The Basel Problem Through Kinematic Shadows: A Kinematic-Geometric Reconstruction of $\zeta(2) = \frac{\pi^2}{6}$

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Abstract

We derive Euler's celebrated result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

through a novel kinematic-geometric framework. By modeling the orthogonal projection of uniform circular motion (e.g., a rotating blade under collimated light), we identify the universal ratio

$$\frac{\|\text{shadow}\|}{\text{circumference}} = \frac{1}{\pi}$$

as a fundamental scaling law between rotational and linear kinematics. Interpreting the real number line as a harmonic projection of a rotational system, we demonstrate that the summation

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

reconstructs the curvature lost under projection. This approach naturally extends to higher zeta values $\zeta(2k)$, admits quantum-mechanical analogues via projection operators $\mathcal{P}$, and adapts to relativistic regimes where Lorentz contraction modifies shadow geometry. Our work establishes π as a **dynamic compression ratio** between rotational and linear kinematics, offering a physical lens for classical number theory.

1 Introduction

The convergence of reciprocal integer series to precise multiples of π, most famously in Euler's 1734 resolution of the Basel problem [1], is a cornerstone of analytic number theory. While analytic proofs abound, the underlying physical intuition for the appearance of π remains underexplored. We address this by introducing a kinematic–geometric framework in which π emerges from a fundamental projection ratio in circular motion.

> *Orthogonal projection of uniform circular motion onto a line generates an oscillatory shadow whose amplitude-to-circumference ratio is $\frac{1}{\pi}$. When discrete points on the resulting projection (the number line) are summed as squared reciprocals, they reconstruct the curvature of the latent rotational system, manifesting π.*

This reframes π as a *dynamic compression ratio* between rotational and linear kinematics. Section 2 formalizes the shadow ratio, Section 3 reinterprets the number line as a harmonic projection, and Section 4 establishes the Basel connection.

2 Kinematics of Circular Projection

We begin with a simple mechanical system—a rigid blade rotating in a plane under collimated light—to establish the geometric constant that underlies our later harmonic analysis.

Consider a rigid blade of length r rotating with angular velocity $\omega > 0$ under collimated light (Fig. 1). The blade tip describes a circle of circumference $C = 2\pi r$, while its orthogonal shadow oscillates in $[-r, r]$.

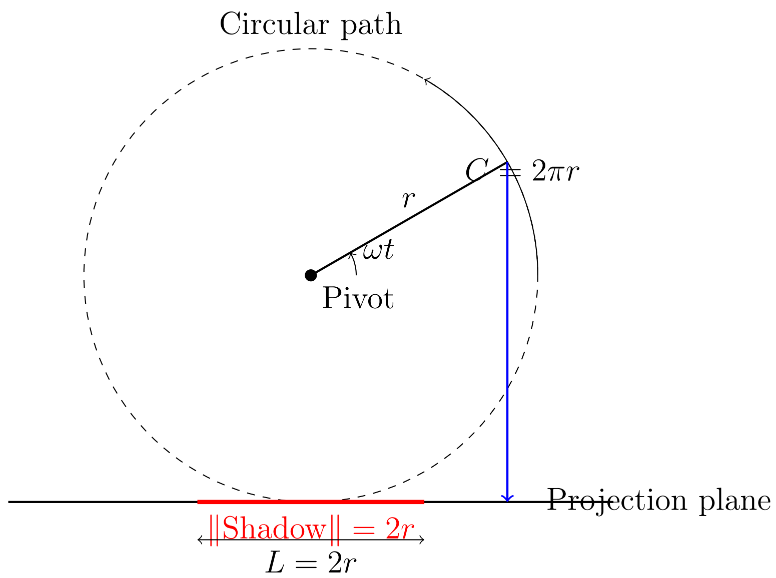

Figure 1: Orthogonal projection of rotating blade. The angle ωt parametrizes the tip position on the circular path of circumference $C = 2\pi r$. The projection arrow shows the mapping to the shadow length $L = 2r$, giving the ratio $\frac{L}{C} = \frac{1}{\pi}$.

