Introduction to arithmetic

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Abstract. This is an introduction to numbers, fractions, percentages and arithmetic. We first discuss what can be done with integers and their quotients, namely basic arithmetic, a look into prime numbers, all sorts of counting results, and with a look into percentages and basic probability too. We then upgrade our knowledge by introducing the real numbers, and exploring what can be done with them, in relation with number theory questions. Then we further upgrade our methods, by introducing and using the complex numbers. Finally, we provide an introduction to modern number theory.

Preface

Number theory is the Queen of Mathematics, who has not wished to deal with numbers in computations, instead of that complicated trigonometry things. This book is an introduction to numbers, and their theory. You will learn from here all you need to know about numbers, fractions and percentages, followed by some basic number theory, also known as basic arithmetic, and then followed by more advanced aspects.

The story of numbers, or at least numbers employed by us humans, is long. Things go back to the Stone Age, where the sighting of a bison was reported with a "Ha" shout, the sighting of two bisons was reported with a "Ya", and of three, with a "Rg". And one day, an interesting thing happened. Gronk came back to camp, from his morning walk, shouting "Rg", and pointing towards the plains. While Kelc and Tay, one coming back from the lake, and the other, from the hill nearby, both started yelling "Ya".

So, which way to go? Times were hard, it was Winter, not much food left, and the more bisons hunted, the better. Big chief started thinking, then drinking, singing and dancing, and in the end, he cut his finger, and wrote on the wall of the cavern:

$$\text{Ya} + \text{Ya} > \text{Rg}$$

And with this, arithmetic was born. They went towards the lake, hunted the Ya + Ya bisons there, and had enough food for the rest of the Winter. Also, during the long Winter nights, they thought some more, and convened for "Uy" to designate the sighting of Ya + Ya bisons. And a few years after, after countless other hunts, they came upon the following fomula, that they wrote on the cavern wall too, and called Theorem:

$$\text{Ha} + \text{Rg} = \text{Uy}$$

So, this was for the beginnings, and many things have happened since, with countless improvings to this bison counting system. Romans in particular came with a system that no one really understands nowadays, I, II, III, IV, V, VI, ..., apart from certain fine intellectuals, and sports fans, but as a matter of telling the whole story, we will employ here that system too, for labeling the parts of the present book.

As already mentioned, this book is an introduction to arithmetic, with the aim of telling you the basic theory of numbers, counting, fractions and percentages, followed by

some basic number theory, also known as basic arithmetic, and then followed by more advanced aspects. The book is organized in four parts, as follows:

Part I deals with numbers, counting, fractions and percentages, all you need to know, with the basics explained, and all sorts of useful tricks and formulae, in the spirit of the above Stone Age Theorem. We will often rely here on intuition, and previous real-life experience with the numbers, and be sometimes a bit philosophical.

Part II deals with the real numbers, and what can be done with them, in relation with arithmetic. In particular, we will get to know more about the prime numbers, via the Euler formula, and other analysis tricks. We will also discuss some more advanced algebraic aspects, such as the finite groups, field theory, and quadratic residues.

Part III deals with the complex numbers, which are something more complicated, and far-reaching, and what can be done with them, again in relation with arithmetic. As main topics here, we will talk about the Cardano formula in degree 3 and 4, higher degree curves, Gauss sums with their sign computed, and the transcendence of e and π.

Part IV goes back to questions from basic arithmetic, notably regarding the prime numbers, with more on the subject, by benefiting from the knowledge of real and complex numbers. We will talk here about Mertens theorems, Chebycheff estimates, the Riemann zeta function, the Prime Number theorem, and the Riemann hypothesis.

In the hope that you will find this book useful, and get to love numbers and their theory. And for more, we will provide some references at the end.

Many thanks to everyone, having helped me to learn about numbers, since childhood and up to nowadays, and still counting. Thanks as well to my cats, it's a bit hard to talk to them because they use quaternions, but I learned from them many things too.

Cergy, August 2025
Teo Banica

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Part I

Numbers, counting

I'm only happy when it rains
I'm only happy when it's complicated
And though I know you can't appreciate it
I'm only happy when it rains

CHAPTER 1

Numbers

1a. Numbers

You certainly know a bit about numbers $1, 2, 3, 4, \ldots$, and we will be here, with this book, for learning more about them. Many things can be said, but instead of starting right away with some complicated mathematics, it is wiser to relax, and go back to these small numbers $1, 2, 3, 4, \ldots$ that you know well, and have some more thinking at them. After all, these small numbers are something quite magic, worth some more thinking. And with the thinking work that we will be doing here being something useful.

So, reviewing the material from elementary school. Shall we start with 7×8, or perhaps with 6×7? I don't know about you, but personally I found these two computations both quite difficult, as a kid, these multiples of 7 are no joke, when learning arithmetic.

In short, thinking well, it is probably wise to leave the multiplications for later, and start more modestly, with the sums. But before doing sums, we must first remember what the numbers themselves are. Which is a very good question, and in answer:

$$|\circ| = \text{one}$$

$$|\circ\circ| = \text{two}$$

$$|\circ\circ\circ| = \text{three}$$

$$|\circ\circ\circ\circ| = \text{four}$$

$$|\circ\circ\circ\circ\circ| = \text{five}$$

$$|\circ\circ\circ\circ\circ\circ| = \text{six}$$

$$|\circ\circ\circ\circ\circ\circ\circ| = \text{seven}$$

$$|\circ\circ\circ\circ\circ\circ\circ\circ| = \text{eight}$$

$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ| = \text{nine}$$

$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = \text{ten}$$

$$\vdots$$

Which sounds a bit boring, so what about stopping there at ten, and also replacing, for purely computational purposes, the various English words above by some mathematical symbols which are easy to draw, say by symbols $1, \ldots, \diamondsuit$, as follows:

$$|\circ| = 1$$
$$|\circ\circ| = 2$$
$$|\circ\circ\circ| = 3$$
$$|\circ\circ\circ\circ| = 4$$
$$|\circ\circ\circ\circ\circ| = 5$$
$$|\circ\circ\circ\circ\circ\circ| = 6$$
$$|\circ\circ\circ\circ\circ\circ\circ| = 7$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ| = 8$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 9$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = \diamondsuit$$

In order to count past ten, without the need of more symbols, we can trick. Let us introduce indeed a new symbol 0, called zero, in the following way:

$$\diamondsuit = 10$$

That is, we are making here the tricky convention that the 1 in 10 stands for ten, and the 0 in 10 stands for nothing. And with this convention, is it quite clear that we can count now beyond 10, a bit in the same way, as follows:

$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 10$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 11$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 12$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 13$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 14$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 15$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 16$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 17$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 18$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 19$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 20$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 21$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 22$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 23$$
$$\vdots$$

Very nice all this, so let us formulate our findings in the following way:

DEFINITION 1.1. *The numbers are sequences of type*

$$n = a_1a_2 \ldots a_k$$

with $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$*, and* $a_1 \neq 0$*. These numbers count, as follows:*

(1) *If a set has* $a_1 \in \{1, \ldots, 9\}$ *objects, the set count is* $n = a_1$*,*
(2) *If a set has* $a_1 \in \{1, \ldots, 9\}$ *chunks of* 10 *objects, followed by* $a_2 \in \{0, \ldots, 9\}$ *objects, the set count is* $n = a_1a_2$*,*
(3) *If a set has* $a_1 \in \{1, \ldots, 9\}$ *chunks of* 100 *objects, followed by* $a_2 \in \{0, \ldots, 9\}$ *chunks of* 10 *objects, and* $a_3 \in \{0, \ldots, 9\}$ *objects, the count is* $n = a_1a_2a_3$*,*

.. and so on, the idea being that we can count any set, no matter how big, in this way.

In mathematical notation, the counting rules above can be summarized as follows, with obvious meanings for the sum and product operations + and $\times$:

$$a_1 = a_1$$

$$a_1a_2 = 10 \times a_1 + a_2$$

$$a_1a_2a_3 = 100 \times a_1 + 10 \times a_2 + a_3$$

$$a_1a_2a_3a_4 = 1000 \times a_1 + 100 \times a_2 + 10 \times a_3 + a_4$$

$$\vdots$$

We conclude that, again in standard mathematical notation, we have the following formula, for an arbitrary number $n = a_1a_2 \ldots a_k$, as in Definition 1.1:

$$a_1a_2 \ldots a_k = 10^{k-1} \times a_1 + 10^{k-2} \times a_2 + \ldots + 10 \times a_{k-1} + a_k$$

With this understood, the problem is now, how these numbers $n = a_1a_2 \ldots a_k$ add? And here, no matter how we approach this question, via counting as in Definition 1.1, or via a mathematical formula with powers of 10 as above, we are led to the conclusion that things are quite clear, up to adding the digits $a \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ themselves.

But, in what regards this latter question, we know the answer to it, learned the hard way in school. So, problem solved, and as our first theorem in this book, we have:

THEOREM 1.2. *The digits $a \in \{1,2,3,4,5,6,7,8,9\}$ add according to the table*

+	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

and then the numbers $n = a_1a_2 \ldots a_k$ add in the obvious way, using this.

PROOF. As already mentioned, this is something that we learned the hard way in school, and which is undoubtedly difficult, so we will not attempt to prove this here, with full details. Here are, however, a few comments about all this:

(1) In what regards the table, that comes by counting sets, by hand, and recording the results. Not to forget, of course, to duly memorize all this afterwards.

(2) As a further remark about the table, observe that this consists of the numbers $1,2,3,4,\ldots$ themselves, filling the / diagonals, in the obvious way. You can call this observation theorem if you want, and in any case, this is certainly something that does happen, and of course I used this myself, when typing in the table. Easy work.

(3) Once the addition table for digits digested, in order to add two arbitrary numbers, $n = a_1a_2 \ldots a_k$ and $m = b_1b_2 \ldots b_s$, we can do this in the following way:

$$\begin{aligned} & a_1a_2 \ldots a_k + b_1b_2 \ldots b_s \\ = \ & (10^{k-1} \times a_1 + 10^{k-2} \times a_2 + \ldots + 10 \times a_{k-1} + a_k) \\ & + (10^{s-1} \times b_1 + 10^{s-2} \times b_2 + \ldots + 10 \times b_{s-1} + b_s) \\ = \ & 10^{k-1} \times a_1 + 10^{s-1} \times b_1 + \ldots\ldots + 10 \times a_{k-1} + 10 \times b_{s-1} + a_k + b_s \\ = \ & 10(10^{k-2} \times a_1 + 10^{s-2} \times b_1 + \ldots\ldots + a_{k-1} + b_{s-1}) + a_k + b_s \end{aligned}$$

Thus, proceeding from right to left, the last digit will obviously be $a_k + b_s$, or rather the last digit of $a_k + b_s$, in case $a_k + b_s \geq 10$, and so on, up to the first digit.

(4) Equivalently, we have here the basic algorithm for addition, obtained by putting $n = a_1a_2 \ldots a_k$ on top of $m = b_1b_2 \ldots b_s$, and summing as in (3), that you know well. □

Getting now to multiplication, things are considerably tougher here, again learned the hard way in school, with the relevant theorem here being as follows:

THEOREM 1.3. *The digits $a \in \{1,2,3,4,5,6,7,8,9\}$ multiply according to the table*

$\times$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

and then the numbers $n = a_1a_2 \ldots a_k$ multiply in the obvious way, using this.

PROOF. As before with Theorem 1.2, this is something that we learned the hard way in school, and which is undoubtedly difficult, so we will not attempt to prove this here, with full details. Here are, however, a few comments about all this:

(1) In what regards the table, again that comes by counting sets, by hand, and recording the results. Not to forget, of course, to duly memorize all this afterwards.

(2) As a further remark about the table, this was harder for me to type in, algorithmically, than the addition one, and what I did is to type in each line, starting from the left, with on line i by adding every time the number i to the previous one. Mathematically, this corresponds to the fact that each line of the table is an arithmetic progression.

(3) Once the multiplication table for digits digested, in order to multiply two arbitrary numbers, $n = a_1a_2 \ldots a_k$ and $m = b_1b_2 \ldots b_s$, we can do this in the following way:

$$\begin{aligned}
& a_1a_2 \ldots a_k \times b_1b_2 \ldots b_s \\
= \ & (10^{k-1} \times a_1 + 10^{k-2} \times a_2 + \ldots + 10 \times a_{k-1} + a_k) \\
& \times (10^{s-1} \times b_1 + 10^{s-2} \times b_2 + \ldots + 10 \times b_{s-1} + b_s) \\
= \ & 10^{k+s-2} \times a_1b_1 + \ldots\ldots + 10 \times a_{k-1}b_s + 10 \times a_kb_{s-1} + a_kb_s \\
= \ & 10(10^{k+s-3} \times a_1b_1 + \ldots\ldots + a_{k-1}b_s + a_kb_{s-1}) + a_kb_s
\end{aligned}$$

Thus, when proceeding from right to left, the last digit will obviously be a_kb_s, or rather the last digit of a_kb_s, in case $a_kb_s \geq 10$, and so on, up to the first digit.

(4) Equivalently, we have the algorithm for multiplication, obtained by putting $n = a_1a_2 \ldots a_k$ on top of $m = b_1b_2 \ldots b_s$, and multiplying as in (3), that you know well. □

And with this, good news, done with the hard mathematics, with the rest of the present book being more or less trivialities, coming from the above.

There is of course some discussion to be made about substractions and divisions too, which are basic operations as well, but we will discuss this later, no hurry with that.

Excuse me, but cat is here meowing something, so let's see what she has to say, before moving forward with our mathematics. So, what is it, cat?

CAT 1.4. *We cats never do computing mistakes, while you humans do a lot of them. Either your brains are not reliable, or you have not properly understood the above.*

Humm, good point, which sort of confirms what I see around me, be that in math teaching, or research, and with myself being of course not excluded from this, at least from time to time. Quite a mystery all this, human brain vs numbers.

So, for reformulating what was said above, and by being this time totally honest about it, we are not really done with the hard mathematics, but with the rest of the present book being more or less trivialities, coming from the above.

1b. Numeration bases

Before moving forward with our mathematics, and as a matter of making sure that we have everything developed on a solid basis, the question is now, is that magical 10 number, that we used in the above as a numeration basis, a really good idea?

That is, you must agree with me that, when thinking a bit, the choice of using 10 as reference number, or "numeration basis", as we use to say in mathematics, in Definition 1.1 looks like something quite arbitrary, that must be clarified.

And so, question now for the two of us to see if what we did so far in this chapter, with numbers, counting, sums and products, can be perhaps enhanced, by using other numbers instead of 10. That is, our precise question is as follows:

QUESTION 1.5. *What about using other numbers instead of* 10 *as numeration basis? Will this simplify our mathematics, numbers, counting, sums, products, or not?*

And good question this is. The answer to it is in fact not obvious, and this even if you know well mathematics, as many of our ancestors did, over the centuries. That is, there was a long debate over the centuries and millenia, about which numeration basis to use, with each one having its own pros and cons, and what we have in the end, 10, for our present human civilization, is in fact not clearly the best answer.

So, difficult question that we have here, mostly likely involving other things than abstract mathematics, and in order to get an idea about what is going on, let us work out some examples. As a first example here, which is something a bit formal, we have:

EXAMPLE 1.6. *Numeration basis two. Here the numbers are sequences of type*

$$n = a_1a_2 \ldots a_k$$

with $a_i \in \{0, 1\}$*, and* $a_1 \neq 0$*, and with the counting going as follows:*

(1) *If a set has* $a_1 = 1$ *objects, the set count is* $n = a_1$*,*
(2) *If a set consists of* $a_1 = 1$ *pairs, followed by* $a_2 \in \{0, 1\}$ *objects, the set count is* $n = a_1a_2$*,*
(3) *If a set consists of* $a_1 = 1$ *quadruplets, followed by* $a_2 \in \{0, 1\}$ *pairs, and then by* $a_3 \in \{0, 1\}$ *objects, the count is* $n = a_1a_2a_3$*,*

.. and so on, the idea being that we can count any set, no matter how big, in this way.

Which sounds quite exciting, doesn't it. More in detail now, here is how the counting in basis two goes, and with this looking like something quite simple:

$$|\circ| = 1$$
$$|\circ\circ| = 10$$
$$|\circ\circ\circ| = 11$$
$$|\circ\circ\circ\circ| = 100$$
$$|\circ\circ\circ\circ\circ| = 101$$
$$|\circ\circ\circ\circ\circ\circ| = 110$$
$$|\circ\circ\circ\circ\circ\circ\circ| = 111$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ| = 1000$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 1001$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 1010$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 1011$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 1100$$
$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 1101$$
$$\vdots$$

Regarding the addition table, this is something ridiculously simple, as follows:

$$\begin{array}{cc} + & 1 \\ 1 & 10 \end{array}$$

As for the multiplication table, this is ridiculously simple too, as follows:

$$\begin{array}{cc} \times & 1 \\ 1 & 1 \end{array}$$

So, shall we use this new system? I would rather say no, on the grounds that what we have in the above seems to require only two neurons for understanding, and we certainly

have more neurons than that. That is, our usual numeration system, using the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and their magic, looks like something more advanced.

Before leaving numeration basis two, however, let us mention that this system is used, successfully, by our friends the computers. But we are smarter than them.

