

Avoiding negative numbers and complex numbers thanks to the study of the geometrical nature of some arithmetical and polynomial problems

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Abstract

In this paper we will see that we can avoid the concepts of negative number and complex number thanks to the study of the underlying vector nature of some arithmetic and polynomial problems. We will see that the geometrical models used until now to represent negative numbers and complex numbers and their operations are not just interpretations or models. Translations, rotations and homotheties are what we need to solve several problems. We will see that what we call "negative numbers" and "complex numbers" are just the solutions of vector calculations and equations. All that is the consequence of the fact that geometrical considerations are unavoidable when we think about debts and gains and when we try to solve some polynomial equations. Those considerations are linked to a geometrical system with symmetries and a center. We will see that thanks to the solutions of those vector equations we can construct paths in the plane. We will also give the vector meaning of the formulas of De Moivre and Euler. An interpretation of the vertical axis linked to gains and losses will also be given.

1 Introduction

The use of negative numbers began in China and India many centuries ago and they were used in Europe from the Middle Age (see [Smith2001]). The study of the equations in Europe led to the apparition of complex numbers because of the square root of negative numbers needed to solve some equations. Raffael Bombelli constructed those numbers in 1572 (see [Katz2009], p 404). Between the end of the 18th and the beginning of the 19th century, several mathematicians as Jean-Robert Argand or Gauss found geometrical interpretations of complex numbers. But, as for negative numbers, many mathematicians were not satisfied with an "imaginary number" or with "complex numbers" even if there are geometrical interpretations. In this paper we will see that actually there are no imaginary numbers nor complex numbers. We will see that several arithmetical and polynomial problems are in fact geometrical problems which lead to solutions that are geometrical operations applied to a unit vector. Those geometrical operations have been considered as complex numbers until now.

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2 Representing gains, losses and debts in Ancient China and India

The Chinese used rods with two colors in order to represent gains and losses, red rods for gains and black for losses. Chinese characters were used also to name the positive and negative quantities. The character 正 (zheng) was one of the ones used for positive quantities and had the meaning of "straight", "correct", and the character 負 (fu) was one of the ones used for negative quantities and implied the idea of "debt" (see [Martzloff2006], p. 202).

We can see the descriptions of the calculations in the "The Nine Chapters on the Mathematical Art" commented by Liu Hui (see [Kangshen2000]). There were rules for addition and subtraction as the nowadays rules (see see [Mumford]). For Liu Hui, "positive and negative numbers have a concrete representation as gains and deficits (de, shi)" (see [Martzloff2006], p. 202).

So they counted 1 unit of gain, two units of gain, etc. And 1 unit of loss, 2 units of loss, etc..

Jean-Claude Martzloff explains that those negative numbers were used only as computational intermediates (see [Martzloff2006], p. 200).

He explains also that "the Chinese from antiquity had a predilection for an analysis of all sorts of phenomena in terms of complementary couples, positive or negative" (see [Martzloff2006], p. 200). Symmetries were also important for them.

In India, negative numbers were used to represent debts (see [Smith2001], [Mattessich1998] and [Brahmagupta]). In India the zero also was introduced. An interpretation of negative numbers as coordinates on a number line occurs in the 12th century (see [Mumford], p. 125).

3 The symmetries with a center in the development of longitudes by the Ancient Greeks

Actually, in the 3rd century BC, Eratosthenes developed a system of latitude and longitude in which his prime meridian passed through Alexandria (see [WikipediaL]). This system was improved later by Hipparchus (2nd century BC) who used a coordinate system in which he took into account the fact of being on east or west. He used degrees and there was a primary meridian from which he determined the fact of being on east or on west (see [WikipediaL]).

So in Ancient Greece there was a development of a system for locating geographical places in which there was a center. And from that center they used units to express the distance from the center, taking into account the two sides, east and west.

4 The geometrical system underlying the calculation of losses and gains

So, there was an underlying idea of symmetry in China. There were losses, a state of equilibrium, and gains. That is geometrical, and more precisely it has to do with a geometrical system with symmetries and a center.

