There Is No Matter: Vacuum Knots as Toroidal Resonant Solitons and the Role of $\alpha_U = k_e A_P$

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Abstract

We develop a mathematically grounded framework in which what is commonly called "matter" consists of finite–energy, knotted excitations of a nonlinear vacuum field. The model is based on a Faddeev–Skyrme–type Lagrangian for a unit vector field $n : \mathbb{R}^{3,1} \to S^2$, coupled to gravity and electromagnetism. Toroidal (Hopf) solitons carry conserved topological charge and exhibit internal "beating" modes. Mass, momentum and spin arise purely from the vacuum stress–energy tensor. The constant $\alpha_U = k_e A_P$ (Coulomb constant times Planck area) plays the role of a vacuum rigidity/constitutive scale, controlling EM couplings and regularization. We lay out explicit assumptions, definitions, energy bounds, and geodesic motion of soliton centers, with citations to classical results (Derrick's theorem, Vakulenko–Kapitanskii bound, moduli space dynamics) [1–8].

1 Setup, Units, and the Constant α_U

Let $\ell_P^2 = A_P = \dfrac{G\hbar}{c^3}$ be the Planck area, and $k_e = \dfrac{1}{4\pi\varepsilon_0}$. Define

$$\alpha_U \equiv k_e A_P = \frac{1}{4\pi\varepsilon_0} \frac{G\hbar}{c^3}. \tag{1}$$

Dimensions. In SI, $[k_e] = \mathrm{N\,m^2/C^2}$ and $[A_P] = \mathrm{m^2}$, hence $[\alpha_U] = \mathrm{N\,m^4/C^2}$. In *Planck units* ($\hbar = c = G = 4\pi\varepsilon_0 = 1$), one has $k_e = 1$, $A_P = 1$ and thus $\alpha_U = 1$ (dimensionless). Throughout, we keep SI bookkeeping but indicate where Planck units render parameters adimensional (see App. A).

2 Vacuum Field and Action

Assumption 1 (Vacuum field and regularity)**.** *There exists a smooth map $n : \mathbb{R}^{3,1} \to S^2$ with $|n| = 1$, approaching a constant at spatial infinity (finite energy class). The spacetime metric is $g_{\mu\nu}$ (signature $-+++$).*

Definition 1 (Action and Lagrangian density). *Consider*

$$S[g, n] = \frac{c^3}{16\pi G} \int R\sqrt{-g}\, d^4x \;+\; \int \mathcal{L}_{\text{vac}}(n, \partial n, g)\sqrt{-g}\, d^4x, \tag{2}$$

$$\mathcal{L}_{\text{vac}} = \frac{\mu^2}{2}\, \partial_\mu n \cdot \partial^\mu n \;-\; \frac{1}{4g^2}\, F_{\mu\nu}F^{\mu\nu} \;+\; \lambda\, \mathcal{V}(n), \tag{3}$$

where $F_{\mu\nu} \equiv n \cdot (\partial_\mu n \times \partial_\nu n)$ is the Faddeev–Skyrme term [3], μ, g, λ are positive constants, and $\mathcal{V}$ is a symmetry–breaking potential (optional).

Remark 1 (Why the Skyrme term). *By Derrick's theorem [1], a purely quadratic sigma model in 3 spatial dimensions cannot support nontrivial static finite–energy solitons; the quartic term $F_{\mu\nu}F^{\mu\nu}$ evades the scaling obstruction, enabling knotted solitons.*

Variation with respect to $g_{\mu\nu}$ gives Einstein's equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\, T_{\mu\nu}[n], \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{vac}}}{\delta g^{\mu\nu}}, \tag{4}$$

so the sole source is *the vacuum n* itself.

3 Finite Energy, Hopf Charge, and Energy Bounds

Assume Minkowski background for this section. Finite energy imposes $n(\mathbf{x}) \to n_\infty$ as $|\mathbf{x}| \to \infty$, compactifying $\mathbb{R}^3$ to S^3 and placing maps in $\pi_3(S^2) \cong \mathbb{Z}$.

Definition 2 (Hopf charge). *The Hopf charge of n is*

$$Q_H[n] = \frac{1}{32\pi^2} \int_{\mathbb{R}^3} \epsilon^{ijk} A_i F_{jk}\, d^3x, \qquad \partial_i A_j - \partial_j A_i = F_{ij}, \tag{5}$$

well–defined up to gauge of A; $Q_H \in \mathbb{Z}$ labels knotting/linking classes [6].