Next on our list, coming natually after numeration basis two, is of course:

EXAMPLE 1.7. *Numeration basis three. Here the numbers are sequences of type*

$$n = a_1 a_2 \ldots a_k$$

with $a_i \in \{0, 1, 2\}$*, and* $a_1 \neq 0$*, and with the counting going as follows:*

(1) *If a set has* $a_1 \in \{1, 2\}$ *objects, the set count is* $n = a_1$*,*
(2) *If a set consists of* $a_1 \in \{1, 2\}$ *triples, followed by* $a_2 \in \{0, 1, 2\}$ *objects, the set count is* $n = a_1 a_2$*,*
(3) *If a set consists of* $a_1 \in \{1, 2\}$ *triples of triples, followed by* $a_2 \in \{0, 1, 2\}$ *triples, and then by* $a_3 \in \{0, 1, 2\}$ *objects, the count is* $n = a_1 a_2 a_3$*,*

.. and so on, the idea being that we can count any set, no matter how big, in this way.

As before, many things can be said here. Here is how the set counting goes:

$$|\circ| = 1$$

$$|\circ\circ| = 2$$

$$|\circ\circ\circ| = 10$$

$$|\circ\circ\circ\circ| = 11$$

$$|\circ\circ\circ\circ\circ| = 12$$

$$|\circ\circ\circ\circ\circ\circ| = 20$$

$$|\circ\circ\circ\circ\circ\circ\circ| = 21$$

$$|\circ\circ\circ\circ\circ\circ\circ\circ| = 22$$

$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 200$$

$$|\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ| = 201$$

$$\vdots$$

Which looks quite fun, observe for instance that we have here a slight advantage with respect to people counting in basis 10, or rather counting in basis 201, as we would say in our basis three, because when it comes to decide whether a set consists of triples, the answer is clear to us, just by looking at the last digit, which must be 0.

Regarding now the addition table, this is something quite fun too, as follows:

$$\begin{array}{ccc} + & 1 & 2 \\ 1 & 2 & 10 \\ 2 & 10 & 11 \end{array}$$

As for the multiplication table, this is again something quite exciting, as follows:

$$\begin{array}{ccc} \times & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 11 \end{array}$$

Time now to draw some conclusions, so, shall we use this new system? I would again say no, again on the grounds that what we have in the above seems to require only few neurons for understanding, and we certainly have more neurons than that.

Coming next, we have numeration basis four, whose theory is as follows:

EXAMPLE 1.8. *Numeration basis four. Here the numbers are sequences of type*

$$n = a_1a_2 \ldots a_k$$

with $a_i \in \{0, 1, 2, 3\}$*, and* $a_1 \neq 0$*, counting in the obvious way, the addition table is*

$$\begin{array}{cccc} + & 1 & 2 & 3 \\ 1 & 2 & 3 & 10 \\ 2 & 3 & 10 & 11 \\ 3 & 10 & 11 & 12 \end{array}$$

the multiplication table is

$$\begin{array}{cccc} \times & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 10 & 12 \\ 3 & 3 & 12 & 21 \end{array}$$

and in practice, this is a sort of a better version of numeration basis two.

To be more precise here, in what regards the last assertion, it is quite clear that everything that we can do, as tricks, in basis two, can be seen as well in basis four. And so, that basis four is more advanced than basis two, due to the more symbols used.

In any case, as before with basis two, all this rather belongs to computer science. So, we will not use this numeration basis, let the computers use it, if they want to.

Coming next, we have something quite interesting, as follows:

EXAMPLE 1.9. *Numeration basis five. Here the numbers are sequences of type*

$$n = a_1 a_2 \ldots a_k$$

with $a_i \in \{0, 1, 2, 3, 4\}$*, and* $a_1 \neq 0$*, counting in the obvious way, the addition table is*

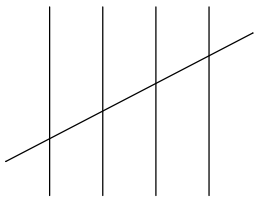
$+$	1	2	3	4
1	2	3	4	10
2	3	4	10	11
3	4	10	11	12
4	10	11	12	13

the multiplication table is

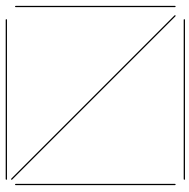
$\times$	1	2	3	4
1	1	2	3	4
2	2	4	11	13
3	3	11	14	22
4	4	13	22	31

and in practice, this is something quite efficient, for counting.

To be more precise here, in what regards the last assertion, there is certainly some truth there, that you might be aware of, because the chunks of five objects are very easy to represent, with a well-known convention for this being as follows:



An alternative convention here, which is widely used as well, is as follows:



Quite interesting all this, and still used on prison walls, and in many other concrete situations. Personally, this is my favorite system, for counting things.

Coming next, we have numeration basis six, which again is something interesting:

EXAMPLE 1.10. *Numeration basis six. Here the numbers are sequences of type*

$$n = a_1 a_2 \ldots a_k$$

with $a_i \in \{0, 1, 2, 3, 4, 5\}$*, and* $a_1 \neq 0$*, counting in the obvious way, the addition table is*

$+$	1	2	3	4	5
1	2	3	4	5	10
2	3	4	5	10	11
3	4	5	10	11	12
4	5	10	11	12	13
5	10	11	12	13	14

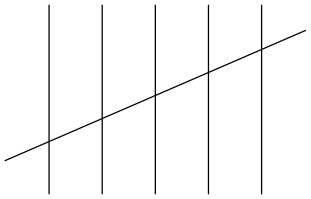
the multiplication table is

$\times$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	10	12	14
3	3	10	13	20	23
4	4	12	20	24	32
5	5	14	23	32	41

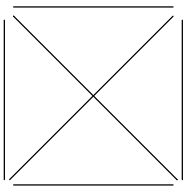
and in practice, this beats both basis two, and basis three.

To be more precise here, in what regards the last assertion, it is pretty much clear that all sorts of tricks from basis two and basis three can be done in basis six too.

In what regards now graphics, the chunks of six objects are quite easy to represent too, with a well-known convention for this being as follows:



An alternative convention here, which is widely used as well, is as follows:



Summarizing, quite interesting numeration basis that we have here, nicely mixing two and three, and that can be successfully used, for various purposes.

Coming next, we have numeration basis seven, which is something fun too:

EXAMPLE 1.11. *Numeration basis seven. Here the numbers are sequences of type*

$$n = a_1a_2 \ldots a_k$$

with $a_i \in \{0, 1, 2, 3, 4, 5, 6\}$*, and* $a_1 \neq 0$*, counting as usual, the addition table is*

$+$	1	2	3	4	5	6
1	2	3	4	5	6	10
2	3	4	5	6	10	11
3	4	5	6	10	11	12
4	5	6	10	11	12	13
5	6	10	11	12	13	14
6	10	11	12	13	14	15

the multiplication table is

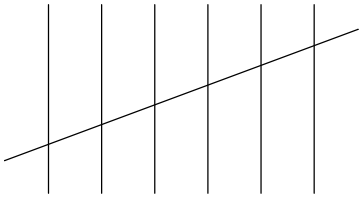
$\times$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	11	13	15
3	3	6	12	15	21	24
4	4	11	15	22	26	33
5	5	13	21	26	34	42
6	6	15	24	33	42	51

and in practice, this solves some of our school 7*-related nightmares.*

To be more precise here, in what regards the last assertion, remember that damn 6×7 and 7×8 computations from school, that we all had big troubles with. Well, in basis seven these two computations take a very simple form, as follows:

$$6 \times 10 = 60 \quad , \quad 10 \times 11 = 110$$

In what regards the graphics, however, not very good news here, because with the square becoming too crowded, we are basically left with ugly pictures, as follows:



And we will stop here with our list of examples. But the question comes now, which system to use? And we have here several schools of thought:

(1) Numeration basis two, or better, four, or even better, eight, or perhaps even sixteen, or why not sixty-four, are something very natural and useful. In practice, and in view of what we can do, and what we can't, the choice is between eight and sixteen.

(2) Numeration basis three, or much better, because even, six, or why now twelve, or twenty-four are something natural and useful too. In practice now, again in view of what we can do, and what we can't, the choice here is between six and twelve.

(3) Finally, we have numeration basis five, or much better, because even, ten. Not very clear what the advantage of using ten would be, but at least, as an interesting observation, at least there is no dillema here, with fifty being barred, as being too big.

So, this was for the story of the bases of numeration, and in what follows we will use, as everyone or almost nowadays, basis ten, somehow for the reasons discussed above.

1c. Basic arithmetic

With counting and numbers understood, let us develop now some basic arithmetic. We have here the following key notion, which often appears in the real life:

DEFINITION 1.12. *We say that b divides a, and we write $b|a$, when*

$$a = bc$$

for some number c. In this case we also use the following notation,

$$c = \frac{a}{b}$$

with this being called fraction, for designating this quotient number c.

All this is quite intuitive, and the fractions are subject to a number of simple formulae, which are all useful, in the real life, which can be summarized as follows:

THEOREM 1.13. *The fractions add according to the following formula,*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

they substract according to the following formula,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

they multiply according to the following formula,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and they divide according to the following formula,

$$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$$

provided that the quotient a/b is a multiple of the quotient c/d.

PROOF. You certainly know this, but for making sure that this is good science, let us see how the mathematical proof, based on Definition 1.12, goes. According to Definition 1.12 we have the following equality, that we will use many times, in what follows:

$$\frac{a}{b}=\frac{ad}{bd}\quad,\quad\forall d$$

(1) In what regards the addition formula, this can be established as follows:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

(2) The proof of the substraction formula is similar, as follows:

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$$

(3) In what regards now the multiplication formula, this comes from:

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)bd=\left(\frac{a}{b}\cdot b\right)\left(\frac{c}{d}\cdot d\right)=ac$$

(4) As for the division formula, this can be proved as follows:

$$\left(\frac{a}{b}:\frac{c}{d}\right)bc=\frac{abc}{b}:\frac{c}{d}=ac:\frac{c}{d}=ad$$

Thus, we are led to the conclusions in the statement. □

And more on this, divisibility of numbers, and on fractions too, in the above sense, and in some generalized sense too, when $b\not|a$, later in this book.

Moving ahead with more arithmetic, we have the following result:

THEOREM 1.14. *Given two numbers a,b, we can talk about their greatest common divisor (a,b), and their least common multiple $[a,b]$. We can write*

$$a=da'\quad,\quad b=db'$$

with a',b' being numbers having no common divisor, and we have:

$$(a,b)=d\quad,\quad[a,b]=da'b'$$

Also, (a,b) and $[a,b]$ are subject to the formula $(a,b)[a,b]=ab$.

PROOF. There are several things going on here, the idea being as follows:

(1) To start with, we can talk indeed about the greatest common divisor $d=(a,b)$ of any two numbers $a,b\in\mathbb{N}$, in several equivalent ways, as follows:

– The simplest way is to argue that since all common divisors $e|a,b$ satisfy $e\leq a,b$, we can pick the greatest such common divisor, $d=(a,b)$.

– Alternatively, we can say that $d=(a,b)$ is the smallest number having the property that any common divisor $e|a,b$ must divide it, $e|d$.

– Yet another approach is by recurrence on $a+b$. Indeed, assuming $a>b$, we can perform the division $a=bn+c$, and then set $(a,b)=(b,c)$, by recurrence.

(2) In practice now, there is some discussion needed here, in order to prove that the above 3 methods yield indeed the same number $d=(a,b)$. But this is best seen via the third method, which produces the same numbers as the first and second methods.

(3) So, this was for the story of the greatest common divisor (a,b), and in what regards the least common multiple $[a,b]$, the story here is similar, and we will leave the details as an exercise. Alternatively, if looking for a quick formal proof here, we can define $[a,b]$ by starting with (a,b), and using the formula $(a,b)[a,b]=ab$, discussed below.

(4) Next, with $d=(a,b)$, we can decompose our two numbers as follows:

$$a=da' \quad , \quad b=db'$$

We have then the following implications, coming from definitions:

$$(a,b)=d \implies (da',db')=d \implies (a',b')=1$$

Thus, we managed to write a,b as in the statement, and with this done, it is clear that we must have $[a,b]=da'b'$, so we have both formulae in the statement, namely:

$$(a,b)=d \quad , \quad [a,b]=da'b'$$

(5) Finally, observe that we have the following formula:

$$ab=d^2a'b'=d\times da'b'=(a,b)[a,b]$$

Thus, we are led to the conclusions in the statement. □

We can basically do the same with three numbers, as follows:

THEOREM 1.15. *Given three numbers a,b,c, we can talk about their greatest common divisor (a,b,c), and their least common multiple $[a,b,c]$, and if we write*

$$a=da' \quad , \quad b=db' \quad , \quad c=dc'$$

with $(a',b',c')=1$, and then further decompose each pair (a',b'), (a',c'), (b',c'), by using their respective greatest common divisors, we are led to a decomposition as follows,

$$a=dpqx \quad , \quad b=dpry \quad , \quad c=dqrz$$

and in terms of this decomposition, we have the following formulae:

$$(a,b,c)=d \quad , \quad [a,b,c]=dpqrxyz$$

Also, $(a,b,c)^2[a,b,c]$ divides abc, but with these numbers being different, in general.

PROOF. As before, the fact that we can talk indeed about greatest common divisors (a,b,c), and about least common multiples $[a,b,c]$ too, is something which is clear. Now if we set $d=(a,b,c)$, we can decompose our three numbers as follows:

$$a=da'\quad,\quad b=db'\quad,\quad c=dc'$$

We have then the following implications, coming from definitions:

$$(a,b,c)=d\implies(da',db',dc')=d\implies(a',b',c')=1$$

Thus, we have managed to write a,b,c as in the statement, and we have:

$$(a,b,c)=d$$

In order to compute now $[a',b',c']$, we can look at the pairs (a',b'), (a',c'), (b',c'), and apply to them the theory that we learned in Theorem 1.14. Indeed, let us set:

$$p=(a',b')\quad,\quad q=(a',c')\quad,\quad r=(b',c')$$

We are led in this way to decompositions as follows, for the numbers a',b',c':

$$a'=pqx\quad,\quad b'=pry\quad,\quad c'=qrz$$

As a conclusion, our original numbers a,b,c decompose as follows:

$$a=dpqx\quad,\quad b=dpry\quad,\quad c=dqrz$$

But with these formulae in hand, the numbers that we were looking for are:

$$(a,b,c)=d\quad,\quad[a,b,c]=dpqrxyz$$

Now when multiplying our numbers a,b,c, we have the following formula:

$$\begin{aligned}abc&=dpqx\cdot dpry\cdot dqrz\\&=d^3(pqr)^2xyz\\&=d^2\cdot dpqrxyz\cdot pqr\\&=(a,b,c)^2\cdot[a,b,c]\cdot pqr\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

In the general case now, that of k numbers, we have the following result:

THEOREM 1.16. *Given numbers $a_1,\ldots,a_k$, we can talk about their greatest common divisor $(a_1,\ldots,a_k)$, and their least common multiple $[a_1,\ldots,a_k]$, and we can write*

$$a_1=da_1'\quad,\quad\ldots\quad,\quad a_k=da_k'$$

with $d=(a_1,\ldots,a_k)$, and with $(a_1',\ldots,a_k')=1$. Also, the product

$$(a_1,\ldots,a_k)^{k-1}[a_1,\ldots,a_k]$$

divides the product $a_1\ldots a_k$, but these numbers are different, in general.

PROOF. There are several things going on here, the idea being as follows:

(1) As before, the fact that we can talk indeed about greatest common divisors, and about least common multiples, is something which is clear from definitions.

(2) Also as before, we can divide our numbers $a_1, \ldots, a_k$ by their common divisor $d = (a_1, \ldots, a_k)$, and we reach to a decomposition as follows, with $(a'_1, \ldots, a'_k) = 1$:

$$a_1 = da'_1 \quad , \quad \ldots \quad , \quad a_k = da'_k$$

(3) However, and here comes the point, when it comes to suitably decomposing our numbers $a_1, \ldots, a_k$, or rather their reduced versions $a'_1, \ldots, a'_k$, by using their various common divisors, as we did in Theorem 1.14 at $k = 2$, and in Theorem 1.15 at $k = 3$, things become considerably more complicated at $k = 4$ and higher. We can only recommend here doing some computations at $k = 4$, in order to understand what the difficulty is.

(4) Summarizing, we cannot say much, as a continuation of this, along the lines of what we did before at $k = 2, 3$, and the only obvious thing that can be said, completing our theorem, is the last assertion, which is something that we know, from Theorem 1.15. □

Moving on, still talking divisibility, here is a key result:

THEOREM 1.17. *Given two numbers satisfying $(a, b) = 1$, we can write*

$$ae + bf = 1$$

for certain integers $e, f \in \mathbb{Z}$.

PROOF. This might sound a bit surprising, but give me any two numbers which are prime to each other, say 7 and 10, and after thinking a bit, here is what I find:

$$7 \times 3 - 10 \times 2 = 1$$

So, let us try to prove now the result, in general. For this purpose, let us look at:

$$b, 2b, 3b, \ldots, ab$$

This is a certain collection of a numbers, and our claim is that the remainders of these numbers, modulo a, are different. Indeed, this is something coming from:

$$\begin{aligned} cb = db(a) &\iff a|(c - d)b \\ &\iff a|c - d \\ &\iff c = d \end{aligned}$$

Thus, our a numbers above are distinct modulo a, and so we have, still modulo a:

$$\big\{b, 2b, 3b, \ldots, ab\big\} = \big\{1, 2, 3, \ldots, a\big\}$$

But this does the job, because we get a certain $f \in \{1, \ldots, a\}$ such that:

$$bf = 1(a)$$

Thus we must have $ae + bf = 1$, for a certain $e \in \mathbb{Z}$, as desired. □

Here is a useful generalization of the above result:

THEOREM 1.18. *Given numbers satisfying* $(a_1, \ldots, a_k) = 1$*, we can write*

$$a_1e_1 + a_2e_2 + \ldots + a_ke_k = 1$$

for certain integers $e_1, \ldots, e_k \in \mathbb{Z}$*.*

PROOF. We already know from Theorem 1.17 that this holds at $k = 2$, and in general, the result will follow from this, the idea being as follows:

(1) Let us first see how the proof goes at $k = 3$. Given three numbers a, b, c having no common divisor, $(a, b, c) = 1$, let us write them as in Theorem 1.15, as follows:

$$a = pqx \quad , \quad b = pry \quad , \quad c = qrz$$

Now let us look at the sums of the following type, with $e, f, g \in \mathbb{Z}$:

$$ae + bf + cg = pqxe + pryf + qrzg$$

– As a first observation, since we have $(qx, ry) = 1$, we know from Theorem 1.17 that the quantities of type $qxe + ryf$ will range over the whole $\mathbb{Z}$.