There was a geometrical underlying structure of negative numbers, but, as far as we know, Indians and Chinese were not aware of that. As we saw before, the use of negative numbers as coordinates in India was seen just as an "interpretation" later, in the 12th century.

In Ancient Greece there was a geographical system with a center and two sides, with symmetry, as we have seen before, but from that there was no a development of rules of calculation with

mathematical objects analogous to what has been developed later in China and India¹. It means that the geometry of symmetries with respect to a center was quite developed in Ancient Greece.

It is now clear that there is a link between negative numbers and the number line thanks to Descartes' representation of numbers. But in the number line, "negative numbers" are just representations. They are positions in the number line. More precisely, they are labels of points in the number line.

"-5" is a label, for example. It's not a really a number. We should talk about units of debt, for example: 1 unit of debt, 2 units of debt. etc. Those quantities of losses or gains can be represented as in figure 1:

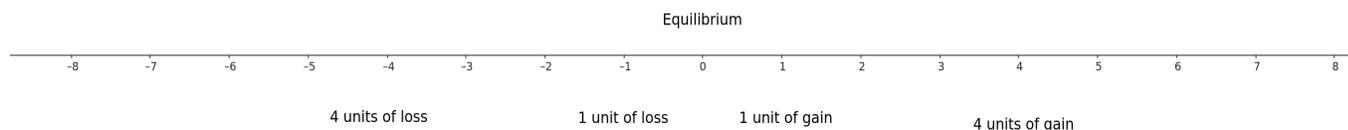


Figure 1: Representation of the number of units of gains and losses

So there are no really "negative numbers". We count, with natural numbers, the number of units of debts, or the numbers of units of length from the middle of the line to the left. And we use labels as "-5" to indicate a position on the line. We use it also for counting the units of debts. Obviously, we can also use magnitudes on the line. For exemple "-4.25" represents also a point on the line between the points represented by "-5" and "-4". And the calculations that we perform are in fact movements on that line from one point to another.

There has been studies about it made by cognitive scientists (see [LakoffNúñez2000]) who recall the underlying geometrical nature of negative numbers and the importance of that fact for education.

It is important to note also that until the XVIIIe century, many calculations involving numbers were linked to geometrical considerations. For example Euclid uses lines in his "Elements", books VII-X, when he presents elements of Number Theory (see [Euclid]). And before the algebrization of Calculus, the questions were closer to geometrical considerations about curves (see [LubetFriedelmeyer2014]).

To sum up, at the beginning human beings count material elements and develop the system of what we call now natural numbers. This is an abstraction. And, because of the need of the measure of lengths, volumes, etc., there are fractions. That is another abstraction. Fractions are also used in commerce. So, commerce was linked from the begining to gemetrical considerations. And then those numbers with fractions are used to work with losses and gains. Losses and gains are linked to the number line. Geographical considerations lead also to work with symmetries and a center. And the link with geometrical considerations is not lost. And, for debts and gains, it implies the use of a geometrical system with a line, a center and symmetries.

And this gemometrical system can be used also to work with measurements in physics (temperature, etc.).

¹During the 3rd century CE Diophantus wrote in his book "Arithmetica" that the equation $4x + 20 = 0$ was "absurd" because the solution would be "-4". In spite of the fact the he didn't develop a system as in China and India, he gave a rule of multiplication which allows to multiply, for example, $(x - 1)(x - 2)$ by saying that a number to be substracted multiplied by a number to be substracted gives a number to be added (see [Smith2001]).

So, that is why the complex numbers are linked to geometrical operations. In fact there are no complex numbers as we will see below. What we have are just geometrical operations who appear because of the use of negative labels when we try to solve certain polynomial equations. And we will see also that some of those operations lead to movements in a plane.

