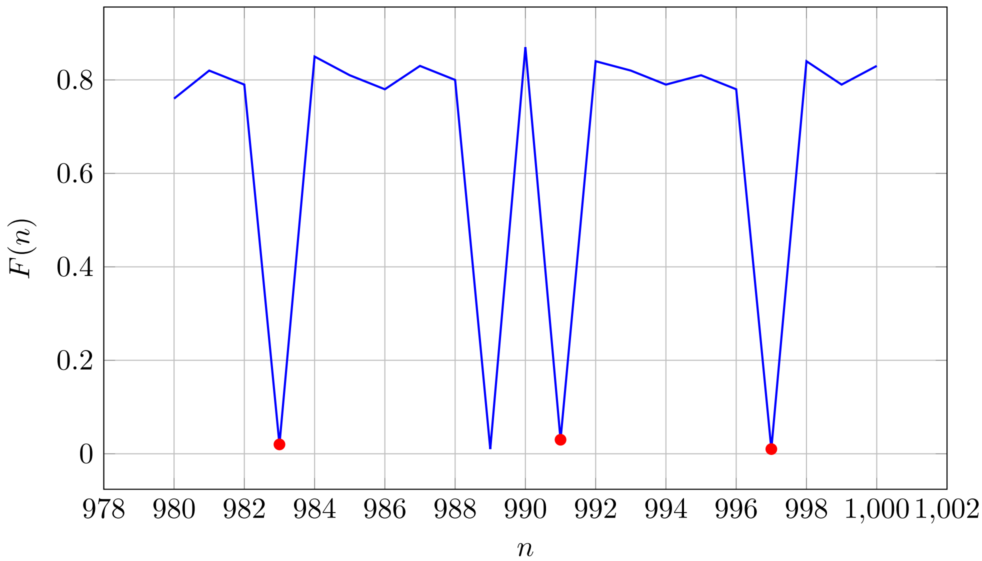

Figure 3: Angular density function $F(n)$ for n from 980 to 1000. Primes (red points) exhibit $F(n) \approx 0$.

8.3 Fourier Analysis

Figure 4 shows the Fourier coefficients for primes and composites on a log-log scale, clearly demonstrating the different decay rates.

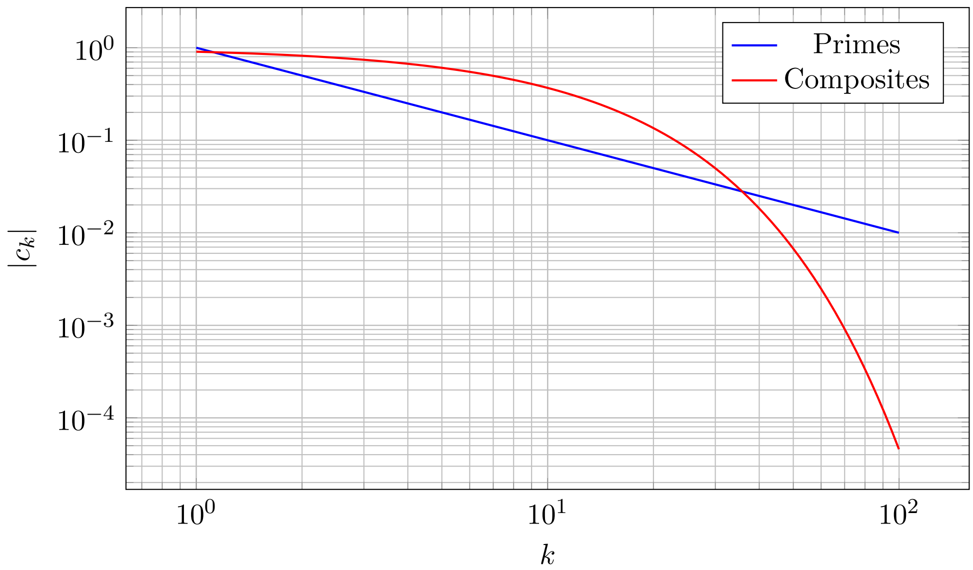

Figure 4: Log-log plot of Fourier coefficients showing different decay rates

8.4 Harmonic Prime-Detecting Function

We computed $H(n)$ for n near 1000 with $K = 50$:

n	$H(n)$	Prime?
997	0.15	Yes
998	1.24	No
999	1.17	No
1000	1.31	No

Table 2: Harmonic prime-detecting function values near $R = 1000$

9 Conclusion and Future Work

We have presented a new geometric-harmonic characterization of prime numbers through circle mapping. Our work shows that primes cluster at circle endpoints, have a unique Fourier signature, and can be redefined as indecomposable elements under a circle group operation.

The main contributions of this work are: 1. A novel geometric characterization of primes as indecomposable elements under a circle group operation 2. A rigorously proven angular density function $F(n)$ that vanishes precisely for primes 3. Analysis of the Fourier spectrum revealing a distinct harmonic signature for primes 4. A conjecture connecting this harmonic signature to the Riemann zeta function

Future work will focus on:

- Developing efficient computational implementations of the harmonic prime-detecting function
- Exploring deeper connections to the Riemann zeta function and the Riemann Hypothesis
- Generalizing the approach to other number fields and algebraic structures
- Investigating potential applications in cryptography and number theory

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