– Next, and getting now towards what we want to prove, we conclude from this that the quantities of type $pqxe + pryf$ range over the whole $p\mathbb{Z}$.

– But then, since we have $(p, qrz) = 1$, we conclude, again by using Theorem 1.17, that the above sums $pqxe + pryf + qrzg$ range over the whole $\mathbb{Z}$.

(2) Thus, theorem proved at $k = 3$, and the proof in general is similar, by recurrence on k. To be more precise, it is technically convenient to look at a slightly more general statement, saying that given $a_1, \ldots, a_k$, we can always find $e_1, \ldots, e_k \in \mathbb{Z}$ such that:

$$a_1e_1 + a_2e_2 + \ldots + a_ke_k = (a_1, \ldots, a_k)$$

But with this picture in hand, it is quite clear that the above arguments used at $k = 3$ will apply in general, and will give the result, by recurrence on $k \in \mathbb{N}$. □

Summarizing, we have up and working a useful theory of greatest common divisors, and least common multiples, but obviously this is just the tip of the iceberg, and many interesting questions remain open. We will be back to them, later in this book.

Moving ahead, we will be mostly interested in congruence questions, based on:

DEFINITION 1.19. *We say that* $a, b \in \mathbb{Z}$ *are congruent modulo* $c \in \mathbb{Z}$*, and write*

$$a = b(c)$$

when c *divides* $b - a$*.*

A first interesting question concerns solving $a = 0(n)$, with n fixed and small. There is a bit of recursivity that can be used, in relation with this, as shown by:

$$6|a \iff 2|a \text{ and } 3|a$$
$$10|a \iff 2|a \text{ and } 5|a$$
$$12|a \iff 3|a \text{ and } 4|a$$
$$14|a \iff 2|a \text{ and } 7|a$$
$$15|a \iff 3|a \text{ and } 5|a$$
$$18|a \iff 2|a \text{ and } 9|a$$
$$20|a \iff 4|a \text{ and } 5|a$$
$$21|a \iff 3|a \text{ and } 7|a$$
$$22|a \iff 2|a \text{ and } 11|a$$
$$24|a \iff 3|a \text{ and } 8|a$$

In general, based on these observations, the idea is that by writing $n = n_1 \dots n_k$ with the factors n_i having no common divisor, we just have to solve this question for certain special values of n, excluding $n = 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, \dots$

These special values of n are called "powers of primes", and many things can be said about them. More on them later in this chapter, and then later in this book.

In practice, the first such numbers, powers of primes, are as follows:

$$q = 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, \dots$$

And in what regards solving $a = 0(q)$, with respect to these powers of primes, there are many useful tricks here, which can be summarized as follows:

THEOREM 1.20. *Given a positive integer $a = a_1 \dots a_k$, we have:*

(1) $2|a$ *when* $2|a_k$.
(2) $3|a$ *when* $3|\sum a_i$.
(3) $4|a$ *when* $4|a_{k-1}a_k$.
(4) $5|a$ *when* $5|a_k$.
(5) $8|a$ *when* $8|a_{k-2}a_{k-1}a_k$.
(6) $9|a$ *when* $9|\sum a_i$.
(7) $11|a$ *when* $11|\sum(-1)^i a_i$.
(8) $16|a$ *when* $16|a_{k-3}a_{k-2}a_{k-1}a_k$.

PROOF. Here the $q = 2^r, 5$ assertions follow from $10 = 2 \times 5$, the $q = 3, 9$ assertions follow from $10 = 9 + 1$, and the $q = 11$ assertion follows from $10 = 11 - 1$. □

All the above is certainly useful, in the daily life, but what is annoying is that for the missing values, $q = 7, 13$, nothing much intelligent, of the same level of simplicity, can be done. However, as mathematicians, we have solutions for everything, as shown by:

THEOREM 1.21. *Assuming that we have convinced mankind to change the numeration basis from* 10 *to* 14*, given a positive integer* $a = a_1 \ldots a_k$*, we have:*

(1) $2|a$ *when* $2|a_k$.
(2) $3|a$ *when* $3|\sum(-1)^i a_i$.
(3) $4|a$ *when* $4|a_{k-1}a_k$.
(4) $5|a$ *when* $5|\sum(-1)^i a_i$.
(5) $7|a$ *when* $7|a_k$.
(6) $8|a$ *when* $8|a_{k-2}a_{k-1}a_k$.
(7) $9|a$ *when* $9|\sum(-1)^i a_i$.
(8) $13|a$ *when* $13|\sum a_i$.
(9) $16|a$ *when* $16|a_{k-3}a_{k-2}a_{k-1}a_k$.

PROOF. Here the $q = 2^r, 7$ assertions follow from $14 = 2 \times 5$, the $q = 3, 5, 9$ assertions follow from $14 = 15 - 1$, and the $q = 13$ assertion follows from $14 = 13 + 1$. □

In short, good news, we have solved indeed the $q = 7, 13$ problems, but as a caveat, we have now $q = 11$ not working. And is this worth it or not, up to you to decide here, and launch an online petition if enthusiastic about it.

1d. Prime numbers

Time now to get into prime numbers, which will be a main theme of discussion, in this book. How many primes do you know? The more the better, and those under 100 are mandatory, at the beginner level, here they are, in all their beauty:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

$$53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

Actually those between 101 and 200 are mandatory too, here they are:

$$101, 103, 107, 109, 113, 127, 131, 137, 139, 149$$

$$151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199$$

But then, can we ignore those between 201 and 300. These are as follows:

$$211, 223, 227, 229, 233, 239, 241$$

$$251, 257, 263, 269, 271, 277, 281, 283, 293$$

Not to forget the primes between 301 and 400, which are as follows:

$$307, 311, 313, 317, 331, 337, 347, 349$$

$$353, 359, 367, 373, 379, 383, 389, 397$$

And we have kept the best for the end, primes between 401 and 500, which are:

$$401, 409, 419, 421, 431, 433, 439, 443, 449$$

$$457, 461, 463, 467, 479, 487, 491, 499$$

So, these are our beasts, in arithmetic, and try to get familiar with them, and learn some of their powers too, because these prime powers are very useful too.

We have already met prime numbers in the above, when talking divisibility, and even used some of their basic properties, that you were certainly very familiar with, but time now to review all this, on a more systematic basis, with proofs and everything.

First, as a definition for the prime numbers, we have:

DEFINITION 1.22. *The prime numbers are the integers $p > 1$ satisfying*

(1) *p does not decompose as $p = ab$, with $a, b > 1$.*
(2) *$p|ab$ implies $p|a$ or $p|b$.*
(3) *$a|p$ implies $a = 1, p$.*

with each of these properties uniquely determining them.

Here the equivalence between the conditions (1,2,3) is something intuitive and standard, which can be deduced by using our common divisor technology developed above. Observe also that we have ruled out $0, 1$ from being primes, and you may of course have a bit of thinking at this, and at $0, 1$ in general, but not too much, stay with us.

Still speaking things that we know, already used in the above, we have:

THEOREM 1.23. *Any integer $n > 1$ decomposes uniquely as*

$$n = p_1^{a_1} \ldots p_k^{a_k}$$

with $p_1 < \ldots < p_k$ primes, and with exponents $a_1, \ldots, a_k \geq 1$.

PROOF. This is something very standard, related to the equivalent conditions (1,2,3) in Definition 1.22, which formally comes by recurrence on n. As an interesting exercise here, work out this for all the integers $n \leq 100$, with no calculators allowed. □

As a first result now about the prime numbers themselves, we have:

THEOREM 1.24. *There is an infinity of prime numbers.*

PROOF. Indeed, assuming that we have finitely many prime numbers are $p_1, \ldots, p_k$, we can set $n = p_1 \ldots p_k + 1$, and this number n cannot factorize, contradiction. □

In practice, we can obtain the prime numbers as follows:

THEOREM 1.25. *The set of prime numbers P can be obtained as follows:*

(1) *Start with $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$*
(2) *Mark the first number, 2, as prime, and remove its multiples.*
(3) *Mark the new first number, 3, as prime, and remove its multiples.*
(4) *Mark the new first number, 5, as prime, and remove its multiples.*
(5) *And so on, with at each step a new prime number found.*

PROOF. This algorithm for finding the primes, which is very old, and called "sieve method", is something obvious, with the first steps being as follows:

$$\begin{array}{cccccccccccccccccccc}
\underline{2} & 3 & \not{4} & 5 & \not{6} & 7 & \not{8} & 9 & \not{10} & 11 & \not{12} & 13 & \not{14} & 15 & \not{16} & 17 & \not{18} & 19 & \not{20} \\
 & \underline{3} & & 5 & & 7 & & \not{9} & & 11 & & 13 & & \not{15} & & 17 & & 19 & \\
 & & & \underline{5} & & 7 & & & & 11 & & 13 & & & & 17 & & 19 & \\
 & & & & & \underline{7} & & & & 11 & & 13 & & & & 17 & & 19 & \\
 & & & & & & & & & \underline{11} & & 13 & & & & 17 & & 19 & \\
 & & & & & & & & & & & \underline{13} & & & & 17 & & 19 & \\
 & & & & & & & & & & & \vdots & & & & & & &
\end{array}$$

Thus, we are led to the conclusion in the statement. □

The above algorithm, while mathematically rather trivial, is something quite fascinating, because it suggests all sorts of mechanical ways of dealing with the primes, via analysis and physics and engineering. Let us record this as a conjecture:

CONJECTURE 1.26. *A good analyst, physicist and engineer would probably have no troubles in elucidating everything about primes, using the sieve method.*

And we will end the present opening chapter with this. Mystery.

1e. Exercises

This was a very basic chapter, save for some subtleties with the numeration bases, that you might not have been aware of, and as exercises on this, we have:

EXERCISE 1.27. *Review if needed the usual rules for addition and multiplication.*

EXERCISE 1.28. *Spend a full day in doing all your math computations in basis* 2*.*

EXERCISE 1.29. *Spend the day after in doing all your computations in basis* 3*.*

EXERCISE 1.30. *And then, spend the day after in doing all computations in basis* 6*.*

EXERCISE 1.31. *Clarify all the details, in relation with* $abc = (a, b, c)^2[a, b, c]pqr$*.*

EXERCISE 1.32. *Then study what happens, via suitable examples, for* a, b, c, d*.*

EXERCISE 1.33. *Find some clever criteria for divisibility by* 7*, and by* 13*.*

EXERCISE 1.34. *Count, before sleeping, the prime numbers listed in the above.*

As bonus exercise, read a bit about atoms, molecules, cells, animals, the origin of mankind, then civilizations, and numbers. It is important to be aware of all this.

CHAPTER 2

Counting

2a. Sets, counting

We know from chapter 1 that the numbers appear by counting sets, and time now to get more systematically into this. So, our objects of study with be the sets E, and the numbers N will play a secondary role, appearing as cardinalities of these sets:

$$N = |E|$$

Observe that this formalism is more general than that of the usual numbers, because we have as well infinite sets. And here, there is a lot of diversity, because some infinite sets might be "thicker" than some other, and we will see examples of this in a moment. However, in the lack of some general theory here, but more on this coming later too, we agree to use the notation $|E| = \infty$ for all these sets. Thus, in the above, we have:

$$N \in \mathbb{N} \cup \{\infty\}$$

Finally, as already mentioned above, the usual numbers $N \in \mathbb{N}$ will play a secondary role, in what follows. So, what we will be doing here will be quite independent of what we did with numbers, in chapter 1. However, we will allow ourselves to use the various discoveries about numbers $N \in \mathbb{N}$ from chapter 1, such as using their decimal form.

Let us start our discussion about sets with something philosophical:

FACT 2.1. *God only invented the empty set $\emptyset$, and then $\mathbb{N}$ naturally came afterwards, somehow by inventing itself, according to the following scheme:*

$$|\emptyset| = 0$$

$$|\{\emptyset\}| = 1$$

$$|\{\emptyset, \{\emptyset\}\}| = 2$$

$$|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| = 3$$

$$|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\}| = 4$$

$$\vdots$$

Alternatively, according this time to physicists, God only invented the Big Bang, and everything including $\mathbb{N}$ naturally came afterwards, with time passing by.

As a comment here, quite nice all this, but why did such things happen indeed, creation of $\mathbb{N}$, via mathematics or physics, according to the above mechanisms. Science has of course no explanation to this. So, Fact 2.1 remains something philosophical.

With this discussed, and getting now towards more concrete science, many things can be said about sets, and their counting. We first have the following result:

THEOREM 2.2. *A finite set E has*

$$|P(E)| = 2^{|E|}$$

possible subsets.

PROOF. This is something quite intuitive, the idea being as follows:

(1) In the case $|E| = 0$, meaning $E = \emptyset$, we have $2^0 = 1$ subsets, namely:

$$\emptyset$$

(2) In the case $|E| = 1$, meaning $E = \{a\}$, we have $2^1 = 2$ subsets, as follows:

$$\emptyset \quad , \quad \{a\}$$

(3) In the case $|E| = 2$, meaning $E = \{a, b\}$, we have $2^2 = 4$ subsets, as follows:

$$\emptyset \quad , \quad \{a\} \quad , \quad \{b\} \quad , \quad \{a, b\}$$

(4) Next, at $|E| = 3$, where $E = \{a, b, c\}$, we have $2^3 = 8$ subsets, as follows:

$$\emptyset \ , \ \{a\} \ , \ \{b\} \ , \ \{c\} \ , \ \{a, b\} \ , \ \{b, c\} \ , \ \{a, c\} \ , \ \{a, b, c\}$$

(5) In the general case now, the simplest is to say that the choice of a subset $G \subset E$ requires a binary choice for each of the elements $a \in E$, either in, or out. Now since these binary choices will multiply, we are led to the formula in the statement. □

Here is a useful generalization of the above result:

THEOREM 2.3. *Given two finite sets E, F, we have*

$$|E \to F| = |F|^{|E|}$$

possible functions $f : E \to F$.

PROOF. This is quite clear too, generalizing the above, but let us do this slowly:

(1) In the case $|F| = 0$, meaning $F = \emptyset$, there is no function $f : E \to \emptyset$, and so our formula is indeed true, with its verification taking the following form:

$$|E \to \emptyset| = 0 = 0^{|E|}$$

As an interesting observation here, our reasoning works fine for $E = \emptyset$ too, with the convention $0^0 = 0$ at the end. But this might be something quite misleading, for instance when doing analysis, so please forget right away this $0^0 = 0$ finding, do it for me.

(2) In the case $|F| = 1$, meaning $F = \{g\}$, there is only one function $f : E \to \{g\}$, namely the constant one, $f(e) = g$, for any $e \in E$. Thus, our cardinality formula in the statement is once again true, with its verification taking now the following form:

$$|E \to \{g\}| = 1 = 1^{|E|}$$

(3) More interestingly now, in the case $|F| = 2$, meaning $F = \{g, h\}$, there are as many functions $f : E \to \{g, h\}$ as there are subsets $G \subset E$, because such functions $f : E \to \{g, h\}$ must be of the following form, for a certain subset $G \subset E$:

$$f(e) = \begin{cases} g & (e \in G) \\ h & (e \notin G) \end{cases}$$

Thus, in this case the formula in the statement holds again, but this time coming as a consequence of Theorem 2.2, as follows:

$$|E \to \{g, h\}| = |P(E)| = 2^{|E|}$$

(4) Summarizing, things going on nice for us, so far. Getting now to uncharted territory, let us study the case $|F| = 3$, meaning $F = \{g, h, k\}$. Here the functions $f : E \to \{g, h, k\}$ must be of the following form, for certain disjoint subsets $G, H \subset E$:

$$f(e) = \begin{cases} g & (e \in G) \\ h & (e \in H) \\ k & \text{otherwise} \end{cases}$$

However, counting the disjoint subsets $G, H \subset E$ does not look exactly obvious.

(5) So, what to do. As a solution, we still have some degree of flexibility coming from the parameter $|E|$, so let us use that. At $|E| = 1$ our formula does hold, as follows:

$$|\{e\} \to \{g, h, k\}| = |e \to g, e \to h, e \to k| = 3 = 3^1$$

At $|E| = 2$ our formula holds as well, with the verification being as follows:

$$\begin{aligned} & |\{d, e\} \to \{g, h, k\}| \\ = \ & |(d, e \to g), (d \to g, e \to h), (d \to g, e \to k), \ldots\ldots, (d, e \to k)| \\ = \ & 9 \\ = \ & 3^2 \end{aligned}$$

And we will stop here with our study at $|F| = 3$, because the next verification would amount in counting $3^3 = 27$ things, and we don't want to do that. Summarizing, our formula seems to hold at $|F| = 3$, but the problem is, how to prove it.