5 Working with negative labels and vectors

During the last 2 centuries, mathematicians have developed the techniques of the euclidean vector space. Those techniques with vectors have integrated the use of negative and positive numbers, with their different operations, in order to express movements and directions. We are going to use those techniques to show the resolution of some problems, but, as we saw above, instead of negative numbers we will talk about negative labels. We will use vectors to perform movements on the number line.

Obviously, the aim of the presentation of the vector techniques is to show the underlying nature of calculations for debts, gains, etc. It's clear that, for practical reasons, we can always work with negative and positive labels, as "-4" or "+5", with the well known arithmetical rules.

For elementary problems as "John has 5 apples and loses 3. How many apples does John have?" we have the following calculation, which corresponds to quantities of material objects:

$$5 - 3 = 2$$

For those calculations we use natural numbers. And with them we can subtract a smaller number from a bigger number.

A problem arise when we want to perform, for example, the following calculation:

$$150 - 200 = ?$$

This calculation corresponds, for example, to the following problem: John has \$150 and buys a machine for \$200. What is his financial situation after that? This problem has no solution with natural numbers. In fact John borrows from a bank \$50 to buy the machine. After that, he is in a financial situation of indebtedness with respect to the bank. And it in his bank account it is usually represented by "-50".

As we saw below "negative numbers" were introduced to perform calculations about debts. And John passed from a state of equilibrium with respect to the bank to state of indebtedness. And the mathematical description of the situation is intrinsically geometric.

So that problem can not be solved with natural numbers. In fact it is a geometrical problem. And we will solve it with the techniques of the euclidean vector space. Instead of the calculation above, we will perform the following calculation:

$$\begin{pmatrix} 150 \\ 0 \end{pmatrix} - \begin{pmatrix} 200 \\ 0 \end{pmatrix}$$

And this is the important point: we pass from a calculation involving natural numbers to a calculation involving points and vectors.

The calculation

$$150 - 200$$

is a schema and the vector equation will follow a part of his structure.

So, we pass from an arithmetical problem to a geometrical problem.

We will see more details about this calculation. The fact the he has, at the begining,\$150 correspond to a point in the line:

$$A = (150, 0)$$

Then he spends \$150 and \$50 (from the bank). So, first, we work now with a translation of A of 150 units towards the left side. For this translation T , we use a vector \vec{v} . In that vector, the component "-5" is a label, as we saw above:

$$\vec{OA} = \begin{pmatrix} 150 \\ 0 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} -150 \\ 0 \end{pmatrix}$$

So, in order to find $T_{\vec{v}}(A)$ we calculate:

$$\vec{OA} + \vec{v} = \begin{pmatrix} 150 \\ 0 \end{pmatrix} + \begin{pmatrix} -150 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So

$$T_{\vec{v}}(A) = (0, 0)$$

After that, he spends \$50 from the bank. We use a new translation towards the left:

$$\vec{w} = \begin{pmatrix} -50 \\ 0 \end{pmatrix}$$

So, in order to find $T_{\vec{w}}((0, 0))$ we calculate::

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \vec{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -50 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -50 \\ 0 \end{pmatrix}$$

We arrive at a point B on the line

$$B = (-50, 0)$$

So finally he has a debt of \$50. As we have seen before, he actually has 50 units of debt and it is represented also by the label "-50".

We have seen some arithmetical operations. For the multiplications, sometimes we have to perform the calculation:

$$3 \times (-4)$$

Obviously, there is no solution with natural numbers. It will become a vector calculation. "3" represents a point in the number line. Its distance to the origin is 3. We want to repeat or reproduce

4 times that distance, but in the opposite direction. This is the description of a homothety H . And we will use also a rotation (see [LakoffNúñez2000], p. 90). Firstly, we will have to perform a rotation of 180° of a translation vector. And then it will undergo a homothety of 4. So we will have to perform the following vectorial calculation:

$$C = (3, 0)$$

$$\vec{OC} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\lambda = 4$$

$$\vec{OD} = H_4 \circ R(180^\circ)(\vec{OC})$$

So we arrive at a point D on the number line:

$$D = (-12, 0)$$

6 Solving first-degree equations without natural solutions

Many equations can be solved using natural numbers. For example:

$$2x + 4 = 10$$

Its solution is $x = 3$.

