On a Volterra Analogue of Grunert's Operational Formula

Abdelhay Benmoussa
Anssis Primary School
Tiqqi, Agadir Ida Outanane 80324
Morocco
abdelhay.benmoussa@taalim.ma

Abstract

Let $V f(x) = \int_0^x f(t)\,dt$ denote the Volterra operator. We derive an explicit expansion for the iterated operator $(xV)^n$ in terms of powers of V:

$$(xV)^n = \sum_{k=0}^{n-1} (-1)^k a(n-1,k)\, x^{n-k} V^{n+k},$$

where $a(n, k)$ are the Bessel coefficients (OEIS A001498). This identity may be viewed as an integral analogue of the classical Grunert's operational formula

$$(xD)^n = \sum_{k=0}^{n} S(n,k)\, x^k D^k,$$

where $S(n, k)$ are the Stirling numbers of the second kind. We also obtain a closed integral representation for $(xV)^n$ and give two applications illustrating the operator identity.

1 Introduction

The Euler differential operator

$$xD, \qquad D = \frac{d}{dx},$$

plays a central role in special functions, combinatorics, and operational calculus. Its powers satisfy the classical Grunert formula [6, 1, 4]:

$$(xD)^n = \sum_{k=0}^{n} S(n,k)\, x^k D^k, \tag{1}$$

where $S(n, k)$ are the Stirling numbers of the second kind.

In this note, we pose a natural but, to the best of our knowledge, unexplored question: what happens to this expansion when the differentiation operator D is replaced by the Volterra operator

$$Vf(x) = \int_0^x f(t)\,dt?$$

To address this problem, we define the operator $(xV)^n$ recursively by

$$(xV)^0 f(x) := f(x), \qquad (xV)^n f(x) := x\,V\big((tV)^{n-1} f\big)(x), \quad n \geq 1.$$

Unlike differentiation, the Volterra operator increases the analytic order of a function through successive integrations. Hence, the structure of $(xV)^n$ differs substantially from that of $(xD)^n$. Nevertheless, one may anticipate an analogous combinatorial expansion:

$$(xV)^n = \sum_k c(n,k)\, x^{\alpha(n,k)} V^{\beta(n,k)},$$

for suitable coefficients $c(n,k)$ and integer exponents $\alpha(n,k)$, $\beta(n,k)$. Surprisingly, these coefficients are not related to Stirling numbers but to a classical sequence known as the *Bessel coefficients* (OEIS A001498; see also [2, 3]). They arise naturally when iterated integrations are analyzed in the same spirit as iterated differentiations. We recall their basic properties in the next section and use them to express $(xV)^n$ in a closed operational form.

2 Bessel coefficients and main result

The *Bessel coefficients* $a(n,k)$ are the coefficients of the classical *Bessel polynomials* $y_n(x)$:

$$y_n(x) = \sum_{k=0}^{n} a(n,k)\, x^k = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \left(\frac{x}{2}\right)^k,$$

so that

$$a(n,k) = \frac{(n+k)!}{2^k k!(n-k)!}, \qquad 0 \leq k \leq n.$$

They satisfy the recurrence relation [3]:

$$a(n,k) = a(n-1,k) + (n-k+1)\, a(n,k-1), \qquad a(0,0) = 1, \tag{2}$$

and the symmetry property

$$a(n,n-1) = a(n,n), \qquad n \geq 1. \tag{3}$$

$n\backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3	3			
3	1	6	15	15		
4	1	10	45	105	105	
5	1	15	105	420	945	945

Table 1: Values of the Bessel coefficients $a(n,k)$ for $0 \le k \le n \le 5$.