(6) In answer, the best is to go back to the proof of Theorem 2.2, and adapt the arguments there. But with this idea in mind, things become clear. Indeed, the choice of a function $f : E \to F$ requires a $|F|$-choice for each of the elements $e \in E$. Now since these $|F|$-choices will multiply, we are led to the formula in the statement. □

Getting now to the infinite sets, we have here, as a basic result:

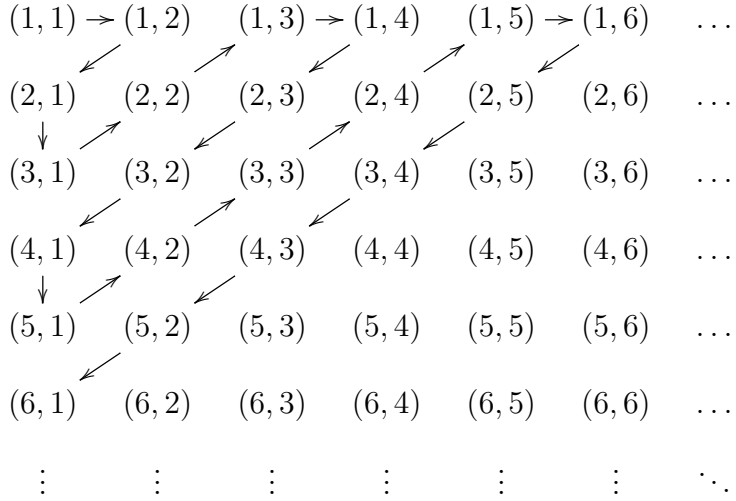
THEOREM 2.4. *The following happen:*

(1) *The set* $\mathbb{N} \times \mathbb{N}$ *is countable.*
(2) *In fact, if two sets* E, F *are countable, so is their product* $E \times F$*.*

PROOF. We can use here a diagonal procedure, in the obvious way. Consider indeed the following table, containing all pairs of type (a, b), with $a, b > 0$:

$$\begin{array}{cccccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & \ldots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & \ldots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) & \ldots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) & \ldots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) & \ldots \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can then snake our way inside this table, in the following way, starting from top left, and we count in this way all the pairs in the table:



Thus, theorem proved for integers, and for $E \times F$ the same argument works. □

As a second result now about infinite sets, which is more subtle, we have:

THEOREM 2.5. *The functions $f : \mathbb{N} \to \{0, 1\}$ are not countable. Thus, we have*

$$2^\infty > \infty$$

strict inequality, at the level of cardinalities of infinite sets.

PROOF. In what regards the first assertion, assume by contradiction that the functions $f : \mathbb{N} \to \{0, 1\}$ are countable. Thus, we can list these functions, as follows:

$$\begin{array}{lllllll} f_0 & = & f_0(0) & f_0(1) & f_0(2) & \ldots \\ f_1 & = & f_1(0) & f_1(1) & f_1(2) & \ldots \\ f_2 & = & f_2(0) & f_2(1) & f_2(2) & \ldots \\ \vdots & & \vdots & \vdots & \vdots & \ddots \end{array}$$

Now consider the following function, with the conventions $\hat{0} = 1$ and $\hat{1} = 0$:

$$f = \hat{f}_0(0) \quad \hat{f}_1(1) \quad \hat{f}_2(2) \quad \ldots$$

This function is then obviously not on our list, which is a contradiction, as desired. Thus, first assertion proved, and in what regards $2^\infty > \infty$, this is just an interpretation of what we found, in view of the formula $|E \to F| = |F|^{|E|}$ from Theorem 2.3. □

In relation with the above, we are led to the following interesting question:

QUESTION 2.6. *Is there any infinite set X whose cardinality $w = |X|$ satisfies*

$$\infty < w < 2^\infty$$

strict inequalities, with the above conventions for the meaning of ∞ and 2^∞?

And more on this, which is actually something quite scary, in relation with mathematical logic, and with real numbers and other numbers too, later in this book.

Back now to generalities, let us examine more operations on sets. We have:

THEOREM 2.7. *We have the following formula,*

$$\left(\bigcup_i A_i \right)^c = \bigcap_i A_i^c$$

as well as the following formula,

$$\left(\bigcap_i A_i \right)^c = \bigcup_i A_i^c$$

with c standing as usual for the complements.

PROOF. This is something elementary, the idea being as follows:

(1) In what regards the first assertion, this is the mathematical formulation of the obvious fact that "x does not belong to any of the A_i precisely when x does not belong to any of the A_i". Or at least, this is how I personally view this formula.

(2) Similar story for the second assertion, with this being the mathematical formulation of the fact that "x does not belong to at least one of the A_i precisely when x does not belong at least to one of the A_i". With this being, again, something trivial.

(3) Finally, in case where you ever find one of (1,2) more difficult than the other, here are some equivalences that might be of interest for you, with $B_i = A_i^c$:

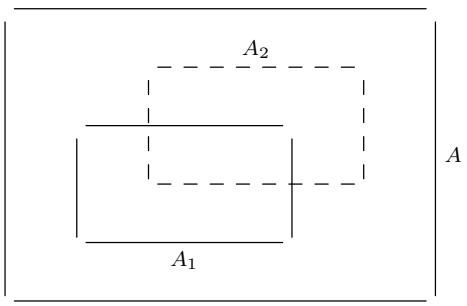
$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c \iff \left(\bigcup_i A_i\right)^{cc} = \left(\bigcap_i A_i^c\right)^c$$
$$\iff \bigcup_i A_i = \left(\bigcap_i A_i^c\right)^c$$
$$\iff \bigcup_i B_i^c = \left(\bigcap_i B_i\right)^c$$

Thus, (1) and (2) are in fact equivalent, so if one is easy, so is the other. □

Getting now to counting unions or intersections as above, we have here the inclusion-exclusion principle, which is something very useful, that we would like to explain now.

In order to explain this, let us first study the case of just two subsets, $A_1, A_2 \subset A$. We have the following result, regarding such a configuration:

PROPOSITION 2.8. *Two subsets of a given set $A_1, A_2 \subset A$ are best represented by the corresponding Venn diagram,*



and we have the following counting formula regarding them,

$$|(A_1 \cup A_2)^c| = |A| - |A_1| - |A_2| + |A_1 \cap A_2|$$

called inclusion-exclusion principle for two sets.

PROOF. This is something quite self-explanatory, the idea being as follows:

(1) To start with, we can draw indeed a Venn diagram as above, and with this Venn diagram in hand, the inclusion-exclusion principle is something clear.

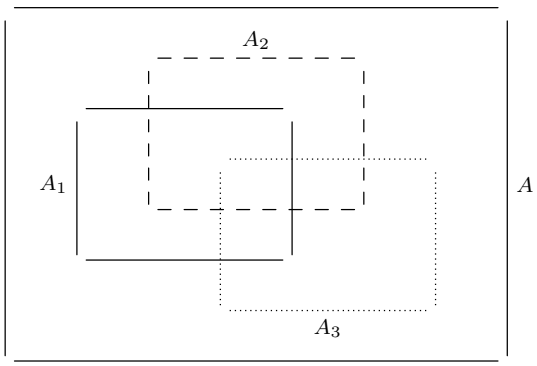
(2) However, shall we really trust diagrams. So, here is as well an abstract proof for the inclusion-exclusion principle, not using diagrams of any kind:

– In order to count $(A_1 \cup A_2)^c$, we certainly have to start with $|A|$, and then remove both $|A_1|$, $|A_2|$, because the elements of A_1, A_2 do not belong to $(A_1 \cup A_2)^c$.

– But then, thinking well, we have to put back $|A_1 \cap A_2|$, because we removed twice the elements of $A_1 \cap A_2$. Thus, we are led to the above formula. □

In the case of three subsets $A_1, A_2, A_3 \subset A$ we have a similar result, as follows:

PROPOSITION 2.9. *Three subsets of a given set $A_1, A_2, A_3 \subset A$ are best represented by the corresponding Venn diagram,*



and we have the following counting formula regarding them,

$$\begin{aligned} |(A_1 \cup A_2 \cup A_3)^c| &= |A| - |A_1| - |A_2| - |A_3| \\ &\quad + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &\quad - |A_1 \cap A_2 \cap A_3| \end{aligned}$$

called inclusion-exclusion principle for three sets.

PROOF. This is again something quite self-explanatory, the idea being as follows:

(1) To start with, we can draw indeed a Venn diagram as above, and with this Venn diagram in hand, the inclusion-exclusion principle is something clear.

(2) However, as before, shall we really trust diagrams. So, here is as well an abstract proof for the inclusion-exclusion principle, not using diagrams of any kind:

– In order to count $(A_1 \cup A_2 \cup A_3)^c$, we have to start with $|A|$, and then remove $|A_1|$, $|A_2|$, $|A_3|$, because the elements of A_1, A_2, A_3 do not belong to $(A_1 \cup A_2 \cup A_3)^c$.

– But then, thinking well, we have to put back $|A_1 \cap A_2|$, $|A_1 \cap A_3|$, $|A_2 \cap A_3|$, because we removed twice the elements of $A_1 \cap A_2$, $A_1 \cap A_3$, $A_2 \cap A_3$.

– And then, thinking some more, we have to remove $|A_1 \cap A_2 \cap A_3|$, because with our various manipulations above, the elements of $A_1 \cap A_2 \cap A_3$ were counted once, then removed 3 times, and then put back 3 times, which overall amounts in counting them once. And since these elements do not belong to the set $(A_1 \cup A_2 \cup A_3)^c$ that we are counting, we have to remove them, and we are led to the above formula. □

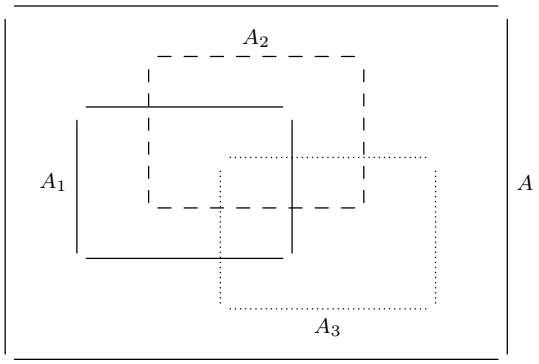
Quite boring all this, you would say, but getting now to the next case, that of four subsets $A_1, A_2, A_3, A_4 \subset A$, something interesting happens here, as follows:

PROPOSITION 2.10. *Four subsets of a given set $A_1, A_2, A_3, A_4 \subset A$ are quite difficult to represent on a Venn diagram, but we still have the following formula regarding them,*

$$\begin{aligned} & |(A_1 \cup A_2 \cup A_3 \cup A_4)^c| \\ = \ & |A| - |A_1| - |A_2| - |A_3| - |A_4| \\ & + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| \\ & - |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4| \\ & + |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

called inclusion-exclusion principle for four sets.

PROOF. In what regards the first assertion, this is obviously something subjective, and I challenge you to find on the following diagram some sort of nice room for an extra set A_4, and with a Nobel Prize in Economics being offered for solving this question:



Regarding now the second assertion, we can use here the same method as in the proof of Proposition 2.8 and Proposition 2.9, and with the comment here that, retrospectively, it was indeed a good idea to work out that proofs, without using Venn diagrams:

– In order to count $(A_1 \cup A_2 \cup A_3 \cup A_4)^c$, we have to start with $|A|$, and then remove each $|A_i|$, because the elements of each A_i do not belong to $(A_1 \cup A_2 \cup A_3 \cup A_4)^c$.

– But then, thinking well, we have to put back each $|A_i \cap A_j|$, because in our count so far, we removed twice the elements of each $A_i \cap A_j$.

– And then, thinking some more, we have to remove each $|A_i \cap A_j \cap A_k|$, because with our various manipulations above, the elements of $A_i \cap A_j \cap A_k$ were counted once, then removed 4 times, and then put back 4 times, which overall amounts in counting them once. And since these elements do not belong to the set $(A_1 \cup A_2 \cup A_3 \cap A_4)^c$ that we are counting, we have to remove them, and we are led to the above formula.

– And then, we are in fact still not done, because in what regards the elements of $A_1 \cap A_2 \cap A_3 \cap A_4$, we counted them once, then removed them 4 times, then added them 6 times, then removed them 4 times, which amounts in counting them $1-4+6-4=-1$ times. Thus, we have to put them back, as to have them counted 0 times, as needed. □

Lots of fun that we are having here, obviously, set theory is something more exciting than what we expected. In general, the inclusion-exclusion principle is as follows:

THEOREM 2.11. *We have the following formula,*

$$\left|\left(\bigcup_i A_i\right)^c\right| = |A| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots$$

called inclusion-exclusion principle.

PROOF. This is indeed quite clear, by thinking a bit, as before, as follows:

(1) In order to count $(\cup_i A_i)^c$, we certainly have to start with $|A|$.

(2) Then, we obviously have to remove each $|A_i|$, and so remove $\sum_i |A_i|$.

(3) But then, we have to put back each $|A_i \cap A_j|$, and so put back $\sum_{i<j} |A_i \cap A_j|$.

(4) Then, we must remove each $|A_i \cap A_j \cap A_k|$, so remove $\sum_{i<j<k} |A_i \cap A_j \cap A_k|$.

⋮

(5) And so on, which leads to the formula in the statement. □

Observe that, for various practical purposes, the inclusion-exclusion principle can be written in a more compact form, by using a double sum, as follows:

$$\left|\left(\bigcup_i A_i\right)^c\right| = \sum_{s\geq 0} (-1)^s \sum_{i_1<\ldots<i_s} |A_{i_1} \cap \ldots \cap A_{i_s}|$$

We will be back to the inclusion-exclusion principle and its applications later. The problem indeed is that, with our mathematics so far, we cannot do many things.

2b. Binomial formula

Time now to go into some truly interesting mathematics. As our first true counting theorem, solving a problem which often appears in real life, we have:

THEOREM 2.12. *The number of possibilities of choosing k objects among n objects is*

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

called binomial number, where $n!=1\cdot 2\cdot 3\ldots(n-2)(n-1)n$, called "factorial n".

PROOF. Imagine a set consisting of n objects. We have n possibilities for choosing our 1st object, then $n-1$ possibilities for choosing our 2nd object, out of the $n-1$ objects left, and so on up to $n-k+1$ possibilities for choosing our k-th object, out of the $n-k+1$ objects left. Since the possibilities multiply, the total number of choices is:

$$\begin{aligned}N&=n(n-1)\ldots(n-k+1)\\&=n(n-1)\ldots(n-k+1)\cdot\frac{(n-k)(n-k-1)\ldots 2\cdot 1}{(n-k)(n-k-1)\ldots 2\cdot 1}\\&=\frac{n(n-1)\ldots 2\cdot 1}{(n-k)(n-k-1)\ldots 2\cdot 1}\\&=\frac{n!}{(n-k)!}\end{aligned}$$

However, when thinking well, the number N that we computed is in fact the number of possibilities of choosing k ordered objects among n objects. Thus, we must divide everything by the number M of orderings of the k objects that we chose:

$$\binom{n}{k}=\frac{N}{M}$$

In order to compute now the missing number M, imagine a set consisting of k objects. There are k choices for the object to be designated #1, then $k-1$ choices for the object to be designated #2, and so on up to 1 choice for the object to be designated #k. We conclude that we have $M=k(k-1)\ldots 2\cdot 1=k!$, and so:

$$\binom{n}{k}=\frac{n!/(n-k)!}{k!}=\frac{n!}{k!(n-k)!}$$

Thus, we are led to the conclusion in the statement. □

The binomial numbers, as constructed above, are quite fascinating objects, and the more you know about them, the better your mathematics will be. Trust me here.

To start with, here are some basic formulae for binomial coefficients that you should definitely memorize, and pull right away, when needed in your computations:

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$$

Here are as well some numerics, with $n = k, k+1, k+2, \ldots$ in each case, that you should know well too, and pull out instantly, when needed in your computations:

$$\binom{n}{2} = 1, 3, 6, 10, 15, 21, 28, \ldots$$

$$\binom{n}{3} = 1, 4, 10, 20, 35, 56, \ldots$$

$$\binom{n}{4} = 1, 5, 15, 35, 70, \ldots$$

Finally, talking numerics, as an important adding to Theorem 2.12, we have:

CONVENTION 2.13. *By definition we have the formula*

$$0! = 1$$

as for the following binomial coefficient computation to work,

$$\binom{n}{n} = \frac{n!}{n!0!} = \frac{n!}{n! \times 1} = 1$$

in agreement with what Theorem 2.12 says, requiring $\binom{n}{n} = 1$.

Going ahead now with more mathematics and less philosophy, with Theorem 2.12 complemented by this convention being in final form, we have:

THEOREM 2.14. *We have the binomial formula*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

valid for any two numbers $a, b \in \mathbb{N}$.

PROOF. We have to compute the following quantity, with n terms in the product:

$$(a+b)^n=(a+b)(a+b)\ldots(a+b)$$

When expanding, we obtain a certain sum of products of a,b variables, with each such product being a quantity of type a^kb^{n-k}. Thus, we have a formula as follows:

$$(a+b)^n=\sum_{k=0}^{n}C_ka^kb^{n-k}$$

In order to finish, it remains to compute the coefficients C_k. But, according to our product formula, C_k is the number of choices for the k needed a variables among the n available a variables. Thus, according to Theorem 2.12, we have the following formula:

$$C_k=\binom{n}{k}$$

We are therefore led to the formula in the statement. □

Theorem 2.14 is something quite interesting. At small values of n we obtain:

$$(a+b)^0=1$$
$$(a+b)^1=a+b$$
$$(a+b)^2=a^2+2ab+b^2$$
$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$
$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$
$$\vdots$$

Now observe that in these formulae, what matters are the coefficients $\binom{n}{k}$, which form a triangle. So, it is enough to memorize this triangle, and this can be done by using:

THEOREM 2.15. *The Pascal triangle, formed by the binomial coefficients* $\binom{n}{k}$,

$$1$$
$$1\ ,\ 1$$
$$1\ ,\ 2\ ,\ 1$$
$$1\ ,\ 3\ ,\ 3\ ,\ 1$$
$$1\ ,\ 4\ ,\ 6\ ,\ 4\ ,\ 1$$
$$1\ ,\ 5\ ,\ 10\ ,\ 10\ ,\ 5\ ,\ 1$$
$$1\ ,\ 6\ ,\ 15\ ,\ 20\ ,\ 15\ ,\ 6\ ,\ 1$$
$$\vdots$$

has the property that each entry is the sum of the two entries above it.