But if we have the following equation:

$$10 + 3x = 4$$

we see that we will have to solve $3x = 4 - 10$. But $4 - 10$ can not be calculated with natural numbers.

So, with the schema of the first equation, we will have a vector equation. And the question will be:

Starting from the point $(10, 0)$, which geometrical operation will lead to the point $(4, 0)$?

So the solution won't be a number, but a geometrical operation.

As the addition appears in the first equation, we see that the operation involves a translation.

We also see that there is a factor 3, so the vector of the translation will undergo a homothety H_3 .

So we get the following vectorial equation:

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

So, after the vectorial operations we get:

$$\vec{x} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

And finally the solution is the following translation:

$$T_{H_3(\vec{x})}$$

So, to reach the point (4,0) from the point (10,0) we must to perform:

$$T_{H_3(\vec{x})}((10,0))$$

7 Solving an equation of degree 2 or higher with no natural solutions

Many equations of degree 2 or higher can be solved just using natural numbers. For example:

$$2 + x^2 = 11$$

In order to find its solution we must to calculate $\sqrt{9}$. So the solution is $x = 3$.

But when we work with negative labels and with vectors, we know that the multiplication of two negative labels is a positive label. We know that $(-3)(-3) = 9$. So (-3) is also a solution of that equation.

The underlying vector explanation is that we look for a geometrical operation, performed two times on a basis vector, which will give us a translation vector. And that translation vector will lead us from the point (2, 0) to the point (11, 0).

We get the following vectorial equation where X stands for an unknown geometrical operation:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} + X \circ X \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 11 \\ 0 \end{pmatrix}$$

So the basis vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ will undergo a homothety H_3 and then rotation of 180° . Our solution is:

$$X = R(180^\circ) \circ H_3$$

So the translation vector is obtained applying twice that solution to the basis vector:

$$X \circ X \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = X \left(\begin{pmatrix} -3 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

We have sometimes equations as the following one:

$$1 + x + x^2 = 0$$

When we try to calculate the discriminant we find:

$$\Delta = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$$

So there is no natural solution. But we can find a geometrical solution. And the question will be:

Starting from the point (1,0), which geometrical operation X will lead to the origin O by following the steps indicated by the vectorial equation?

We get the following vectorial equation where X stands for an unknown geometrical operation:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + X\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + X \circ X\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Here are the steps: the vector basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ undergoes the operation X and we get a translation vector used to reach a second point, then the same vector basis undergoes twice the operation X and we get another translation vector, and from the second point we reach the origin 0.

We know, thanks of two centuries of studies about that kind of equations, that the solution is linked to a composition of a rotation and a homothety. But what we are showing here is that the solution is not a complex number but a geometrical operation, and more precisely a composition of a homothety and a rotation. So X has the following form, where $\lambda > 0$ and $-180^\circ \leq \theta \leq 180^\circ$:

$$X = H_\lambda \circ R(\theta)$$

In order to solve that equation, we will firstly use a matrix, and we get:

$$X = \lambda \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

So the vectorial equation becomes:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So we get:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + \lambda^2 \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To continue, we must to solve a parametric equation:

$$\begin{cases} 1 + \lambda \cos(\theta) + \lambda^2 \cos(2\theta) = 0 \\ \lambda \sin(\theta) + \lambda^2 \sin(2\theta) = 0 \end{cases}$$

Knowing that $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$ and that $\cos(2\theta) = 2 \cos^2(\theta) - 1$, we get:

$$\begin{cases} 1 + \lambda \cos(\theta) + \lambda^2(2 \cos^2(\theta) - 1) = 0 \\ \lambda \sin(\theta) + \lambda^2 2 \sin(\theta) \cos(\theta) = 0 \end{cases}$$

