The Mirror Wave Function of Prime Numbers: An Unexpected Discovery

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(Revised and Corrected Version)

July 31, 2025

Abstract

We explore the application of the **Discrete Fourier Transform (DFT)** to a class of modular wave functions. For an integer $q \geq 2$, a Dirichlet character χ modulo q, and an integer p coprime to q, we introduce the modular wave function $\psi_p(x) = \chi(p)e^{i\frac{2\pi p^{-1}}{q}x}$, where p^{-1} is the modular inverse of p modulo q.

We rigorously demonstrate that the DFT of $\psi_p(x)$ is a **Kronecker delta peak** with a value of $\chi(p)\sqrt{q}$, located precisely at the frequency $k = p^{-1} \pmod q$, and zero everywhere else. This result illustrates a direct and elegant connection between modular inverses and spectral analysis, showing how arithmetic structures can be encoded and detected using signal processing tools.

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1 Introduction

Number theory and harmonic analysis meet in the study of Dirichlet L-functions, whose non-trivial zeros are related to the distribution of prime numbers. Here, we introduce a modular wave function, $\psi_p(x)$, which encodes the modular properties of an integer p coprime to a modulus q.

Its Discrete Fourier Transform reveals a spectacular property: a peak with an amplitude of $|\chi(p)|\sqrt{q}$ at $k = p^{-1} \pmod q$. This "mirror symmetry" suggests a new approach to probing prime numbers via spectral tools, with potential implications in cryptography and number theory.

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2 Definitions and Main Theorem

Let $q \geq 2$ be an integer. The **Discrete Fourier Transform (DFT)** of a function $f(x)$ for $x \in \{0, 1, \ldots, q-1\}$ is defined by:

$$\tilde{f}(k) = \frac{1}{\sqrt{q}} \sum_{x=0}^{q-1} f(x) e^{-i\frac{2\pi k}{q}x}, \quad k \in \{0, 1, \ldots, q-1\}.$$

The modular wave function is defined as:

$$\psi_p(x) = \chi(p) e^{i\frac{2\pi p^{-1}}{q}x},$$

where χ is a primitive Dirichlet character modulo q, p is an integer such that $\gcd(p, q) = 1$, and p^{-1} is the modular inverse of p modulo q.

[Spectral Localization Theorem] For any $q \geq 2$, primitive Dirichlet character χ modulo q, and integer p with $\gcd(p, q) = 1$, the DFT $\tilde{\psi}_p(k)$ is non-zero if and only if $k \equiv p^{-1} \pmod{q}$. More precisely:

$$\tilde{\psi}_p(k) = \begin{cases} \chi(p)\sqrt{q} & \text{if } k \equiv p^{-1} \pmod{q} \\ 0 & \text{if } k \not\equiv p^{-1} \pmod{q} \end{cases}$$

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3 Proof

Consider the DFT:

$$\tilde{\psi}_p(k) = \frac{\chi(p)}{\sqrt{q}} \sum_{x=0}^{q-1} e^{i\frac{2\pi}{q}(p^{-1}-k)x}$$

Case 1: $k \equiv p^{-1} \pmod{q}$ In this case, the integer $p^{-1} - k$ is a multiple of q. Consequently, the exponent $i\frac{2\pi}{q}(p^{-1} - k)x$ is a multiple of $i2\pi$, which implies that $e^{i\frac{2\pi}{q}(p^{-1}-k)x} = 1$. The sum is then:

$$\sum_{x=0}^{q-1} 1 = q.$$

Thus, we have:

$$\tilde{\psi}_p(p^{-1}) = \frac{\chi(p)}{\sqrt{q}} \cdot q = \chi(p)\sqrt{q}.$$

The amplitude of this peak is $|\tilde{\psi}_p(p^{-1})| = |\chi(p)\sqrt{q}| = \sqrt{q}$, since $|\chi(p)| = 1$.

Case 2: $k \not\equiv p^{-1} \pmod{q}$ The sum is a geometric series of q terms with a ratio $r = e^{i\frac{2\pi}{q}(p^{-1}-k)}$. Since $k \not\equiv p^{-1} \pmod{q}$, the ratio $r \neq 1$. The sum is given by the formula:

$$\sum_{x=0}^{q-1} r^x = \frac{1-r^q}{1-r}.$$

Let's calculate the numerator:

$$1 - r^q = 1 - \left(e^{i\frac{2\pi}{q}(p^{-1}-k)}\right)^q = 1 - e^{i2\pi(p^{-1}-k)}.$$

Since $(p^{-1} - k)$ is an integer, $e^{i2\pi(p^{-1}-k)} = 1$. The numerator is therefore $1 - 1 = 0$. As the denominator $1 - r$ is non-zero, the entire sum is equal to zero.

$$\tilde{\psi}_p(k) = \frac{\chi(p)}{\sqrt{q}} \cdot 0 = 0.$$

This completes the proof.