PROOF. In practice, the theorem states that the following formula holds:

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

There are many ways of proving this formula, all instructive, as follows:

(1) Brute-force computation. We have indeed, as desired:

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k}&=\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\&=\frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right)\\&=\frac{(n-1)!}{(k-1)!(n-k-1)!}\cdot\frac{n}{k(n-k)}\\&=\binom{n}{k}\end{aligned}$$

(2) Algebraic proof. We have the following formula, to start with:

$$(a+b)^n=(a+b)^{n-1}(a+b)$$

By using the binomial formula, this formula becomes:

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}=\left[\sum_{r=0}^{n-1}\binom{n-1}{r}a^rb^{n-1-r}\right](a+b)$$

Now let us perform the multiplication on the right. We obtain a certain sum of terms of type a^kb^{n-k}, and to be more precise, each such a^kb^{n-k} term can either come from the $\binom{n-1}{k-1}$ terms $a^{k-1}b^{n-k}$ multiplied by a, or from the $\binom{n-1}{k}$ terms a^kb^{n-1-k} multiplied by b. Thus, the coefficient of a^kb^{n-k} on the right is $\binom{n-1}{k-1}+\binom{n-1}{k}$, as desired.

(3) Combinatorics. Let us count k objects among n objects, with one of the n objects having a hat on top. Obviously, the hat has nothing to do with the count, and we obtain $\binom{n}{k}$. On the other hand, we can say that there are two possibilities. Either the object with hat is counted, and we have $\binom{n-1}{k-1}$ possibilities here, or the object with hat is not counted, and we have $\binom{n-1}{k}$ possibilities here. Thus $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$, as desired. □

There are many more things that can be said about binomial coefficients, with all sorts of interesting formulae, and we will be back to this later, on a regular basis. The idea will always be the same, namely that in order to find such formulae you have a choice between algebra and combinatorics, a bit as in the above, and that when it comes to formal proofs, the brute-force computation method is something useful too.

In practice, the best is to master all 3 techniques. Among others, you will have in this way 3 different methods, for making sure that your formulae are correct indeed.

2c. Binomial coefficients

Getting now to more advanced things regarding the binomial coefficients, let us formulate, as a complement to the various particular cases discussed before:

DEFINITION 2.16. *The central binomial coefficients are the following numbers,*

$$D_n = \binom{2n}{n}$$

which are not to be confused with the middle binomial coefficients,

$$E_n = \binom{n}{[n/2]}$$

with $[.]$ *standing as usual for the integer part.*

Observe that we can recover the central binomial coefficients as particular cases of the middle binomial coefficients, due to the following trivial formula:

$$D_n = E_{2n}$$

However, in practice, the central binomial coefficients D_n are the truly interesting quantities, and the middle binomial coefficients E_n remain something secondary. Regarding now the numerics for the central binomial coefficients, these are as follows:

$$D_n = 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, \ldots$$

This sequence is actually something quite fascinating, and if you are a number theory nerd, and hope so are you, one of the first things that you will discover, by playing with it, is that these central binomial coefficients factorize as follows:

$$\begin{aligned} D_n &= 1 \times 1, 2 \times 1, 3 \times 2, 4 \times 5, 5 \times 14, 6 \times 42, \\ &\quad 7 \times 132, 8 \times 429, 9 \times 1430, 10 \times 4862, \ldots \end{aligned}$$

Thus, we are led in this way to the following conjecture:

CONJECTURE 2.17. *The central binomial coefficients factorize as*

$$D_n = (n+1)C_n$$

with $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$ *being certain integers.*

However, this is something which is not trivial to prove, with bare hands, and we will leave it for later in this chapter, once we will know more things.

More modestly now, but along the same lines, let us attempt to work out some basic arithmetic properties of the general binomial coefficients. We have here the following result, which is something very useful, in practice, for various purposes:

THEOREM 2.18. *Given a prime $p \geq 2$, the exponent of p inside $n!$ is*

$$a_n = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$$

and so the exponent of p inside $\binom{n}{k}$ is given by the formula

$$b_{n,k} = a_n - a_k - a_{n-k}$$

with $[.]$ standing as usual for the integer part.

PROOF. This is indeed something quite self-explanatory, the idea being as follows:

(1) Regarding the notations that are used in the statement, given an integer $q \in \mathbb{N}$, consider the arithmetic progression formed by it, which is as follows:

$$0, q, 2q, 3q, 4q, \ldots$$

Now let us come with a second integer, $n \in \mathbb{N}$. This must fit somewhere, with respect to the above sequence, and so we can find a number $e \in \mathbb{N}$ such that:

$$eq \leq n < (e+1)q$$

With this done, we make the following convention, used in the statement:

$$\left[\frac{n}{q}\right] = e$$

(2) Now let us prove the result. Regarding the first assertion, which is the main one, we must compute the exponent of p inside the following product:

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots n$$

But here, as a first contribution to the exponent, we have a 1 factor for each of the multiples of p of type $0 < pa \leq n$, and so this first contribution is given by:

$$c_1 = \left[\frac{n}{p}\right]$$

Next, as a second contribution, we must add a supplementary 1 factor for each of the multiples of p^2 of type $0 < p^2a \leq n$, and this second contribution is given by:

$$c_2 = \left[\frac{n}{p^2}\right]$$

But then, we have as well a third contribution, coming by adding a supplementary 1 factor for each of the multiples of p^3 of type $0 < p^3a \leq n$, which is given by:

$$c_3 = \left[\frac{n}{p^3}\right]$$

And so on, and we are led in this way to the formula in the statement, namely:

$$a_n = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$$

By the way, observe that this sum is finite, stopping where $p^k > n$.

(3) In what regards now the second assertion, we must compute the exponent of p inside the binomial coefficients, which are given by the following formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

But this gives right away $b_{n,k} = a_n - a_k - a_{n-k}$, as claimed. □

There are many interesting illustrations for the above result. We will be back to such things later in this book, when doing more advanced arithmetic.

As an interesting application of the theory of the binomial coefficients, we can now present a useful application of the inclusion-exclusion principle. Let us start with something well-known, and quite intiutive, often met in the real life, as follows:

DEFINITION 2.19. *A permutation of $\{1, \ldots, N\}$ is a bijection, as follows:*

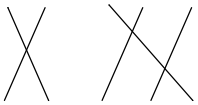
$$\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$$

The set of such permutations is denoted S_N.

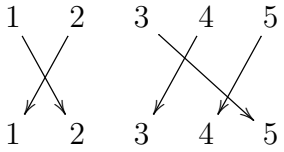
There are many possible notations for the permutations, the basic one consisting in writing the numbers $1, \ldots, N$, and below them, their permuted versions:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

Another method, which is certainly faster, and which is actually my personal favorite, is by denoting the permutations as diagrams, acting from top to bottom:

$\sigma =$ 

To be more precise, this diagram stands as a shorthand for the following diagram:

$\sigma =$ 

Here are now some basic properties of the permutations, which are all quite intuitive, coming from definitions, and which are good to know:

THEOREM 2.20. *The permutations have the following properties:*

(1) *There are $N!$ of them.*
(2) *They are stable by composition, $(\sigma, \tau) \to \sigma\tau$.*
(3) *They are stable by inversion too, $\sigma \to \sigma^{-1}$.*

PROOF. This is something elementary, the idea being as follows:

(1) In order to construct a permutation $\sigma \in S_N$, we have:

– N choices for the value of $\sigma(N)$.

– $(N-1)$ choices for the value of $\sigma(N-1)$.

– $(N-2)$ choices for the value of $\sigma(N-2)$.

$\vdots$

– and so on, up to 1 choice for the value of $\sigma(1)$.

Thus, we have $N!$ choices for any element $\sigma \in S_N$, as claimed in (1).

(2) This is clear, because composing bijections gives you a bijection.

(3) This is clear too, because the inverse of a bijection is a bijection. □

Many other things can be said, about permutations. With this discussed, here is now the application of the inclusion-exclusion principle that we were having in mind:

THEOREM 2.21. *The number of permutations $\sigma \in S_N$ having no fixed points is*

$$K = N! - \frac{N!}{1!} + \frac{N!}{2!} - \ldots + (-1)^{N-1}\frac{N!}{(N-1)!} + (-1)^N\frac{N!}{N!}$$

and the probability $P = K/N!$ seems to converge to an interesting number.

PROOF. Let us prove the formula in the statement, and for the interpretation afterwards, which obviously is something going above our present level of mathematics, and is more of an advertisement for things to come later in this book, we will see later.

(1) Consider the following sets of permutations, one for each $i \in \{1, \ldots, N\}$:

$$S_N^i = \left\{\sigma \in S_N \middle| \sigma(i) = i\right\}$$

The set of permutations having no fixed points is then given by:

$$X_N = \left(\bigcup_i S_N^i\right)^c$$

Thus, we can in principle compute $K = |X_N|$ by using inclusion-exclusion.

(2) In practice now, in order to compute $K = |X_N|$, consider as well the following sets, depending on indices $i_1 < \ldots < i_k$, obtained by taking intersections:

$$S_N^{i_1 \ldots i_k} = S_N^{i_1} \bigcap \ldots \bigcap S_N^{i_k}$$

Observe that we have the following formula:

$$S_N^{i_1\ldots i_k}=\left\{\sigma\in S_N\Big|\sigma(i_1)=i_1,\ldots,\sigma(i_k)=i_k\right\}$$

Now by arguing as in the proof of Theorem 2.20, or simply by applying that Theorem 2.20 to the set $\{1,\ldots,N\}-\{i_1,\ldots,i_k\}$, we have the following formula:

$$|S_N^{i_1\ldots i_k}|=(n-k)!$$

(3) We can apply now the inclusion-exclusion principle, and we obtain, as claimed:

$$\begin{aligned}
K&=|X_N|\\
&=\left|\left(\bigcup_i S_N^i\right)^c\right|\\
&=|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^i\cap S_N^j|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1}\cup\ldots\cup S_N^{i_N}|\\
&=|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^{ij}|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1\ldots i_N}|\\
&=\sum_{k=0}^N(-1)^k\sum_{i_1<\ldots<i_k}|S_N^{i_1\ldots i_k}|\\
&=\sum_{k=0}^N(-1)^k\sum_{i_1<\ldots<i_k}(N-k)!\\
&=\sum_{k=0}^N(-1)^k\binom{N}{k}(N-k)!\\
&=\sum_{k=0}^N(-1)^k\frac{N!}{k!}
\end{aligned}$$

(4) Finally, regarding the last assertion, obviously we are punching there above our weight, but since all this is quite exciting, here are some details, and more later. According to our computation above, and modulo some fraction mathematics that we will explain later, the probability for a random permutation $\sigma\in S_N$ to have no fixed points is:

$$P=\frac{K}{N!}=\sum_{k=0}^N\frac{(-1)^k}{k!}$$

Now the point is that, with $N\to\infty$, the numbers on the right converge to $1/e$, with $e=2.7182\ldots$ being the well-known number from analysis. We will be back to this. □

2d. Further counts

We would like to count now loops on graphs, with this being a quite interesting question. Think for instance percolation, when making coffee, each droplet of water will have to make its way through the coffee particles, and this is how making coffee works.

So, as a first philosophical question, what are the simplest graphs X, that we can try to do some loop computations for? And here, we have 3 possible answers, as follows:

FACT 2.22. *The following are graphs X, with a distinguished vertex $0 \in X$:*

(1) *The circle graph, having N vertices, with 0 being one of the vertices.*
(2) *The segment graph, having N vertices, with 0 being the vertex at left.*
(3) *The segment graph, having $2N + 1$ vertices, with 0 being in the middle.*

So, let us start with these. For the circle, the computations are quite non-trivial, and you can try doing some, in order to understand what I am talking about. The problem comes from the fact that loops of length $k = 0, 2, 4, 6, \ldots$ are quite easy to count, but then, once we pass $k = N$, the loops can turn around the circle or not, and they can even turn several times, and so on, and all this makes the count too complicated. In addition, again due to loop turning, when N is odd, we have as well loops of odd length.

As for the two segment graphs, here the computations look again complicated, and even more complicated than for the circle, because, again, once we pass $k = N$ many things can happen, and this makes the count too complicated. And here, again you can try doing some computations, in order to understand what I am talking about.

All this is obviously not very good news, and so, as question for us, shall we give up, in waiting for more advanced techniques, for dealing with such questions?

Well, instead of giving up, let us look face-to-face at the difficulties that we met. We are led this way, after analyzing the situation, to the following thought:

THOUGHT 2.23. *The difficulties that we met, with the circle and the two segments, come from the fact that our loops are not "free to move",*

(1) *for the circle, because these can circle around the circle,*
(2) *for the segments, obviously because of the endpoints,*

and so our difficulties will dissapear, and we will be able to do our exact loop count, once we find a graph X where the loops are truly "free to move".

Thinking some more, all this definitely buries the first interval graph, where the vertex 0 is one of the endpoints. However, we can still try to recycle the circle, by unwrapping it, or extend our second interval graph up to ∞. But in both cases what we get is the graph $\mathbb{Z}$ formed by the integers. So, let us formulate the following definition:

DEFINITION 2.24. *An infinite graph is the same thing as a finite graph, but now with an infinite number of vertices, usually taken countable:*

$$|X| = \infty$$

As a basic example, we have the graph $\mathbb{Z}$*. We also have the graph* $\mathbb{N}$*.*

Leaving aside $\mathbb{N}$, which looks more complicated, let us try now to count the length k paths on $\mathbb{Z}$, based at 0. At $k = 1$ we have 2 such paths, ending at -1 and 1, and the count results can be pictured as follows, with everything being self-explanatory:

$$\circ - \circ - \circ - \bullet - \circ - \circ - \circ$$
$$1 \qquad 1$$

At $k = 2$ now, we have 4 paths, one of which ends at -2, two of which end at 0, and one of which ends at 2. The results can be pictured as follows:

$$\circ - \circ - \circ - \bullet - \circ - \circ - \circ$$
$$1 \qquad 2 \qquad 1$$

At $k = 3$ now, we have 8 paths, the distribution of the endpoints being as follows:

$$\circ - \circ - \circ - \circ - \bullet - \circ - \circ - \circ - \circ$$
$$1 \qquad 3 \qquad 3 \qquad 1$$

As for $k = 4$, here we have 16 paths, the distribution of the endpoints being as follows:

$$\circ - \circ - \circ - \circ - \circ - \bullet - \circ - \circ - \circ - \circ - \circ$$
$$1 \qquad 4 \qquad 6 \qquad 4 \qquad 1$$

And good news, we can see in the above the Pascal triangle, namely:

$$1$$
$$1 \ , \ 1$$
$$1 \ , \ 2 \ , \ 1$$
$$1 \ , \ 3 \ , \ 3 \ , \ 1$$
$$1 \ , \ 4 \ , \ 6 \ , \ 4 \ , \ 1$$
$$1 \ , \ 5 \ , \ 10 \ , \ 10 \ , \ 5 \ , \ 1$$
$$\vdots$$

Thus, eventually, we found the simplest graph ever, finite or not, namely $\mathbb{Z}$, and we have the following beautiful result about it:

THEOREM 2.25. *The paths on $\mathbb{Z}$ are counted by the binomial coefficients. In particular, the $2k$-paths based at 0 are counted by the numbers*

$$D_k = \binom{2k}{k}$$

which are the central binomial coefficients.

PROOF. This follows from the above discussion. Indeed, we certainly have the Pascal triangle, and the rest is just a matter of finishing. There are many possible ways here, a straightforward one being that of arguing that the number E_k^l of length k loops $0 \to l$ is subject, due to the binary choice at the end, to the following recurrence relation:

$$E_k^l = E_{k-1}^{l-1} + E_{k-1}^{l+1}$$

But this is exactly the recurrence for the Pascal triangle, as desired. We will leave clarifying all the details here as an instructive exercise for you, reader. □

As a second example now, let us try to count the loops of $\mathbb{N}$, based at 0. This is something less obvious, and at the experimental level, the result is as follows:

PROPOSITION 2.26. *The Catalan numbers C_k, counting the loops on $\mathbb{N}$ based at 0,*

$$C_k = \#\Big\{0 - i_1 - \ldots - i_{2k-1} - 0\Big\}$$

are numerically $1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

PROOF. To start with, we have indeed $C_1 = 1$, the only loop here being $0 - 1 - 0$. Then we have $C_2 = 2$, due to two possible loops, namely:

$$0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0$$

Then we have $C_3 = 5$, the possible loops here being as follows:

$$0 - 1 - 0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 0 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 3 - 2 - 1 - 0$$

In general, the same method works, with $C_4 = 14$ being left to you, as an exercise, and with C_5 and higher to me, and I will be back with the solution, in due time. □

Obviously, computing the numbers C_k is no easy task, and finding the formula of C_k, out of the data that we have, does not look as an easy task either. So, we will do what combinatorists do, let me teach you. The first step is to relax, then to look around, not with the aim of computing your numbers C_k, but rather with the aim of finding other objects counted by the same numbers C_k. With a bit of luck, among these objects some will be easier to count than the others, and this will eventually compute C_k.