So,

$$\begin{cases} 1 + \lambda \cos(\theta) + 2\lambda^2 \cos^2(\theta) - \lambda^2 = 0 \\ \lambda \sin(\theta) + 2\lambda^2 \sin(\theta) \cos(\theta) = 0 \end{cases}$$

Then, from the second equation we get:

$$\lambda \sin(\theta)(1 + 2\lambda \cos(\theta)) = 0$$

So, looking for non-trivial solutions, we solve:

$$1 + 2\lambda \cos(\theta) = 0$$

Then, working with negative labels:

$$\cos(\theta) = -\frac{1}{2\lambda}$$

We use that result in the first equation of the parametric equation and we get:

$$1 + \lambda\left(-\frac{1}{2\lambda}\right) + 2\lambda^2\left(-\frac{1}{2\lambda}\right)^2 - \lambda^2 = 0$$

So,

$$1 - \frac{1}{2} + \frac{1}{2} - \lambda^2 = 0$$

So,

$$1 - \lambda^2 = 0$$

As $\lambda > 0$, the only solution is $\lambda = 1$. This means that the operation X we are looking for has no homothety. It's just a rotation. and now we will find θ .

We saw above that

$$1 + 2\lambda \cos(\theta) = 0$$

So,

$$1 + 2 \cos(\theta) = 0$$

Then

$$\cos(\theta) = -\frac{1}{2}$$

So, the principal solutions are $\theta_1 = 120^\circ$ and $\theta_2 = -120^\circ$. And finally we have 2 solutions for X:

$$X_1 = R(120^\circ)$$

$$X_2 = R(-120^\circ)$$

But for the further developments, we will work rather with simple vectors and matrices in order to construct polynomials. And now we will use an analogue method to the completing the square. We take the vectorial equation seen above:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + X\begin{pmatrix} 1 \\ 0 \end{pmatrix} + X \circ X\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We rewrite it in the following way, knowing that X is a matrix:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + X \begin{pmatrix} 1 \\ 0 \end{pmatrix} + X^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then we rewrite it as this to have perfect square trinomial, where I_2 stands for the identity matrix of size 2:

$$\frac{3}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4} I_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + X \begin{pmatrix} 1 \\ 0 \end{pmatrix} + X^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thanks to the rules of distribution of matrix, we get:

$$\frac{3}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\frac{1}{2}I_2 + X)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So,

$$(\frac{1}{2}I_2 + X)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{3}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We rewrite it as:

$$(\frac{1}{2}I_2 + X)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Let $Y = \frac{1}{2}I_2 + X$, so we get the equation:

$$Y^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Thanks to what we have seen before, we know that Y must be a composition of a rotation and a homothety, such that if it is applied twice to the vector basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get $\frac{3}{4} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. So this time we need a rotation of 90° composed with a homothety $H_{\frac{\sqrt{3}}{2}}$, and also a rotation of -90° with a

homothety $H_{\frac{\sqrt{3}}{2}}$. So $Y_1 = H_{\frac{\sqrt{3}}{2}} \circ R(90^\circ)$.

So,

$$(\frac{1}{2}I_2 + X_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = H_{\frac{\sqrt{3}}{2}} \circ R(90^\circ) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

And then,

$$X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + H_{\frac{\sqrt{3}}{2}} \circ R(90^\circ) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

After the rotation of 90° of the vector basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we can rewrite it as:

$$X_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\sqrt{3}}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now we can describe the operation X_1 : it is the operation such that if it is applied to the vector basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get the vector $\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$.