Theorem 1 (Vakulenko–Kapitanskii bound [2]). *There exists $C(\mu, g) > 0$ such that any finite–energy configuration obeys*

$$E[n] \;\geq\; C(\mu, g)\, |Q_H[n]|^{3/4}. \tag{6}$$

Proposition 1 (Existence & stability (conditional)). *For suitable μ, g, λ and admissible $\mathcal{V}$, the energy functional associated to (3) admits, in each $Q_H \neq 0$ sector, local minima represented by toroidal (knot) solitons ("Hopfions") that are dynamically stable under small perturbations.*[1]

4 Mass, Momentum, Spin from the Vacuum Tensor

Definition 3 (Mass, momentum, and spin). *For a static finite–energy solution n_0, define rest mass*

$$M = \frac{1}{c^2} \int_{\mathbb{R}^3} T_{00}[n_0]\, d^3x, \tag{7}$$

linear momentum $P_i = \int T_{0i}[n]\, d^3x$ and angular momentum

$$J_k = \epsilon_{kij} \int x_i\, T_{0j}[n]\, d^3x. \tag{8}$$

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[1]Full existence theorems for absolute minimizers in each Q_H sector are subtle; see numerical and analytical evidence in [3–6].

Proposition 2 (Mass and spin are vacuum integrals). *M, P_i, J_k depend only on $T_{\mu\nu}[n]$ from (4); no additional "matter" sector is required. Hence inertial and gravitational mass coincide by construction.*

5 Expanded Proofs: Derrick Scaling and the Vakulenko–Kapitanskii Bound

Throughout this section we work on $\mathbb{R}^3$ with finite–energy configurations $n : \mathbb{R}^3 \to S^2$ in the class $n(\mathbf{x}) \to n_\infty$ as $|\mathbf{x}| \to \infty$, and we set

$$X \;=\; \int_{\mathbb{R}^3} |\nabla n|^2 \, dx, \qquad Y \;=\; \int_{\mathbb{R}^3} |F|^2 \, dx, \qquad F_{ij} \equiv n \cdot (\partial_i n \times \partial_j n), \quad |F|^2 := \tfrac{1}{2} F_{ij} F_{ij}.$$

The (static) energy associated with (3) reads

$$E[n] \;=\; \frac{\mu^2}{2} X \;+\; \frac{1}{4g^2} Y. \tag{9}$$

5.1 Derrick scaling (necessity of the Skyrme term)

Theorem 2 (Derrick test in 3D, cf. [1]). *Let $E_2[n] = \frac{\mu^2}{2} \int |\nabla n|^2$ be the pure sigma–model energy in $d = 3$. For $n_\lambda(\mathbf{x}) := n(\mathbf{x}/\lambda)$ one has*

$$E_2[n_\lambda] \;=\; \lambda \, E_2[n].$$

Hence E_2 has no nontrivial stationary points under scaling; no static finite–energy solitons exist in $d = 3$ with the quadratic term alone.

Proof. Compute $\partial_i n_\lambda(\mathbf{x}) = (1/\lambda) \, \partial_i n(\mathbf{x}/\lambda)$. Then

$$E_2[n_\lambda] = \frac{\mu^2}{2} \int_{\mathbb{R}^3} \frac{1}{\lambda^2} |\nabla n(\mathbf{x}/\lambda)|^2 \, d^3x = \frac{\mu^2}{2} \int_{\mathbb{R}^3} \frac{1}{\lambda^2} |\nabla n(\mathbf{y})|^2 \, \lambda^3 d^3y = \lambda \, E_2[n].$$

Since $dE_2[n_\lambda]/d\lambda|_{\lambda=1} = E_2[n] \neq 0$ unless n is trivial, there is no stationary point. $\quad\square$

Remark 2. *For the Skyrme term $E_4[n] := \frac{1}{4g^2} \int |F|^2$, the same scaling gives $E_4[n_\lambda] = \lambda^{-1} E_4[n]$. Thus the full energy (9) scales as $E(\lambda) = A \lambda + B \lambda^{-1}$ with $A = \frac{\mu^2}{2} X$, $B = \frac{1}{4g^2} Y$, which* does *possess a finite minimizer at $\lambda_* = \sqrt{B/A}$, with*

$$E[n] \;\geq\; \min_{\lambda > 0} \{A\lambda + B\lambda^{-1}\} \;=\; 2\sqrt{AB} \;=\; \frac{\mu}{\sqrt{2}\, g} \sqrt{XY}. \tag{10}$$

This is the basic coercivity that stabilizes solitons against scaling.

5.2 The Vakulenko–Kapitanskii bound $E \gtrsim |Q_H|^{3/4}$

We now derive the standard lower bound $E \geq \kappa \, |Q_H|^{3/4}$ for the Faddeev–Skyrme energy in terms of the Hopf charge

$$Q_H[n] \;=\; \frac{1}{32\pi^2} \int_{\mathbb{R}^3} \epsilon^{ijk} \, A_i F_{jk} \, d^3x, \qquad \nabla \times \mathbf{A} = \mathbf{F}, \quad \nabla \cdot \mathbf{A} = 0, \tag{11}$$

where we view F_{ij} as the components of the *magnetic field* $\mathbf{F}$ and choose the Coulomb gauge for $\mathbf{A}$ (Biot–Savart inverse curl).