This was for the strategy. In practice now, we first have the following result:

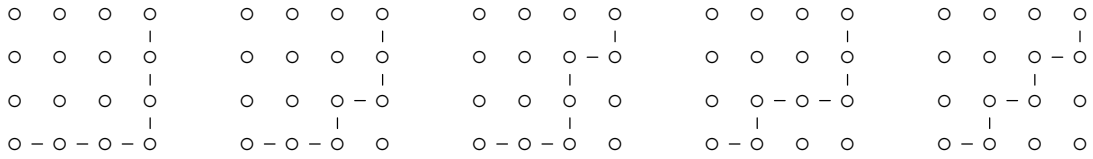
THEOREM 2.27. *The Catalan numbers C_k count:*

(1) *The length $2k$ loops on $\mathbb{N}$, based at 0.*
(2) *The noncrossing pairings of $1, \ldots, 2k$.*
(3) *The noncrossing partitions of $1, \ldots, k$.*
(4) *The length $2k$ Dyck paths in the plane.*

PROOF. All this is standard combinatorics, the idea being as follows:

(1) To start with, in what regards the various objects involved, the length $2k$ loops on $\mathbb{N}$ are the length $2k$ loops on $\mathbb{N}$ that we know, and the same goes for the noncrossing pairings of $1, \ldots, 2k$, and for the noncrossing partitions of $1, \ldots, k$, the idea here being that you must be able to draw the pairing or partition in a noncrossing way.

(2) Regarding now the length $2k$ Dyck paths in the plane, these are by definition the paths from $(0, 0)$ to (k, k), marching North-East over the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, by staying inside the square $[0, k] \times [0, k]$, and staying as well under the diagonal of this square. As an example, here are the 5 possible Dyck paths at $n = 3$:



(3) Thus, we have definitions for all the objects involved, and in each case, if you start counting them, as we did in Proposition 2.26 with the loops on $\mathbb{N}$, you always end up with the same sequence of numbers, namely those found in Proposition 2.26:

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$$

(4) In order to prove now that (1-4) produce indeed the same numbers, many things can be said. The idea is that, leaving aside mathematical brevity, and more specifically abstract reasonings of type $a = b, b = c \implies a = c$, what we have to do, in order to fully understand what is going on, is to etablish $\binom{4}{2} = 6$ equalities, via bijective proofs.

(5) But this can be done, indeed. As an example here, the noncrossing pairings of $1, \ldots, 2k$ from (2) are in bijection with the noncrossing partitions of $1, \ldots, k$ from (3), via

fattening the pairings and shrinking the partitions. We will leave the details here as an instructive exercise, and exercise as well, to add (1) and (4) to the picture.

(6) However, matter of having our theorem formally proved, I mean by me professor and not by you student, here is a less elegant argument, which is however very quick, and does the job. The point is that, in each of the cases (1-4) under consideration, the numbers C_k that we get are easily seen to be subject to the following recurrence:

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

The initial data being the same, namely $C_1 = 1$ and $C_2 = 2$, in each of the cases (1-4) under consideration, we get indeed the same numbers. □

Now we can pass to the second step, namely selecting in the above list the objects that we find the most convenient to count, and count them. This leads to:

THEOREM 2.28. *The Catalan numbers are given by the formula*

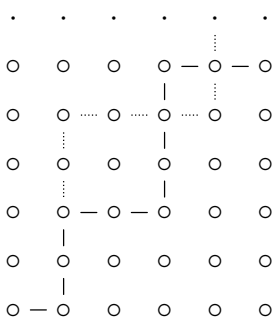
$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

with this being best seen by counting the length $2k$ Dyck paths in the plane.

PROOF. This is something quite tricky, the idea being as follows:

(1) Let us count indeed the Dyck paths in the plane. For this purpose, we use a trick. Indeed, if we ignore the assumption that our path must stay under the diagonal of the square, we have $\binom{2k}{k}$ such paths. And among these, we have the "good" ones, those that we want to count, and then the "bad" ones, those that we want to ignore.

(2) So, let us count the bad paths, those crossing the diagonal of the square, and reaching the higher diagonal next to it, the one joining $(0, 1)$ and $(k, k+1)$. In order to count these, the trick is to "flip" their bad part over that higher diagonal, as follows:



(3) Now observe that, as it is obvious on the above picture, due to the flipping, the flipped bad path will no longer end in (k, k), but rather in $(k-1, k+1)$. Moreover, more is true, in the sense that, by thinking a bit, we see that the flipped bad paths are precisely

those ending in $(k-1,k+1)$. Thus, we can count these flipped bad paths, and so the bad paths, and so the good paths too, and so good news, we are done.

(4) To finish now, by putting everything together, we have:

$$\begin{aligned} C_k &= \binom{2k}{k} - \binom{2k}{k-1} \\ &= \binom{2k}{k} - \frac{k}{k+1}\binom{2k}{k} \\ &= \frac{1}{k+1}\binom{2k}{k} \end{aligned}$$

Thus, we are led to the formula in the statement. □

Note in passing that we have proved here Conjecture 2.17. Many other things can be said about the Catalan numbers. We will be back to this.

2e. Exercises

This was yet another very standard chapter, and as exercises on this, we have:

EXERCISE 2.29. *Learn more about $2^\infty > \infty$, and the continuum hypothesis.*

EXERCISE 2.30. *Meditate at, or of course solve, the Venn diagram problem for 4 sets.*

EXERCISE 2.31. *Find some further interesting applications of inclusion-exclusion.*

EXERCISE 2.32. *Compute and memorize the coefficients $\binom{n}{k}$, for all $n \leq 10$.*

EXERCISE 2.33. *Learn more about the arithmetic properties of the numbers $\binom{n}{k}$.*

EXERCISE 2.34. *Learn more about permutations, and their various properties.*

EXERCISE 2.35. *Do some further counts for the circle, and the segment graphs.*

EXERCISE 2.36. *Learn more about the Catalan numbers, as much as you can.*

As bonus exercise, find and start reading a nice book on combinatorics.

CHAPTER 3

Fractions

3a. Fractions

Time now for some more complicated mathematics, going beyond what we know about the positive integers. We will be talking here about the mathematics of fractions.

We denote as before by $\mathbb{N}$ the set of positive integers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, with $\mathbb{N}$ standing for "natural". As before, quite often we will need negative numbers too, and we denote by $\mathbb{Z}$ the set of all integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, with $\mathbb{Z}$ standing for "zahlen", which is German for "numbers". Also, we use the following notations:

$$\mathbb{N}^* = \mathbb{N} - \{0\} \quad , \quad \mathbb{Z}^* = \mathbb{Z} - \{0\}$$

We recall from chapter 1 that for doing advanced mathematics with the integers we must study their quotients, and we are led in this way to the following definition:

DEFINITION 3.1. *Given an integer dividing another integer,*

$$b|a$$

we can talk about the corresponding quotient c, given by $a = bc$, which is denoted

$$c = \frac{a}{b}$$

and is called fraction.

The fractions are subject to a number of formulae, that we explained in chapter 1, which are all useful, in the real life. For addition and substraction, the formulae are:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad , \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

As for multiplication and division, here the formulae are as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad , \quad \frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$$

Also as explained in chapter 1, we can order such fractions, with the relevant formula here, for the positive fractions, being something very intuitive, as follows:

$$\frac{a}{b} < \frac{c}{d} \iff ad < bc$$

The point now is that we can talk about fractions even when $b|a$ fails, in the obvious way. And, with this convention, the above formulae still hold. Let us start with:

DEFINITION 3.2. *The rational numbers are the quotients of type*

$$r = \frac{a}{b}$$

with $a, b \in \mathbb{Z}$, and $b \neq 0$, identified according to the usual rule for quotients, namely:

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

We denote the set of rational numbers by $\mathbb{Q}$, standing for "quotients".

So, this is the definition of the rational numbers, which will take us some time, in order to properly understand. Generally speaking, the idea will be as follows:

(1) We will usually treat, based on a number of abstract results that we will prove, the rational numbers as usual numbers, that is, as integers.

(2) With the remark of course that the rational numbers are not necessarily integers. It is only that their arithmetic is quite similar to that of the integers.

Which might seem quite abstract. So welcome to algebra, where such methods, generalizing objects and then saying "results for them follow as before" is commonplace.

As a first observation, we have inclusions as follows, coming from definitions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

Getting now to operations and algebra, we must first talk about addition and substraction, and then about multiplication and division. Thus, many things to be done here, and we will do this slowly. As a first result, regarding the addition, we have:

THEOREM 3.3. *We can add the rational numbers $r = a/b$ according to the usual rule for quotients, namely*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and with this convention, we have the following formulae,

$$(r + s) + t = r + (s + t) \quad , \quad r + s = s + r$$

called associativity and commutativity of the addition operation.

PROOF. We can certainly define an operation as in the statement, and with this done, our operation is indeed associative, as shown by the following computation:

$$\begin{aligned}\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} &= \frac{ad+bc}{bd}+\frac{e}{f}\\ &= \frac{(ad+bc)f+bde}{bdf}\\ &= \frac{adf+bcf+bde}{bdf}\\ &= \frac{adf+b(cf+de)}{bdf}\\ &= \frac{a}{b}+\frac{cf+de}{df}\\ &= \frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)\end{aligned}$$

As for the commutativity of the sum, this is clear from definitions, as shown by:

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= \frac{ad+bc}{bd}\\ &= \frac{cb+da}{db}\\ &= \frac{c}{d}+\frac{a}{b}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Next, we must talk about substraction. The result here is similar, as follows:

PROPOSITION 3.4. *We can invert the rational numbers $r=a/b$ according to the usual rule for quotients, namely*

$$-\frac{a}{b}=\frac{-a}{b}$$

and with this convention, we have the following formula:

$$r+(-r)=(-r)+r=0$$

More generally, we can substract the rationals according to the usual rule

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

and with this convention, all the basic formulae that we know for fractions still hold.

PROOF. We can certainly define an inversion operation as in the statement, and with this done, the first formula in the statement holds indeed, as shown by:

$$\frac{a}{b}+\left(-\frac{a}{b}\right)=\frac{a}{b}+\frac{-a}{b}=\frac{ab-ab}{b}=\frac{0}{b}=0$$

As for the second formula in the statement, its proof is similar, as follows:

$$\left(-\frac{a}{b}\right)+\frac{a}{b}=\frac{-a}{b}+\frac{a}{b}=\frac{-ab+ab}{b}=\frac{0}{b}=0$$

Regarding now the substraction operation in the statement, that comes, somewhat by definition, from the following computation:

$$\frac{a}{b}-\frac{c}{d}=\frac{a}{b}+\left(-\frac{c}{d}\right)=\frac{a}{b}+\frac{-c}{d}=\frac{ad-bc}{bd}$$

Finally, in what regards the last assertion, this is something rather informal, and we will leave doing some computations here, with general rational numbers instead of fractions, in order to make sure that everything works fine, as an exercise. □

Summarizing, everything fine with the addition of rationals. Next, we must talk about multiplication. The result here is similar to Theorem 3.3, as follows:

THEOREM 3.5. *We can multiply the rational numbers $r=a/b$ according to the usual rule for quotients, namely*

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

and with this convention, we have the following formulae,

$$(rs)t=r(st)\quad,\quad rs=sr$$

called associativity and commutativity of the multiplication operation.

PROOF. We can certainly define an operation as in the statement, and with this done, our operation is indeed associative, as shown by the following computation:

$$\begin{aligned}\left(\frac{a}{b}\cdot\frac{c}{d}\right)\frac{e}{f}&=\frac{ac}{bd}\cdot\frac{e}{f}\\&=\frac{ace}{bdf}\\&=\frac{a}{b}\cdot\frac{ce}{df}\\&=\frac{a}{b}\left(\frac{c}{d}\cdot\frac{e}{f}\right)\end{aligned}$$

As for the commutativity property, this is clear from definitions, as shown by:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}=\frac{ca}{db}=\frac{c}{d}\cdot\frac{a}{b}$$

Thus, we are led to the conclusions in the statement. □

Next, we must talk about division. The result here is again routine, as follows:

PROPOSITION 3.6. *We can invert the nonzero rational numbers $r = a/b$ according to the usual rule for quotients, namely*

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

and with this convention, we have the following formula:

$$r \cdot r^{-1} = r^{-1} \cdot r = 1$$

More generally, we can divide the rationals according to the usual rule

$$\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$$

and with this convention, all the basic formulae that we know for fractions still hold.

PROOF. We can certainly define an inversion operation as in the statement, and with this done, the first formula in the statement holds indeed, as shown by:

$$\frac{a}{b}\left(\frac{a}{b}\right)^{-1} = \frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1$$

As for the second formula in the statement, its proof is similar, as follows:

$$\left(\frac{a}{b}\right)^{-1}\frac{a}{b} = \frac{b}{a} \cdot \frac{a}{b} = \frac{ba}{ab} = 1$$

Regarding now the division operation in the statement, that comes, somewhat by definition, from the following computation:

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b}\left(\frac{c}{d}\right)^{-1} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Finally, in what regards the last assertion, this is something rather informal, and we will leave doing some computations here, with general rational numbers instead of fractions, in order to make sure that everything works fine, as an exercise. □

As a comment here, in the division formula from Proposition 3.6 we assume of course $c \neq 0$. It is possible to go beyond this, with the introduction of the infinity symbols $\pm\infty$, which are subject to a number of useful formulae, such as:

$$\frac{1}{\pm\infty} = 0$$

However, more on such things later, when we will be doing analysis.

3b. More fractions

As a conclusion now, things fine with both the addition and the multiplication, and you might probably think that we are done with all this algebra. Well, you must be kidding. Many other basic algebraic things remain to be done, such as:

THEOREM 3.7. *The addition and multiplication of rationals are subject to*

$$r(s+t) = rs + rt \quad , \quad (r+s)t = rt + st$$

called distributivity formulae.

PROOF. The verification of the first formula goes as follows:

$$\begin{aligned}
\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) &= \frac{a}{b} \cdot \frac{cf + de}{df} \\
&= \frac{acf + ade}{bdf} \\
&= \frac{acf}{bdf} + \frac{ade}{bdf} \\
&= \frac{ac}{bd} + \frac{ae}{bf} \\
&= \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}
\end{aligned}$$

As for the second formula, this follows from the first one, and commutativity:

$$\begin{aligned}
(r+s)t &= t(r+s) \\
&= tr + ts \\
&= rt + st
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Summarizing, many operations that we have here, and job for us to get familiar with all this, via practice, tricks and so on. Here is my favorite trick for fractions, which is something quite trivial, but I will call this Theorem, because this is perhaps the matematical formula that I use the most, in my complicated, daily quantum physics work:

THEOREM 3.8. *We have the following substraction formula,*

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

valid for any $n \in \mathbb{N}$. As illustrations for this, we have

$$1 - \frac{1}{2} = \frac{1}{2} \quad , \quad \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad , \quad \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad , \quad \frac{1}{4} - \frac{1}{5} = \frac{1}{20} \quad \ldots$$

and with the knowledge of these latter formulae being mandatory too.

PROOF. This is something trivial, but since we called our result Theorem, as mathematicians do, let us pull out now a complete proof, also as mathematicians do. We have the following computation, based on our general formula for substracting fractions:

$$\begin{aligned}\frac{1}{n}-\frac{1}{n+1} &= \frac{1\times(n+1)-1\times n}{n(n+1)}\\ &= \frac{(n+1)-n}{n(n+1)}\\ &= \frac{1}{n(n+1)}\end{aligned}$$

Thus, theorem proved, and for the particular cases at the end, I will leave it to you. The more such particular cases you know well, the better your mathematics will be. □

At a more abstract level now, we have the following result, regarding the sums:

THEOREM 3.9. *The sum of two fractions is always of the following form,*

$$\frac{a}{b}+\frac{c}{d}=\frac{e}{[b,d]}$$

with $e\in\mathbb{Z}$ being a certain number. More generally, the sum of n fractions is of the form

$$\frac{a_1}{b_1}+\ldots+\frac{a_n}{b_n}=\frac{e}{[b_1,\ldots,b_n]}$$

with $e\in\mathbb{Z}$ being a certain number.

PROOF. In what regards the first assertion, we know from chapter 1 that the least common multiple $[b,d]$ appears as follows, for certain integers p,q:

$$[b,d]=bp=dq$$

But with this, we have the following computation, proving the first assertion:

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= \frac{ap}{bp}+\frac{cq}{dq}\\ &= \frac{ap}{[b,d]}+\frac{cq}{[b,d]}\\ &= \frac{ap+cq}{[b,d]}\end{aligned}$$

As for the second assertion, its proof is similar. We know that the least common multiple $[b_1,\ldots,b_n]$ appears as follows, for certain integers $p_1,\ldots,p_n$:

$$[b_1,\ldots,b_n]=b_1p_1=\ldots=b_np_n$$

But with these formulae in hand, we have the following computation:

$$\begin{aligned}\frac{a_1}{b_1}+\ldots+\frac{a_n}{b_n} &= \frac{a_1p_1}{b_1p_1}+\ldots+\frac{a_np_n}{b_np_n}\\ &= \frac{a_1p_1}{[b_1,\ldots,b_n]}+\ldots+\frac{a_np_n}{[b_1,\ldots,b_n]}\\ &= \frac{a_1p_1+\ldots+a_np_n}{[b_1,\ldots,b_n]}\end{aligned}$$

Thus, theorem proved, but as before with many other such things, a lot of practice is needed, meaning working out a lot of exercises, in order to master well this method. □

Summarizing, we have a nice theory of rational numbers, extending well what we knew from before, from chapter 1, regarding the fractions $r = a/b$ with $b|a$.