As we know that $X_1 = H_\lambda \circ R(\theta_1)$, we will have:

$$\lambda = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\theta_1 = \arctan\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$$

So we will have:

$$\lambda = 1$$

$$\theta_1 = 120^\circ$$

With the solution X_1 we can represent the path from the the point $A(1,0)$ to the origin O. It can be also represented as a path from to origin to the origin, through the points A and B. This path is constructed with the addition of the vectors we get in each step of the vector equation. See the figure 2

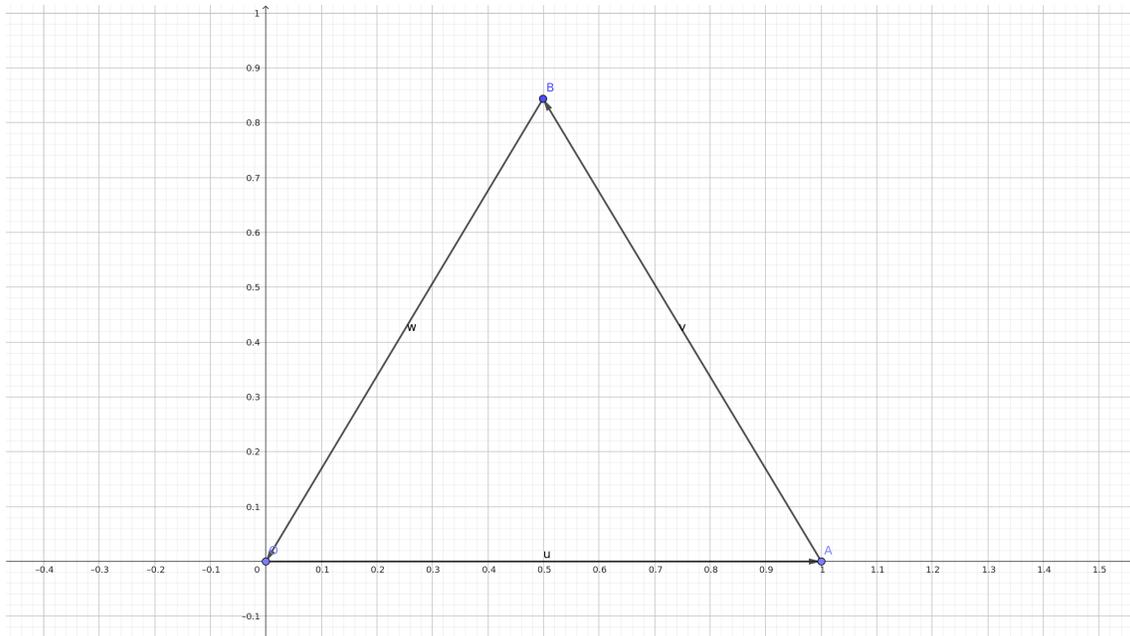


Figure 2: Representation of the path given by a solution to the vector equation

As we have seen before, there will be another solution X_2 with:

$$\lambda = 1$$

$$\theta_1 = -120^\circ$$

As we have seen, negative numbers have a geometrical underlying structure. That is why geometry appears again when we try to solve some polynomial equations.

8 The vector nature of the formulas of Euler and De Moivre

8.1 The formula of De Moivre

As we have seen that, if we apply an operation X , which is a rotation, to the vector basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get:

$$X\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = R(\theta)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

We can rewrite it as this:

$$X\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = R(\theta)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \cos(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now, if we apply n times the operation X to that vector basis, we get:

$$X^n\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = R^n(\theta)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \cos(n\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(n\theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And that corresponds to what De Moivre found with his formula:

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$$

In fact, the underlying nature of the complex number i is linked the vector basis $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The fact of multiplying $\cos(\theta) + i \sin(\theta)$ to itself n times produces $n-1$ rotations, and so a multiplication of θ by n .