Main result

Theorem 1 (Benmoussa's operational formula)**.** *For every integer $n \ge 1$,*

$$(xV)^n = \sum_{k=0}^{n-1} (-1)^k a(n-1,k)\, x^{n-k}\, V^{n+k}, \tag{4}$$

where V^m denotes the m-fold composition

$$V^0 f(x) := f(x), \qquad V^{m+1} f(x) := V\big(V^m f\big)(x). \tag{5}$$

Unlike differentiation, the integral operator V^m admits the classical Cauchy formula for repeated integration:

$$V^m f(x) = \int_0^x \frac{(x-t)^{m-1}}{(m-1)!}\, f(t)\, dt.$$

Substituting into (4) gives a closed-form integral representation:

$$(xV)^n f(x) = \int_0^x \left(\sum_{k=0}^{n-1} \frac{(-1)^k}{2^k k! (n-1-k)!}\, x^{n-k} (x-t)^{n-1+k} \right) f(t)\, dt \tag{6}$$

$$= \frac{x}{2^{n-1}(n-1)!} \int_0^x (x^2 - t^2)^{n-1}\, f(t)\, dt \tag{7}$$

Hence $(xV)^n$ is a Volterra-type integral operator with kernel

$$K(x,t) = \frac{x(x^2-t^2)^{n-1}}{2^{n-1}(n-1)!}.$$

3 Proof

Lemma 2. *For all $m, m' \in \mathbb{N}$ with $m' \ge 1$,*

$$V\big(t^m V^{m'}(f)(t)\big)(x) = \sum_{j=0}^{m} (-1)^j (m)_j\, x^{m-j}\, V^{m'+j+1}(f)(x), \tag{8}$$

where $(m)_j = m(m-1)\cdots(m-j+1)$ is the falling factorial.

Proof. By induction on m. For $m = 0$, (8) reduces to $V(V^{m'} f)(x) = V^{m'+1} f(x)$, consistent with (5). Assume true for m. Using integration by parts:

$$V\big(t^{m+1} V^{m'}(f)(t)\big)(x) = x^{m+1} V^{m'+1}(f)(x) - (m+1)\, V\big(t^m V^{m'+1}(f)(t)\big)(x),$$

the induction hypothesis extends the formula to $m + 1$. □

Theorem 3. *For all $n \geq 1$,*

$$(xV)^n f(x) = \sum_{k=0}^{n-1} (-1)^k a(n-1, k)\, x^{n-k}\, V^{n+k} f(x).$$

Proof. Induction on n. The base case $n = 1$ holds trivially: $(xV)f(x) = x\, V f(x)$.

Assume the formula holds for some $n \geq 1$:

$$(tV)^n f(t) = \sum_{k=0}^{n-1} (-1)^k a(n-1, k)\, t^{n-k}\, V^{n+k} f(t).$$

Then

$$\begin{aligned}(xV)^{n+1} f(x) &= x\, V\big((tV)^n(f)(t)\big)(x) \\ &= x \sum_{k=0}^{n-1} (-1)^k a(n-1, k) V\left[t^{n-k} V^{n+k}(f)(t)\right](x).\end{aligned}$$

Applying Lemma 2 and reindexing gives

$$(xV)^{n+1} f(x) = \sum_{i=0}^{n} (-1)^i a(n, i)\, x^{n+1-i}\, V^{n+1+i} f(x),$$

where the combinatorial identity

$$a(n, i) = \sum_{k=0}^{\min(n-1, i)} (n-k)_{i-k}\, a(n-1, k) \tag{9}$$

can be proven from the recurrence (2) and symmetry (3). □

4 Two applications

In this section, we illustrate the operator identity (4) by evaluating $(xV)^n$ on two classical test functions: power functions and the exponential function. Throughout this section we assume $x > 0$.

Power functions

We first compute $(xV)^n(t^{\alpha-1})(x)$ for $\alpha > 0$ using two different approaches.

Method 1: Closed-form kernel representation.