There is actually one more thing to be talked about, in relation with this, namely the ordering of the fractions. The result here, formulated a bit informally, is as follows:

THEOREM 3.10. *We can order the positive fractions as follows,*

$$\frac{a}{b}<\frac{c}{d}\iff ad<bc$$

and with this, all the basic results about the ordering of numbers extend.

PROOF. This is obviously something a bit informal, but there is a reason for this, there are in fact countless things to be checked here, and my concern as math professor is you getting a bit bored by all this, and starting to attend a chemistry class instead. This being said, as a sample verification, let us attempt to prove the following fact:

$$\frac{a}{b}<\frac{c}{d}\ ,\ \frac{e}{f}<\frac{g}{h}\implies \frac{a}{b}+\frac{e}{f}<\frac{c}{d}+\frac{g}{h}$$

But this can be proved indeed, with some patience, as follows:

$$\begin{aligned}\frac{a}{b}+\frac{e}{f}<\frac{c}{d}+\frac{g}{h} &\iff \frac{af+be}{bf}<\frac{ch+dg}{dh}\\ &\iff (af+be)dh<bf(ch+dg)\\ &\iff adfh+bdeh<bcfh+bdgf\\ &\iff adfh-bcfh<bdgf-bdeh\\ &\iff (ad-bc)fh<bd(gf-eh)\end{aligned}$$

Indeed, what we have at the end does hold, because our assumptions give:

$$(ad-bc)fh<0<bd(gf-eh)$$

So, verification done, and for the rest, exercise of course for you to formulate a more precise version of the statement, and perform the other verifications that are needed. □

Finally, as a last useful piece of abstract mathematics for the fractions, further clarifying a number of manipulations that we did in chapter 2, we have:

THEOREM 3.11. *We can talk about the integer part of fractions,*

$$\left[\frac{a}{b}\right] = c \in \mathbb{Z}$$

with the number $c \in \mathbb{Z}$ being by definition the unique one such that

$$c \leq \frac{a}{b} < c + 1$$

and with this being compatible with what we did before in chapter 2.

PROOF. This is again something informal and compact, and again designed for you to learn a bit about this, and not attend that chemistry class instead. In what regards now the proof, according to chapter 2, and assuming $a, b > 0$, the integer part of a/b was constructed by determining where a fits, with respect to the following progression:

$$0, b, 2b, 3b, 4b, \ldots$$

To be more precise, the number $c = [a/b]$ was given by the following formula:

$$cb \leq a < (c + 1)b$$

Now observe that, by dividing everything by b, this is equivalent to:

$$c \leq \frac{a}{b} < c + 1$$

Thus, we are led to the conclusion in the statement. There are of course far more things that can be said, regarding this, and regarding the fractional part too, given by:

$$\left\{\frac{a}{b}\right\} = \frac{a}{b} - \left[\frac{a}{b}\right]$$

With exercise of course for you, to further explore all this. □

Getting now a bit abstract, the basic operations on the rational numbers, namely sum, product and inversion, tell us that $\mathbb{Q}$ is a field, in the following sense:

DEFINITION 3.12. *A field is a set F with a sum operation $+$ and a product operation $\times$, subject to the following conditions:*

(1) *$a + b = b + a$, $a + (b + c) = (a + b) + c$, there exists $0 \in F$ such that $a + 0 = 0$, and any $a \in F$ has an inverse $-a \in F$, satisfying $a + (-a) = 0$.*
(2) *$ab = ba$, $a(bc) = (ab)c$, there exists $1 \in F$ such that $a1 = a$, and any $a \neq 0$ has a multiplicative inverse $a^{-1} \in F$, satisfying $aa^{-1} = 1$.*
(3) *The sum and product are compatible via $a(b + c) = ab + ac$.*

So, this is the much feared definition of the fields, and more on this later in this chapter. In the meantime, let us record the following result, coming from the above:

THEOREM 3.13. *The rational numbers $\mathbb{Q}$ form a field, with operations given by:*

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}\quad,\quad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

In fact, $\mathbb{Q}$ is the smallest field containing $\mathbb{Z}$, or even the smallest field containing $\mathbb{N}$.

PROOF. This is more or less clear from the above, as follows:

(1) We have indeed a field, with the operations in the statement, with the verification of the various field axioms being something clear, and that we already know well from chapter 1, for the fractions which happen to be integers, that is, $r=a/b$ with $b|a$.

(2) As a comment here, we have opted to include in the statement only the basic operations for fractions, that of the sum, and product. The other operations, regarding substraction, division, inverses, can be all deduced easily from these two operations.

(3) Regarding now the last assertion, when searching for a field containing $\mathbb{N}$, by looking at the equation $c=a-b$ we are led into $\mathbb{Z}$. But then, by looking at the equation $c=a/b$, we are led into $\mathbb{Q}$. Thus, the field that we were looking for is $\mathbb{Q}$. □

Still staying a bit abstract, as a first result about the rationals, in relation with what we like to do the most, since the beginning of this book, namely counting, we have:

THEOREM 3.14. *The field of rational numbers $\mathbb{Q}$ is countable.*

PROOF. This comes from what we know from chapter 2, but always good to talk about that again. Consider the following table, containing all quotients a/b, with $a,b\in\mathbb{N}$:

$$\begin{array}{cccccccc}
1/1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & \ldots\\
2/1 & 2/2 & 2/3 & 2/4 & 2/5 & 2/6 & \ldots\\
3/1 & 3/2 & 3/3 & 3/4 & 3/5 & 3/6 & \ldots\\
4/1 & 4/2 & 4/3 & 4/4 & 4/5 & 4/6 & \ldots\\
5/1 & 5/2 & 5/3 & 5/4 & 5/5 & 5/6 & \ldots\\
6/1 & 6/2 & 6/3 & 6/4 & 6/5 & 6/6 & \ldots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

We can then snake our way inside this table, in the following way, starting from top left, and we count in this way $\mathbb{Q}_+$, with some redundancies:

$$\begin{array}{cccccccc}
1/1 & \to & 1/2 & 1/3 & \to & 1/4 & 1/5 \to & 1/6 \quad \ldots \\
\swarrow & \nearrow & & \swarrow & \nearrow & & \swarrow & \\
2/1 & 2/2 & 2/3 & 2/4 & 2/5 & 2/6 & \ldots & \\
\downarrow & \nearrow & \swarrow & \nearrow & \swarrow & & & \\
3/1 & 3/2 & 3/3 & 3/4 & 3/5 & 3/6 & \ldots & \\
\swarrow & \nearrow & \swarrow & & & & & \\
4/1 & 4/2 & 4/3 & 4/4 & 4/5 & 4/6 & \ldots & \\
\downarrow & \nearrow & \swarrow & & & & & \\
5/1 & 5/2 & 5/3 & 5/4 & 5/5 & 5/6 & \ldots & \\
\swarrow & & & & & & & \\
6/1 & 6/2 & 6/3 & 6/4 & 6/5 & 6/6 & \ldots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots &
\end{array}$$

Thus, after eliminating the redundancies, theorem proved. □

Many other things can be said, as a continuation of the above, notably with the question of explicitly listing the elements $\mathbb{Q}$, if possible in some sort of increasing order. However, as we will soon discover, by doing some analysis, this is not really possible.

3c. Convergence

Time now to do some analysis, with our rational numbers. Here is what you need to know, in order to do mathematical analysis:

DEFINITION 3.15. *We say that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ converges to $x \in \mathbb{Q}$ when:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - x| < \varepsilon$$

In this case, we write $\lim_{n\to\infty} x_n = x$, or simply $x_n \to x$.

This might look quite scary, at a first glance, but when thinking a bit, there is nothing scary about it. Indeed, let us try to understand, how shall we translate $x_n \to x$ into mathematical language. And here, things are quite straightforward, as follows:

(1) The condition $x_n \to x$ tells us that "when n is big, x_n is close to x".

(2) That is, this tells us that "when n is big enough, x_n gets arbitrarily close to x".

(3) But, "n big enough" obviously means $n \geq N$, for some $N \in \mathbb{N}$.

(4) And "x_n arbitrarily close to x" means $|x_n - x| < \varepsilon$, for some $\varepsilon > 0$.

Thus, we are naturally led to the above definition, and end of the story. In fact, as this book will go by, you will get so used to Definition 3.15, to the point of loving it.

Time perhaps for some illustrations. As a basic example for all this, we have:

PROPOSITION 3.16. *We have* $1/n \to 0$.

PROOF. This is obvious, but let us prove it by using Definition 3.15. We have:

$$\begin{aligned}\left|\frac{1}{n}-0\right|<\varepsilon &\iff \frac{1}{n}<\varepsilon\\ &\iff \frac{1}{\varepsilon}<n\\ &\iff \left[\frac{1}{\varepsilon}\right]<n\end{aligned}$$

Thus we can take $N=[1/\varepsilon]+1$ in Definition 3.15, and we are done. □

There are many other examples, and more on this in a moment. Going ahead with more theory, let us complement Definition 3.15 with:

DEFINITION 3.17. *We write* $x_n \to \infty$ *when the following condition is satisfied:*

$$\forall K>0, \exists N\in\mathbb{N}, \forall n\geq N, x_n>K$$

Similarly, we write $x_n\to-\infty$ *when the same happens, with* $x_n<-K$ *at the end.*

Again, this is something very intuitive, coming from the fact that $x_n\to\infty$ can only mean that x_n is arbitrarily big, for n big enough. We will leave some thinking here, along the lines of the discussion following Definition 3.15, as an instructive exercise.

As a basic illustration now for Definition 3.17, we have:

PROPOSITION 3.18. *We have* $n^2\to\infty$.

PROOF. As before, this is obvious, but let us prove it using Definition 3.17. We have the following computation, with a formal meaning for the number $\sqrt{K}$, namely with $n>\sqrt{K}$ meaning in practice $n>r$, for any $r\in\mathbb{Q}$ satisfying $K^2\geq r$:

$$\begin{aligned}n^2>K &\iff n>\sqrt{K}\\ &\iff n>[\sqrt{K}]\end{aligned}$$

Thus we can take $N=[\sqrt{K}]+1$ in Definition 3.17, and we are done. □

We can unify and generalize Proposition 3.16 and Proposition 3.18, as follows:

PROPOSITION 3.19. *We have the following convergence,*

$$n^a \to \begin{cases} 0 & (a < 0) \\ 1 & (a = 0) \\ \infty & (a > 0) \end{cases}$$

with $n \to \infty$.

PROOF. This follows indeed by using the same method as in the proof of Proposition 3.16 and Proposition 3.18, and we will leave the details here to you, reader. □

We have some general results about limits, summarized as follows:

THEOREM 3.20. *The following happen:*

(1) *The limit $\lim_{n\to\infty} x_n$, if it exists, is unique.*
(2) *If $x_n \to x$, with $x \in (-\infty, \infty)$, then x_n is bounded.*
(3) *Assuming $x_n \to x$, any subsequence of x_n converges to x.*

PROOF. All this is elementary, coming from definitions:

(1) Assuming $x_n \to x$, $x_n \to y$ we have indeed, for any $\varepsilon > 0$, for n big enough:

$$|x - y| \leq |x - x_n| + |x_n - y| < 2\varepsilon$$

(2) Assuming $x_n \to x$, we have $|x_n - x| < 1$ for $n \geq N$, and so, for any $k \in \mathbb{N}$:

$$|x_k| < 1 + |x| + \sup\left(|x_1|, \ldots, |x_{n-1}|\right)$$

(3) This is clear indeed from definitions. □

Here are as well some general rules for computing limits:

THEOREM 3.21. *The following happen, with the standard conventions*

$$\infty + \infty = \infty \quad , \quad \infty \cdot \infty = \infty \quad , \quad \frac{1}{\infty} = 0$$

and with the conventions that $\infty - \infty$ and $\infty \cdot 0$ are undefined:

(1) *$x_n \to x$ implies $\lambda x_n \to \lambda x$.*
(2) *$x_n \to x$, $y_n \to y$ implies $x_n + y_n \to x + y$.*
(3) *$x_n \to x$, $y_n \to y$ implies $x_n y_n \to xy$.*
(4) *$x_n \to x$ with $x \neq 0$ implies $1/x_n \to 1/x$.*

PROOF. All this is again elementary, coming from definitions:

(1) This is something which is obvious from definitions.

(2) This follows indeed from the following estimate:

$$|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y|$$

(3) This follows indeed from the following estimate:

$$\begin{aligned}|x_ny_n-xy| &= |(x_n-x)y_n+x(y_n-y)|\\ &\leq |x_n-x|\cdot|y_n|+|x|\cdot|y_n-y|\end{aligned}$$

(4) This is again clear, by estimating $1/x_n-1/x$, in the obvious way. □

As an application of the above rules, we have the following useful result:

THEOREM 3.22. *The $n\to\infty$ limits of quotients of polynomials are given by*

$$\lim_{n\to\infty}\frac{a_pn^p+a_{p-1}n^{p-1}+\ldots+a_0}{b_qn^q+b_{q-1}n^{q-1}+\ldots+b_0}=\lim_{n\to\infty}\frac{a_pn^p}{b_qn^q}$$

with the limit on the right being $\pm\infty$, 0, a_p/b_q, depending on the values of p,q.

PROOF. The first assertion comes from the following computation:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_pn^p+a_{p-1}n^{p-1}+\ldots+a_0}{b_qn^q+b_{q-1}n^{q-1}+\ldots+b_0} &= \lim_{n\to\infty}\frac{n^p}{n^q}\cdot\frac{a_p+a_{p-1}n^{-1}+\ldots+a_0n^{-p}}{b_q+b_{q-1}n^{-1}+\ldots+b_0n^{-q}}\\ &= \lim_{n\to\infty}\frac{a_pn^p}{b_qn^q}\end{aligned}$$

As for the second assertion, this comes from Proposition 3.19. □

Time now to get into some truly interesting mathematics. Let us start with:

DEFINITION 3.23. *Given numbers $x_0,x_1,x_2,\ldots\in\mathbb{Q}$, we write*

$$\sum_{n=0}^{\infty}x_n=x$$

with $x\in[-\infty,\infty]$ when $\lim_{k\to\infty}\sum_{n=0}^{k}x_n=x$.

As before with the sequences, there is some general theory that can be developed for the series, and more on this in a moment. As a first, basic example, we have:

THEOREM 3.24. *We have the "geometric series" formula*

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$$

valid for any $|x|<1$. For $|x|\geq 1$, the series diverges.

PROOF. This is something fundamental, worth a detailed discussion, as follows:

(1) Let us first see what happens for $x=1/2$. Here the formula to be proved is:

$$\sum_{n=0}^{\infty}\frac{1}{2^n}=2$$

But this is something clear, because we can take the interval $[0,2]$, and successively divide at right into halves, and we are led to the above formula.

(2) Next, let us see what happens for $x=1/3$. Here the formula to be proved is:

$$\sum_{n=0}^{\infty}\frac{1}{3^n}=\frac{3}{2}$$

But this can be proved a bit as for $x=1/2$, in a geometric way, exercise for you.

(3) Next, let us see what happens for $x=1/k$. Here the formula to be proved is:

$$\sum_{n=0}^{\infty}\frac{1}{k^n}=\frac{k}{k-1}$$

But this can be proved a bit as for $k=2,3$, again a good exercise for you.

(4) In general now, our first claim, which comes by multiplying and simplifying, is that we have the following formula, for the partial sums of the series:

$$\sum_{n=0}^{k}x^n=\frac{1-x^{k+1}}{1-x}$$

But this proves the first assertion, because with $k\to\infty$ we get:

$$\sum_{n=0}^{k}x^n\to\frac{1}{1-x}$$

As for the second assertion, this is clear as well from our formula above. □

Finally, let us record as well the following result, due to Riemann:

THEOREM 3.25. *We have the following formula,*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\infty$$

with the divergence happening, despite the terms going to 0.

PROOF. The first assertion comes from the following computation:

$$\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots&=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots\\
&\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ldots\\
&=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\\
&=\infty
\end{aligned}$$

As for the second assertion, this is a mere remark, never to be forgotten. □

As an interesting application of our convergence technology, we have:

THEOREM 3.26. *We have the generalized binomial formula*

$$(1+x)^a=\sum_{k=0}^{\infty}\binom{a}{k}x^k$$

with the generalized binomial coefficients being given by

$$\binom{a}{k}=\frac{a(a-1)\ldots(a-k+1)}{k!}$$

valid for any exponent $a\in\mathbb{Z}$, and any $|x|<1$.