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4 Numerical Examples

We validate the theorem with two corrected cases:

1. **Case** $q = 5, p = 2$: Here, the modular inverse is $p^{-1} = 3$ (since $2 \cdot 3 = 6 \equiv 1 \pmod{5}$). The quadratic character gives $\chi(2) = (\frac{2}{5}) = -1$. According to the theorem, the DFT has a peak at $k = 3$ with a value of $\tilde{\psi}_2(3) = \chi(2)\sqrt{5} = -\sqrt{5} \approx -2.236$. For all other values of $k \in \{0, 1, 2, 4\}$, the DFT is rigorously zero.

2. **Case** $q = 13, p = 7$: Here, the modular inverse is $p^{-1} = 2$ (since $7 \cdot 2 = 14 \equiv 1 \pmod{13}$). For the quadratic character, $\chi(7) = (\frac{7}{13}) = -1$. The theorem predicts a peak at $k = 2$ with a value of $\tilde{\psi}_7(2) = \chi(7)\sqrt{13} = -\sqrt{13} \approx -3.606$. The DFT is zero for all other values of k.

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5 Discussion

5.1 Sensitivity to γ

The precise choice of the frequency $\gamma = \frac{2\pi p^{-1}}{q}$ is crucial. If γ is perturbed by an error term, for example $\gamma = \frac{2\pi p^{-1}}{q} + \epsilon$, the spectral peak would lose amplitude and broaden, as predicted by Fourier analysis. The signal would become a "sinc" function rather than a single peak.

5.2 Link with the Zeros of L-functions

The suggested links with the zeros of Dirichlet L-functions are highly speculative and not demonstrated by the theorem presented here. The result remains an elegant and direct application of the DFT to a modular function.

5.3 Potential Applications

This spectral localization could be exploited in cryptography (e.g., for detecting modular inverses) or in spectral analysis to study the properties of prime numbers via signal processing techniques, by searching for specific frequency signatures.

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6 Conclusion

The modular wave function $\psi_p(x) = \chi(p)e^{i\frac{2\pi p^{-1}}{q}x}$ reveals an unexpected symmetry in its DFT, with a unique and perfectly localized peak at $k = p^{-1} \pmod q$. The corrected examples confirm this rigorous result. By using the fundamental principles of Fourier analysis, we have demonstrated how an arithmetic property like the modular inverse can be encoded and spectrally detected.

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7 Popularized Description

Imagine a prime number detector that works like a cosmic radio. Prime numbers, like 2 or 7, are special numbers. In our study, we created a mathematical function, called the "mirror wave function," which associates a kind of vibratory signal with each prime number p. This signal is a wave that rotates at a very precise speed, determined by the modular inverse of the prime number.

When we analyze this signal with the Discrete Fourier Transform (DFT), it's like tuning a radio dial to pick up a specific frequency. For each prime number p and a certain modulus q (for example, $q = 5$ or $q = 13$), the signal emitted by p produces a "sound" (or a peak) on only one unique frequency. This frequency corresponds exactly to its modular inverse, denoted p^{-1}.

For example, for the prime number $p = 2$ with the modulus $q = 5$, the modular inverse is 3 (because $2 \times 3 = 6$, which is equivalent to 1 when you only count up to 5). The Fourier analysis of our signal produces a peak only on frequency 3. All other frequencies are silent. For the prime number $p = 7$ and the modulus $q = 13$, the inverse is 2 (because $7 \times 2 = 14 \equiv 1 \pmod{13}$), and the peak is on frequency 2.

The magic of this process is that the strength of this peak is always the same for all prime numbers of the same modulus, which shows a hidden uniformity. This "spectral music" is not a revolutionary discovery, but an elegant application of existing mathematics, showing that even the most fundamental properties of numbers can be seen in a new light through tools from physics and signal processing. It's like using a very powerful microscope to observe patterns that were always there, but that we had never thought to look for in this way.