[Maxwell–Sobolev estimate] There exists a universal $C_S > 0$ such that, for the Coulomb gauge potential $\mathbf{A}$ associated to $\mathbf{F}$,

$$\|\mathbf{A}\|_{L^6(\mathbb{R}^3)} \ \leq \ C_S \|\mathbf{F}\|_{L^2(\mathbb{R}^3)}. \tag{12}$$

Sketch. In Coulomb gauge, $-\Delta\mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{F}$ in $\mathbb{R}^3$. Elliptic regularity and the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6$ imply $\|\mathbf{A}\|_{L^6} \leq C\|\nabla\mathbf{A}\|_{L^2} \leq C\|\mathbf{F}\|_{L^2}$. $\qquad\square$

[Interpolation] For $\mathbf{F} \in L^1 \cap L^2(\mathbb{R}^3)$,

$$\|\mathbf{F}\|_{L^{6/5}} \ \leq \ \|\mathbf{F}\|_{L^1}^{2/3} \|\mathbf{F}\|_{L^2}^{1/3}. \tag{13}$$

Proof. Use Riesz–Thorin (or Hölder): $1/(6/5) = 5/6 = (1-\theta)/1 + \theta/2$ with $\theta = 2/3$. $\quad\square$

[Pointwise algebraic bound] There exists $c_0 > 0$ such that $|F(\mathbf{x})| \leq c_0 |\nabla n(\mathbf{x})|^2$ pointwise, hence

$$\|\mathbf{F}\|_{L^1} \ \leq \ c_0 \int |\nabla n|^2 \ = \ c_0 \, X. \tag{14}$$

Proof. Since $F_{ij} = n \cdot (\partial_i n \times \partial_j n)$ and $|a \times b| \leq |a|\,|b|$, we have $|F_{ij}| \leq |\partial_i n|\,|\partial_j n|$. Summing over (i,j) yields $|F| \lesssim |\nabla n|^2$. $\qquad\square$

Theorem 3 (Vakulenko–Kapitanskii lower bound [2])**.** *There exists a constant $C_* > 0$ (depending only on universal Sobolev constants) such that*

$$|Q_H[n]| \ \leq \ C_* \, X^{2/3} \, Y^{2/3}. \tag{15}$$

Consequently,

$$E[n] \ \geq \ \frac{\mu}{\sqrt{2}\,g} \, C_*^{-3/4} \, |Q_H[n]|^{3/4}. \tag{16}$$

Proof. Start from (11) and apply Hölder:

$$|Q_H| \ \leq \ \frac{1}{32\pi^2} \int |\mathbf{A}||\mathbf{F}| \ \leq \ C \, \|\mathbf{A}\|_{L^6} \|\mathbf{F}\|_{L^{6/5}} \ \overset{(12),(13)}{\leq} \ C \, \|\mathbf{F}\|_{L^2} \|\mathbf{F}\|_{L^1}^{2/3} \|\mathbf{F}\|_{L^2}^{1/3}.$$

Thus $|Q_H| \leq C \, \|\mathbf{F}\|_{L^1}^{2/3} \|\mathbf{F}\|_{L^2}^{4/3}$. Using (14) and $\|\mathbf{F}\|_{L^2}^2 = Y$ gives

$$|Q_H| \ \leq \ C_* \, X^{2/3} \, Y^{2/3}.$$

Raise both sides to the $3/2$ power: $|Q_H|^{3/2} \leq C_*^{3/2} \, XY$. From (10) we have $E \geq \frac{\mu}{\sqrt{2}\,g}\sqrt{XY} \geq \frac{\mu}{\sqrt{2}\,g} C_*^{-3/4} |Q_H|^{3/4}$, proving (16). $\qquad\square$

Remark 3 (Constants and sharpness)**.** *The explicit value of C_* depends on the Sobolev constant in (12) and on the algebraic constant c_0 in (14); the exponent $3/4$ is sharp for this chain of inequalities. Equality is not attained (no minimizer saturates both Hölder and Sobolev simultaneously) but the bound is universally valid for finite–energy maps in the given homotopy class.*

5.3 Corollaries for the model with α_U

Remark 4. *The role of α_U in EM couplings* (17)–(18) *affects response coefficients but not the topological exponent. In particular, the* existence *of finite–energy knotted configurations and the $E \gtrsim |Q_H|^{3/4}$ bound are controlled by (μ, g) and Sobolev constants; α_U enters when translating these solutions into effective EM observables (polarizabilities, induced currents).*

6 Dynamics: Moduli Space and Geodesic Motion

Let $\{n(\mathbf{x}; \mathbf{a})\}$ be a family of near–minimizing configurations (collective coordinates $\mathbf{a}$: center, orientation, phase).

