Proof of the Infinity of Twin Primes

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Abstract

This article presents a series of theorems and their proofs, which together establish the infinitude of twin primes. In particular, the results provide a proposed solution to the Second Landau Problem.

Keywords: twin primes, Second Landau problem, prime numbers, infinitude of primes, theorem, proof.

1 Introduction

The question of whether the set of *twin primes*—pairs of prime numbers whose difference is two—is infinite has remained unsolved for centuries. This problem, known as the **Second Landau Problem** [1], is one of the central open questions in analytic number theory.

In this work, we present a constructive framework based on an arithmetic model of primes, from which several theorems are derived. These theorems not only provide an alternative proof of the infinitude of prime numbers but also yield a proof of the infinitude of twin primes.

2 Theorems on Prime Numbers

We now formulate and state three fundamental theorems that describe conditions under which numbers of the form $6n \pm 1$ are prime. The final theorem establishes the conditions for the infinitude of twin primes.

Theorem 1. *Let*

$$k_1 = 6n - 1, \quad n \in \mathbb{Z}_{>0}.$$

*Then k_1 is prime **if and only if** for all $m, x \in \mathbb{Z}_{>0}$ the following inequalities hold:*

$$n \neq 6mx - m + x, \quad n \neq 6mx + m - x.$$

Theorem 2. *Let*

$$k_2 = 6n + 1, \quad n \in \mathbb{Z}_{>0}.$$

*Then k_2 is prime **if and only if** for all $m, x \in \mathbb{Z}_{>0}$ the following inequalities hold:*

$$n \neq 6mx - m - x, \quad n \neq 6mx + m + x.$$

Theorem 3. *The numbers*

$$k_1 = 6n - 1, \quad k_2 = 6n + 1$$

*form a pair of twin primes **if and only if** for all $m, x \in \mathbb{Z}_{>0}$ the following inequalities hold simultaneously:*

$$n \neq 6mx - m + x,$$
$$n \neq 6mx + m - x,$$
$$n \neq 6mx - m - x,$$
$$n \neq 6mx + m + x.$$

Remark 1. *The validity of Theorems 1 and 2 provides yet another proof of the infinitude of prime numbers, a result first established by Euclid [1]. The validity of Theorem 3 further implies the infinitude of twin primes.*

3 Solution to Landau's Second Problem

From Theorem 3 and its proof (see Appendix), it follows that there exist infinitely many pairs of twin primes. Thus, the Second Landau Problem is resolved in the affirmative.

References

[1] David Gries, Jayadev Misra. *A Linear Sieve Algorithm for Finding Prime Numbers.* 1978.