From the kernel formula (7), we have

$$(xV)^n(t^{\alpha-1})(x) = \frac{x}{2^{n-1}(n-1)!}\int_0^x (x^2-t^2)^{n-1}t^{\alpha-1}\,dt.$$

Setting $t = xu^{1/2}$, so that $dt = \frac{x}{2}u^{-1/2}du$, gives

$$\begin{aligned}
(xV)^n(t^{\alpha-1})(x) &= \frac{x^{\alpha+2n-1}}{2^n(n-1)!}\int_0^1 (1-u)^{n-1}u^{\frac{\alpha}{2}-1}\,du \\
&= \frac{x^{\alpha+2n-1}}{2^n(n-1)!}\,\mathrm{B}\Big(n,\frac{\alpha}{2}\Big) && \text{(by the Beta function)} \\
&= \frac{x^{\alpha+2n-1}}{2^n(n-1)!}\,\frac{\Gamma(n)\Gamma(\alpha/2)}{\Gamma(n+\alpha/2)} \\
&= \frac{x^{\alpha+2n-1}\Gamma(\alpha/2)}{2^n\,\Gamma(n+\alpha/2)}.
\end{aligned}$$

Method 2: Series expansion.

From the operator identity (6),

$$\begin{aligned}
(xV)^n(t^{\alpha-1})(x) &= \sum_{k=0}^{n-1}\frac{(-1)^k a(n-1,k)}{(n+k-1)!}\,x^{n-k}\int_0^x (x-t)^{n+k-1}t^{\alpha-1}\,dt \\
&= \sum_{k=0}^{n-1}(-1)^k a(n-1,k)\,x^{n-k}\frac{\Gamma(n+k)\Gamma(\alpha)}{\Gamma(\alpha+n+k)} \\
&= x^{\alpha+2n-1}\Gamma(\alpha)\sum_{k=0}^{n-1}\frac{(-1)^k a(n-1,k)}{\Gamma(\alpha+n+k)}.
\end{aligned}$$

Equating the two results yields the identity

$$\frac{\Gamma(\alpha/2)}{2^n\,\Gamma(\alpha)\,\Gamma(n+\alpha/2)} = \sum_{k=0}^{n-1}\frac{(-1)^k a(n-1,k)}{\Gamma(\alpha+n+k)}. \tag{10}$$

Exponential function

Next we evaluate $(xV)^{n+1}(e^t)(x)$. Using (6),

$$(xV)^{n+1}(e^t)(x) = \sum_{k=0}^{n}\frac{(-1)^k a(n,k)}{(n+k)!}\,x^{n+1-k}\int_0^x (x-t)^{n+k}e^t\,dt.$$

The integral evaluates as

$$\int_0^x (x-t)^{n+k} e^t \, dt = e^x \, \gamma(n+k+1, x),$$

where γ is the lower incomplete gamma function. Hence,

$$\begin{aligned}(xV)^{n+1}(e^t)(x) &= e^x \sum_{k=0}^{n} \frac{(-1)^k a(n,k)}{(n+k)!} \, x^{n+1-k} \, \gamma(n+k+1, x) \\ &= e^x \sum_{k=0}^{n} (-1)^k a(n,k) \, x^{n+1-k} \left(1 - e^{-x} \sum_{j=0}^{n+k} \frac{x^j}{j!}\right).\end{aligned}$$

Expanding yields

$$\begin{aligned}(xV)^{n+1}(e^t)(x) &= e^x \sum_{k=0}^{n} (-1)^k a(n,k) \, x^{n+1-k} - \sum_{k=0}^{n} (-1)^k a(n,k) \, x^{n+1-k} \sum_{j=0}^{n+k} \frac{x^j}{j!} \\ &= e^x \sum_{k=0}^{n} (-1)^k a(n,k) \, x^{n+1-k} - \sum_{k=0}^{n} (-1)^{n-k} \frac{(2(n-k)-1)!!}{(2k)!!} \, x^{2k+1}, \qquad (11)\end{aligned}$$

The simplification of the double sum in the second line follows from a computation given in [5].

Evaluating at $x = 1$ yields a Dobiński-type identity for the sequence $a(n) =$ A000806:

$$a(n) = \sum_{k=0}^{n} (-1)^k a(n,k) = \frac{1}{e} \left[(xV)^{n+1}(e^t)(1) + \sum_{k=0}^{n} (-1)^{n-k} \frac{(2(n-k)-1)!!}{(2k)!!} \right].$$

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