Assumption 2 (Adiabatic regime). *Internal modes are gapped relative to collective motion; radiation losses are negligible over the timescale of interest [6].*

Proposition 3 (Geodesic approximation [6,7]). *The effective dynamics of $\mathbf{a}(t)$ is geodesic motion on the moduli space with metric $\mathcal{G}_{ab} = \int d^3x \, \partial_a n \cdot \partial_b n$ (with Skyrme corrections), yielding Newtonian free motion at leading order in flat space. In curved $g_{\mu\nu}$, centers follow spacetime geodesics to leading order, realizing the equivalence principle.*

7 Electromagnetism: Two Consistent Routes and the Scale α_U

7.1 Emergent constitutive route

Define the 2–form

$$\mathcal{H}^{\mu\nu} \equiv \alpha_U \, F^{\mu\nu}[n], \qquad J^\mu \equiv \partial_\nu \mathcal{H}^{\mu\nu}. \tag{17}$$

Then J^μ is automatically conserved. In SI, α_U sets the strength (units in App. A); in Planck units, $\alpha_U = 1$.

7.2 Direct EM coupling

With Maxwell $F_{\mu\nu}[A]$, add

$$\mathcal{L}_{\text{int}} = \alpha_U \, F_{\mu\nu}[n] \, F^{\mu\nu}[A], \tag{18}$$

a gauge–invariant constitutive cross–term. This endows vacuum knots with EM response and effective "charges" via polarization; α_U acts as a vacuum rigidity scale.

8 Gravity from the Vacuum Only

From (4), the gravitational source is $T_{\mu\nu}[n]$; no independent matter term appears. Thus trajectories, lensing, and redshift are predictions of the vacuum configuration alone. Cosmological implications (effective Λ) can be tied to large–scale averaging or to an oscillatory "time–of–time" regulator (Appendix B).

9 Internal "Beating" Modes (Toroidal Resonances)

Linearizing $n(\mathbf{x}, t) = n_0(\mathbf{x}) + \epsilon \, \eta(\mathbf{x}, t)$ yields a self–adjoint spectral problem for the Hessian of $E[n]$ with Skyrme term. Two prominent families correspond to poloidal/toroidal deformations:

$$n(\mathbf{x}, t) \approx n_0(\mathbf{x} - \mathbf{v}t) + \epsilon \Big[u_\theta(\mathbf{x}) \cos \omega_\theta t + u_\phi(\mathbf{x}) \cos \omega_\phi t \Big], \qquad (19)$$

producing observable "beats" that modulate T_{00} and EM response (via (17)–(18)); see numerics in [4–6].

10 What Is (and Is Not) Proved

- **Proved/standard:** Derrick scaling obstruction [1]; existence of energy lower bounds [2]; geodesic approximation framework [6, 7].

- **Strong evidence:** Stable Hopf solitons in Faddeev–Skyrme numerics and variational studies [3–5].

- **Open/conditional:** Full global existence of minimizers for every Q_H with all interactions; exact spin/charge quantization (can invoke Finkelstein–Rubinstein constraints in future work).

11 Predictions and Falsifiability

Scattering of knots, excitation spectra $\omega_{\theta,\phi}$, EM polarizabilities (scaling with α_U), and gravitational geodesic motion of energy centers are all accessible in principle; see §A–B for scaling.

Conclusions

Within this framework, every observable attributed to "matter" (mass, momentum, spin, EM response, gravity) is a functional of a single nonlinear vacuum field. The constant α_U acts as a constitutive rigidity linking Planck geometry and EM scale. What remains is to nail down existence theorems in each topological sector and to match spectra quantitatively.

A Dimensions and Normalizations for α_U

In SI, $[\alpha_U] = \mathrm{N}\,\mathrm{m}^4/\mathrm{C}^2$. In Planck units ($\hbar = c = G = 4\pi\varepsilon_0 = 1$), charges are measured in $q_P = \sqrt{4\pi\varepsilon_0 \hbar c}$ so $k_e = 1$ and $A_P = 1$, hence $\alpha_U = 1$. Thus (17) and (18) become cleanly normalized.

B Time-of-Time Regulator and Effective Λ

If the clock variable is $T(t) = \alpha_U \sin \omega t + A_P \cos \omega t$, then $\langle \dot{T}^2 \rangle = \frac{\omega^2}{2}(\alpha_U^2 + A_P^2)$ provides a bounded regulator for vacuum fluctuations, suggestive of an effective cosmological term

sourced by vacuum rigidity rather than a naive mode sum [8]. A full backreaction treatment is deferred to future work.

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