8.2 The formulas of Euler

We know that:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{k!}x^k + \dots$$

This formula can be seen as a schema for a vector version. We will work with another function called Exp_v . If it is applied to the result of an operation X , we get:

$$Exp_v\left(X\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + X\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \frac{1}{2!}X^2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \frac{1}{3!}X^3\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \dots$$

We will apply that function to $X = H_\lambda \circ R(90^\circ)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$:

$$\begin{aligned}
Exp_v(H_\lambda \circ R(90^\circ))\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + H_\lambda \circ R(90^\circ)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \frac{1}{2!}H_\lambda^2 \circ R^2(90^\circ)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \frac{1}{3!}H_\lambda^3 \circ R^3(90^\circ)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \dots \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + H_\lambda\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + \frac{1}{2!}H_\lambda^2\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) + \frac{1}{3!}H_\lambda^3\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) + \frac{1}{4!}H_\lambda^4\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + \frac{1}{5!}H_\lambda^5\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + \dots \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -\lambda^2 \\ 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 \\ -\lambda^3 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} \lambda^4 \\ 0 \end{pmatrix} + \frac{1}{5!}\begin{pmatrix} 0 \\ \lambda^5 \end{pmatrix} + \frac{1}{6!}\begin{pmatrix} -\lambda^6 \\ 0 \end{pmatrix} + \dots \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + \begin{pmatrix} -\frac{1}{2!}\lambda^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{3!}\lambda^3 \end{pmatrix} + \begin{pmatrix} \frac{1}{4!}\lambda^4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{5!}\lambda^5 \end{pmatrix} + \begin{pmatrix} -\frac{1}{6!}\lambda^6 \\ 0 \end{pmatrix} + \dots \\
&= \begin{pmatrix} 1 - \frac{1}{2!}\lambda^2 + \frac{1}{4!}\lambda^4 - \frac{1}{6!}\lambda^6 + \dots \\ \lambda - \frac{1}{3!}\lambda^3 + \frac{1}{5!}\lambda^5 - \dots \end{pmatrix} \\
&= \begin{pmatrix} \cos(\lambda) \\ \sin(\lambda) \end{pmatrix} \\
&= \cos(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

So we have found the vector expression of what Euler found using ix :

$$e^{ix} = \cos(x) + i \sin(x)$$

As we saw before, the complex number i is linked the vector basis $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

And finally, if $\lambda = \pi$, we get with the vector function :

$$\begin{aligned}
Exp_v(H_\pi \circ R(90^\circ))\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= \cos(\pi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\pi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{aligned}$$

And that corresponds to what Euler found with his formula:

$$e^{i\pi} = -1$$

9 The meaning of the other number line

As we have seen, at the beginning negative labels were created to perform calculations with gains and debts. We have seen that those calculation are intrinsically geometric and we have found a second axis which is also a number line. That number line can be used to describe the gains and losses of a second agent. For exemple, John has \$100 and his spends little by little his money by buying objects in a store. When John losses money, the store gets the money. And there will be a