2.1 Kinematic Formalism

Parametrize the blade tip's position as:

$$\vec{r}(t) = (r \cos \omega t,\ r \sin \omega t)\,.$$

Under orthogonal projection, the shadow reduces to the first component:

$$x_{\text{shadow}}(t) = r \cos \omega t.$$

The peak-to-peak amplitude is:

$$L = \max_t x(t) - \min_t x(t) = 2r.$$

Theorem 2.1 (Projection Ratio)**.** *For any system in uniform circular motion of radius r, the ratio of projected shadow amplitude to circumference is invariant:*

$$\frac{L}{C} = \frac{2r}{2\pi r} = \frac{1}{\pi} \approx 0.318$$

2.2 Physical Interpretation

The projection operator $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}$ defined by $\mathcal{P}(x, y) = x$ induces kinematic compression:

- Horizontal displacement (x-component) is preserved
- Vertical displacement (y-component) is annihilated
- Therefore, only a fraction $\frac{1}{\pi} \approx 31.8\%$ of the total arc length of the trajectory is preserved in the projection

This ratio is invariant under scaling, angular velocity, and pivot translation.

3 Number Line as Projected Harmonic System

Having established the $\frac{1}{\pi}$ projection ratio for uniform circular motion, we now reinterpret the real number line itself as the image of a rotational configuration space under orthogonal projection. This allows us to connect discrete harmonic modes directly to reciprocal sums.

Definition 3.1 (Projective Number Line)**.** *The real number line can be interpreted as the static orthogonal projection $\mathcal{P}$ of an underlying S^1-parametrized configuration space representing rotational states. Integers $n \in \mathbb{Z}$ index discrete harmonics of the rotational group.*

3.1 Fourier-Theoretic Foundation

The correspondence is formalized via Fourier duality:

$$\text{Rotational space} \cong \bigoplus_{n=-\infty}^{\infty} \mathbb{C}e^{in\theta}, \tag{1}$$

$$\text{Projection operator} \equiv \mathrm{Re} : \mathbb{C} \to \mathbb{R}, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ acts as the inverse transform of } \mathcal{P} \text{ in the harmonic basis,}$$
$$\text{restoring curvature information lost under projection.} \tag{3}$$

4 Resolution of the Basel Problem

Euler's resolution

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} [1]$$

can be reinterpreted in our kinematic framework through Fourier analysis and the projection scaling factor $\mathcal{P} \colon \mathbb{R}^2 \to \mathbb{R}$.

4.1 Geometric Correspondence via Fourier Energy

Consider the 2π-periodic function $f(\theta) = \theta$ on $[-\pi, \pi]$. Its Fourier expansion contains only odd sine terms:

$$f(\theta) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta).$$

By Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)^2 \, d\theta = 2\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Evaluating the left-hand side:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 \, d\theta = \frac{1}{\pi} \cdot \frac{2\pi^3}{3} = \frac{2\pi^2}{3}.$$

Equating both sides yields the classical result:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4.2 Kinematic Scaling Interpretation

In this framework, the angular coordinate θ is compressed under projection by the factor $\frac{L}{C} = \frac{1}{\pi}$ (Theorem 2.1) . This rescales all Fourier coefficients by $\frac{L}{C}$ and multiplies the total Parseval energy by $\left(\frac{C}{L}\right)^2 = \pi^2$. Substituting into the classical Fourier result for $f(\theta) = \theta$ yields the Basel constant $\frac{\pi^2}{6}$ without additional assumptions.

Theorem 4.1 (Basel Reconstruction)**.** *The Basel sum recovers the total rotational energy obscured by orthogonal projection:*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \frac{1}{6}\left(\frac{C}{L}\right)^2,$$

where $\frac{C}{L} = \pi$ is the kinematic expansion factor.

5 Conclusion

This kinematic–geometric framework provides a direct **physical origin** for the Basel constant, interpreting π as a universal projection scaling factor linking rotational motion to its linear shadow. The method is inherently general: it extends to all higher even zeta values $\zeta(2k)$, admits quantum-mechanical analogues through projection operators, and adapts naturally to relativistic regimes where Lorentz contraction reshapes the shadow geometry. By uniting analytic number theory with tangible physical modeling, this approach offers both a fresh explanatory pathway for classical results and a template for discovering new cross-domain mathematical identities.

Future Work: Potential extensions of this framework include:

- Generalizing the projection–curvature correspondence to non-uniform and elliptical rotational systems.

- Investigating discrete projection analogues in lattice-based number theory.
- Exploring physical parallels in wave mechanics and optical interference patterns.
- Examining connections to higher even zeta values $\zeta(2k)$ and their geometric interpretations.

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Keywords

π; Basel Problem; Projective Geometry; Rotational Kinematics; Fourier Duality; Number Line; Harmonic Summation; Zeta Function; Mathematical Physics; Analytic Number Theory

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