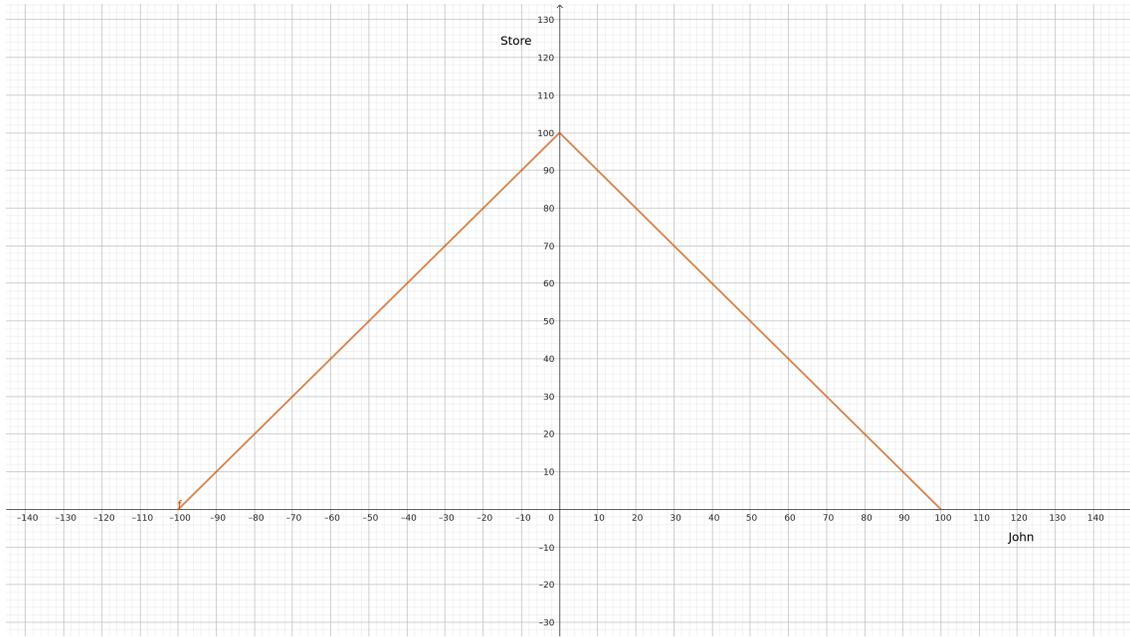


Figure 3: Representation of the gains and losses of 2 agents

moment in which John has no more money, and the store could give him some credit. But in doing that, the store losses money. That situation can be represented as in figure 3:

The other axis can be used to represent the situation in pole vaulting. The pole vaulter has a gain of height as he losses some meters from the standards above which there is the bar. When he is on the other side of the standards, he has a negative position from the standards and he losses height. That situation can be represented as in figure 4 (in meters):

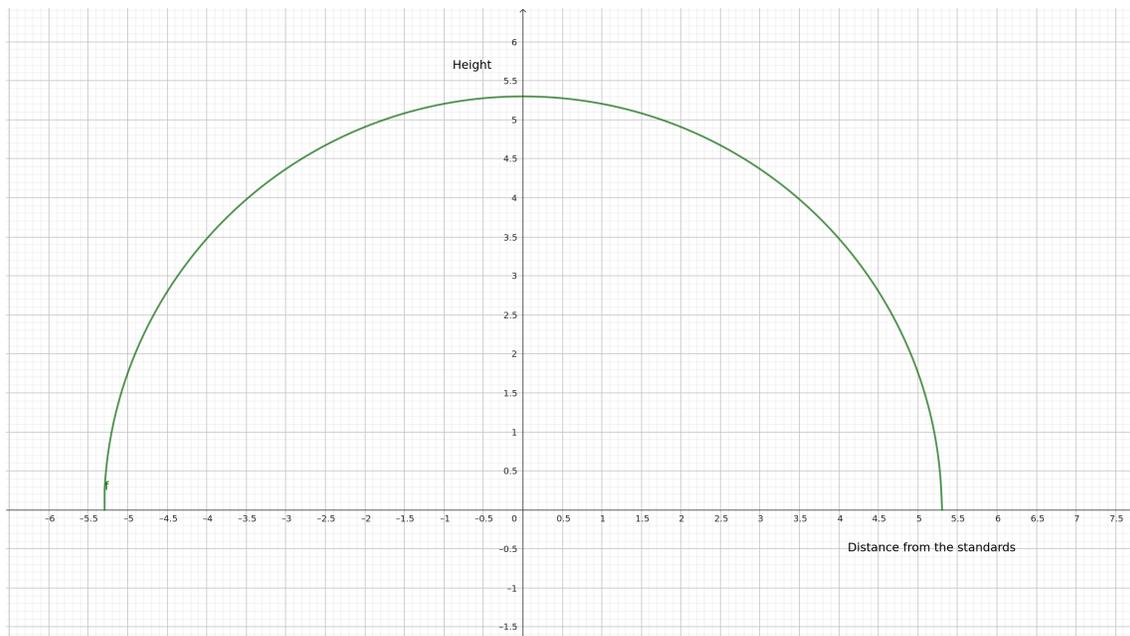


Figure 4: Representation of the gains and losses of distance and height

10 Conclusion

So we have seen that there are no negative numbers nor complex numbers. There are actually geometrical operations which are the solutions of certain arithmetical and polynomial problems. And that is a part of a geometrical system with symmetries and a center.

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