PROOF. This is something quite tricky, the idea being as follows:

(1) For exponents $a\in\mathbb{N}$, this is something that we know well from chapter 2, and which is valid for any $x\in\mathbb{Q}$, coming from the usual binomial formula, namely:

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$$

(2) For the exponent $a=-1$ this is something that we know too, from the above, coming from the following formula, valid for any $|x|<1$:

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots$$

Indeed, this is exactly our generalized binomial formula at $a=-1$, because:

$$\binom{-1}{k}=\frac{(-1)(-2)\ldots(-k)}{k!}=(-1)^k$$

(3) Let us discuss now the general case $a\in-\mathbb{N}$. With $a=-n$, and $n\in\mathbb{N}$, the generalized binomial coefficients are given by the following formula:

$$\begin{aligned}\binom{-n}{k}&=\frac{(-n)(-n-1)\ldots(-n-k+1)}{k!}\\&=(-1)^k\frac{n(n+1)\ldots(n+k-1)}{k!}\\&=(-1)^k\frac{(n+k-1)!}{(n-1)!k!}\\&=(-1)^k\binom{n+k-1}{n-1}\end{aligned}$$

Thus, our generalized binomial formula at $a=-n$, and $n\in\mathbb{N}$, reads:

$$\frac{1}{(1+t)^n}=\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{n-1}t^k$$

(4) In order to prove this formula, it is convenient to write it with $-t$ instead of t, in order to get rid of signs. The formula to be proved becomes:

$$\frac{1}{(1-t)^n}=\sum_{k=0}^{\infty}\binom{n+k-1}{n-1}t^k$$

We prove this by recurrence on n. At $n=1$ this formula definitely holds, as explained in (2) above. So, assume that the formula holds at $n\in\mathbb{N}$. We have then:

$$\begin{aligned}\frac{1}{(1-t)^{n+1}}&=\frac{1}{1-t}\cdot\frac{1}{(1-t)^n}\\&=\sum_{k=0}^{\infty}t^k\sum_{l=0}^{\infty}\binom{n+l-1}{n-1}t^l\\&=\sum_{s=0}^{\infty}t^s\sum_{l=0}^{s}\binom{n+l-1}{n-1}\end{aligned}$$

On the other hand, the formula that we want to prove is:

$$\frac{1}{(1-t)^{n+1}}=\sum_{s=0}^{\infty}\binom{n+s}{n}t^k$$

Thus, in order to finish, we must prove the following formula:

$$\sum_{l=0}^{s}\binom{n+l-1}{n-1}=\binom{n+s}{n}$$

(5) In order to prove this latter formula, we proceed by recurrence on $s\in\mathbb{N}$. At $s=0$ the formula is trivial, $1=1$. So, assume that the formula holds at $s\in\mathbb{N}$. In order to prove the formula at $s+1$, we are in need of the following formula:

$$\binom{n+s}{n}+\binom{n+s}{n-1}=\binom{n+s+1}{n}$$

But this is the Pascal formula, that we know from chapter 2, and we are done. □

Quite interestingly, the formula in Theorem 3.26 holds for other exponents too, but this is something non-trivial, whose proof will have to wait until chapter 5 below.

We will be back to more analysis in chapter 5, when talking real numbers. The point indeed is that many sequences or series of rationals seem to converge, but to some numbers that are not rational. Well, these mysterious limits, that we do not know about yet, are the real numbers, that we will introduce and study in detail in chapter 5.

3d. Fields, algebra

With analysis understood, time now for some more algebra. Remember from our discussion regarding fields and $\mathbb{Q}$, that can be coupled if necessary with the general discussion regarding numbers and $\mathbb{N}$, from the beginning of chapter 2, that we have:

PRINCIPLE 3.27. *$\mathbb{Q}$ is the simplest field.*

However, and here comes our point, purely mathematically speaking, this is not exactly true, because, by a strange twist of fate, the numbers $0, 1$, whose presence in a field is mandatory, $0, 1 \in F$, can form themselves a field, with addition as follows:

$$1 + 1 = 0$$

To be more precise, according to our field axioms, we certainly must have:

$$0 + 0 = 0 \times 0 = 0 \times 1 = 1 \times 0 = 0$$

$$0 + 1 = 1 + 0 = 1 \times 1 = 1$$

Thus, everything regarding the addition and multiplication of $0, 1$ is uniquely determined, except for the value of $1 + 1$. And here, you would say that we should normally set $1 + 1 = 2$, with $2 \neq 0$ being a new field element, but the point is that $1 + 1 = 0$ is something natural too, this being the addition modulo 2:

$$1 + 1 = 0(2)$$

And, what we get in this way is a field, denoted as follows:

$$\mathbb{F}_2 = \{0, 1\}$$

Let us summarize this finding, along with a bit more, obtained by suitably replacing our 2, used for addition, with an arbitrary prime number p, as follows:

THEOREM 3.28. *The following happen:*

(1) *$\mathbb{Q}$ is the simplest field having the property $1 + \ldots + 1 \neq 0$, in the sense that any field F having this property must contain it, $\mathbb{Q} \subset F$.*
(2) *The property $1 + \ldots + 1 \neq 0$ can hold or not, and if not, the smallest number of terms needed for having $1 + \ldots + 1 = 0$ is a certain prime number p.*
(3) *$\mathbb{F}_p = \{0, 1, \ldots, p - 1\}$, with p prime, is the simplest field having the property $1 + \ldots + 1 = 0$, with p terms, in the sense that this implies $\mathbb{F}_p \subset F$.*

PROOF. All this is basic number theory, the idea being as follows:

(1) This is clear, because $1 + \ldots + 1 \neq 0$ tells us that we have an embedding $\mathbb{N} \subset F$, and then by taking inverses with respect to $+$ and $\times$ we obtain $\mathbb{Q} \subset F$.

(2) Again, this is clear, because assuming $1+\ldots+1=0$, with $p=ab$ terms, chosen minimal, we would have a formula as follows, which is a contradiction:

$$(\underbrace{1+\ldots+1}_{a\ terms})(\underbrace{1+\ldots+1}_{b\ terms})=0$$

(3) This follows a bit as in (1), with the copy $\mathbb{F}_p\subset F$ consisting by definition of the various sums of type $1+\ldots+1$, which must cycle modulo p, as shown by (2). □

Getting back now to our philosophical discussion regarding numbers, what we have in Theorem 3.28 is not exactly good news, suggesting that, on purely mathematical grounds, there is a certain rivalry between $\mathbb{Q}$ and $\mathbb{F}_p$, as being the simplest field.

So, which of these two fields shall we study here, say as having been created first? Not an easy question, but as an answer to this, let us update Principle 3.27 as follows:

PRINCIPLE 3.29 (update). *Ignoring what pure mathematics might say, and trusting instead physics and chemistry, we will choose to trust in $\mathbb{Q}$, as being the simplest field.*

In short, welcome to science, and with this being something quite natural for us, mathematics and science being the topic of the present book.

This being said, in view of the above, let us study a bit more the fields $\mathbb{F}_p$, with this being good learning, trust me. At $p=2$, the situation is as follows:

PROPOSITION 3.30. *The integers modulo 2 add according to the table*

$$\begin{array}{cc} + & 1\\ 1 & 0\end{array}$$

and multiply according to the table

$$\begin{array}{cc} \times & 1\\ 1 & 1\end{array}$$

and they form a finite field, $\mathbb{F}_2=\{0,1\}$.

PROOF. This is indeed something self-explanatory, that we know from the above. As an important comment here, observe the similarity with the tables from chapter 1, for numeration basis 2. Indeed, the addition table there was as follows:

$$\begin{array}{cc} + & 1\\ 1 & 10\end{array}$$

As for the multiplication table there, that was as follows:

$$\begin{array}{cc} \times & 1\\ 1 & 1\end{array}$$

And we will leave some further thinking here, as an instructive exercise. □

Coming next, at $p=3$ the situation is as follows:

PROPOSITION 3.31. *The integers modulo* 3 *add according to the table*

$$\begin{array}{ccc} + & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array}$$

and multiply according to the table

$$\begin{array}{ccc} \times & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{array}$$

and they form a finite field, $\mathbb{F}_3 = \{0, 1, 2\}$.

PROOF. This is again something self-explanatory, that we know from the above. Observe again the similarity with the tables from chapter 1, for numeration basis 3. □

Let us study as well the case $p = 4$. Not a prime, with the result here being:

PROPOSITION 3.32. *The integers modulo* 4 *add according to the table*

$$\begin{array}{cccc} + & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{array}$$

and multiply according to the table

$$\begin{array}{cccc} \times & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 0 & 2 \\ 3 & 3 & 2 & 1 \end{array}$$

and they form a beast $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ *which is not a field.*

PROOF. This is again something self-explanatory, that we know from the above. Observe again the similarity with the tables from chapter 1, for numeration basis 4. □

The above result is quite concerning, and the obvious problem is, what is that beast $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ that we found. We will answer this question later, but in the meantime, let us record, for future reference, what is wrong with $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, namely:

$$2 \times 2 = 0$$

Coming next, at $p = 5$ the situation is as follows:

PROPOSITION 3.33. *The integers modulo* 5 *add according to the table*

$$\begin{array}{cccccc} + & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{array}$$

and multiply according to the table

$$\begin{array}{cccccc} \times & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 1 & 3 \\ 3 & 3 & 1 & 4 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{array}$$

and they form a finite field, $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.

PROOF. This is again something self-explanatory, that we know from the above. Observe again the similarity with the tables from chapter 1, for numeration basis 5. □

At $p = 6$ now, not a prime, the result here is as follows:

PROPOSITION 3.34. *The integers modulo* 6 *add according to the table*

$$\begin{array}{ccccccc} + & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{array}$$

and multiply according to the table

$$\begin{array}{ccccccc} \times & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 0 & 2 & 4 \\ 3 & 3 & 0 & 3 & 0 & 3 \\ 4 & 4 & 2 & 0 & 4 & 2 \\ 5 & 5 & 4 & 3 & 2 & 1 \end{array}$$

and they form a beast $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ *which is not a field.*

PROOF. This is again something self-explanatory, that we know from the above. Observe again the similarity with the tables from chapter 1, for numeration basis 6. □

Finally, at $p = 7$, which is a prime again, the result is as follows:

PROPOSITION 3.35. *The integers modulo* 7 *add according to the table*

$+$	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

and multiply according to the table

$\times$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

and they form a finite field, $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$.

PROOF. This is again something self-explanatory, that we know from the above. Observe again the similarity with the tables from chapter 1, for numeration basis 7. □

Summarizing, quite fun all this, obviously related to numeration bases, and to many other interesting things regarding the essence of numbers, that we met in chapter 1.

But the question that you might surely have in mind is, what to do with $p = 4, 6$ and other composite numbers? As a lazy answer here, let us formulate:

DEFINITION 3.36. *A commutative ring is a set* R *with a sum operation* $+$ *and a product operation* $\times$*, subject to the following conditions:*

(1) $a + b = b + a$, $a + (b + c) = (a + b) + c$, *there exists* $0 \in R$ *such that* $a + 0 = 0$.
(2) *Any* $a \in R$ *has an inverse* $-a \in R$, *satisfying* $a + (-a) = 0$.
(3) $ab = ba$, $a(bc) = (ab)c$, *there exists* $1 \in R$ *such that* $a1 = a$.
(4) *The sum and product are compatible via* $a(b + c) = ab + ac$.

That is, this is something as a field, save for the existence of multiplicative inverses.

Which is a lazy definition indeed, as mathematicians usually pull out when stuck with something complicated, because with this we can formulate right away, as a theorem:

THEOREM 3.37. *The following happen:*

(1) *The integers modulo* $p \in \mathbb{N}$ *form a commutative ring* $\mathbb{Z}_p$.
(2) *When* p *is prime, this ring is a field, denoted* $\mathbb{F}_p$.

PROOF. This is clear indeed from definitions, because the integers modulo $p \in \mathbb{N}$ satisfy all the properties from Definition 3.36, trivially. □

Getting back now to our question above, what to do with $p = 4, 6$ and other composite numbers, there is a non-lazy answer to it too, stating that in the case where p is the power of a prime, we can manufacture a certain finite field $\mathbb{F}_p$. But, more on this later.

Moving ahead now, with some more arithmetic, many things can be done with $\mathbb{Q}$, but getting straight to the point, one thing that fails is solving $x^2 = 2$:

THEOREM 3.38. *The field $\mathbb{Q}$ does not contain a square root of 2:*

$$\sqrt{2} \notin \mathbb{Q}$$

In fact, among integers, only the squares, $n = m^2$ with $m \in \mathbb{N}$, have square roots in $\mathbb{Q}$.

PROOF. This is something very standard, the idea being as follows:

(1) In what regards $\sqrt{2}$, assuming that $r = a/b$ with $a, b \in \mathbb{N}$ prime to each other satisfies $r^2 = 2$, we have $a^2 = 2b^2$, and so $a \in 2\mathbb{N}$. But then by using again $a^2 = 2b^2$ we obtain $b \in 2\mathbb{N}$ as well, which contradicts our assumption $(a, b) = 1$.

(2) Along the same lines, any prime number $p \in \mathbb{N}$ has the property $\sqrt{p} \notin \mathbb{Q}$, with the proof here being as the above one for $p = 2$, by congruence and contradiction.

(3) More generally, our claim is that any $n \in \mathbb{N}$ which is not a square has the property $\sqrt{n} \notin \mathbb{Q}$. Indeed, we can argue here that our number decomposes as $n = p_1^{a_1} \ldots p_k^{a_k}$, with $p_1, \ldots, p_k$ distinct primes, and our assumption that n is not a square tells us that one of the exponents $a_1, \ldots, a_k \in \mathbb{N}$ must be odd. Moreover, by extracting all the obvious squares from n, we can in fact assume $a_1 = \ldots = a_k = 1$. But with this done, we can set $p = p_1$, and the congruence argument from (2) applies, and gives $\sqrt{n} \notin \mathbb{Q}$, as desired. □

We can talk if we want about fields like $\mathbb{Q}[\sqrt{2}]$, as follows:

THEOREM 3.39. *The following set, with $\sqrt{2}$ formally solving $x^2 = 2$, is a field,*

$$\mathbb{Q}[\sqrt{2}] = \left\{ a + b\sqrt{2} \middle| a, b \in \mathbb{Q} \right\}$$

and the same happens for any $\mathbb{Q}[\sqrt{n}]$, with $n \neq m^2$ being not a square.

PROOF. All the field axioms are clearly satisfied, except perhaps for the inversion axiom. But this axiom is satisfied too, due to the following formula:

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}$$

Observe that the denominator is nonzero, due to $a^2/b^2 \neq 2$, that we know from Theorem 3.38. As for the case of $\mathbb{Q}[\sqrt{n}]$, this is similar, again by using Theorem 3.38. □

Many other things can be said, about quadratic fields. We will be back to this.

3e. Exercises

This was a quite standard chapter, and as exercises on this, we have:

EXERCISE 3.40. *Further review, if needed, the basic theory of fractions.*

EXERCISE 3.41. *How to avoid redundancies, in our method for counting* $\mathbb{Q}_+$*?*

EXERCISE 3.42. *Train in writing the symbol* ε*, and the other Greek letters as well.*

EXERCISE 3.43. *Compute some explicit limits of quotients of polynomials.*

EXERCISE 3.44. *Can you sum* $\sum_n x^n$ *geometrically, when* x *is rational?*

EXERCISE 3.45. *Further work on the generalized binomial formula, say at* $a \in \mathbb{Z}/2$*.*

EXERCISE 3.46. *Learn more about the finite fields, their properties, and structure.*

EXERCISE 3.47. *Have a further look as well at the quadratic fields.*

As bonus exercise, shop at the market. This is how fractions are best learned.

CHAPTER 4

Percentages

4a. Percentages

As explained earlier in this book, and which is something key for mathematics, the best way to deal with the integers is by using their decimal form:

$$n = a_1 \ldots a_k$$

The point now is that, in what regards the fractions, we can write them in decimal form too. Indeed, in the case where our fraction is of the special form $n/10$, with n as above, we can use the following convention, with a dot appearing on the right:

$$\frac{n}{10} = a_1 \ldots a_{k-1}.a_k$$

Next, in the case where our fraction is of the special form $n/100$, with n as above, we can use the following convention, also with a dot appearing on the right:

$$\frac{n}{100} = a_1 \ldots a_{k-2}.a_{k-1}a_k$$

And so on, and we are led in this way to the following notion:

DEFINITION 4.1. *We can write any fraction of the form $n/10^l$ in decimal form, according to the following convention:*

$$\frac{a_1 \ldots a_k b_1 \ldots b_l}{10^l} = a_1 \ldots a_k.b_1 \ldots b_l$$

Moreover, this writing is bijective, in the sense that any expression $a_1 \ldots a_k.b_1 \ldots b_l$ comes from such a fraction $n/10^l$, as above.

For many practical purposes, the case $l = 2$, so $10^l = 100$, and $n \in \{0, 1, 2, \ldots, 100\}$, is of particular interest. In this case the corresponding fraction is called percentage, and is written in the following simplified way, with the symbol % standing for $/100$:

$$\frac{n}{100} = n\%$$

With this understood, let us discuss now what can be done with the arbitrary fractions. These can be written as well in decimal form, with the result here being as follows:

THEOREM 4.2. *We can write any rational number $r = p/q$ in decimal form, with this decimal form being finite or not,*

$$r = \pm a_1a_2 \ldots a_k.b_1b_2b_3 \ldots$$

with the rule $999 \ldots = 1$, at the end, according to the following convention:

$$r = \pm \left(a_1a_2 \ldots a_k + \frac{b_1}{10} + \frac{b_2}{100} + \frac{b_3}{1000} + \ldots \right)$$

Moreover, the decimal expressions that we can obtain in this way are periodic, and with this taken into account, our writing is bijective.

PROOF. Many things going on here, the idea being as follows:

(1) Let us first prove that any rational number $r = p/q$ can be written in decimal form, as in the statement. We can assume $r > 0$, and we want to write r as:

$$r = a_1a_2 \ldots a_k.b_1b_2b_3 \ldots$$

But this can be done by using the integer part. Indeed, we can first set:

$$a_1 \ldots a_k = [r]$$

Next, let us look for b_1. This must be a digit $b_1 \in \{0, \ldots, 9\}$, subject to:

$$a_1 \ldots a_k.b_1 \leq r < a_1 \ldots a_k.(b_1 + 1)$$

According to our convention from Definition 4.1, these inequalities read:

$$\frac{a_1 \ldots a_kb_1}{10} \leq r < \frac{a_1 \ldots a_kb_1 + 1}{10}$$

Thus, the digit b_1 that we are looking for must appear according to:

$$a_1 \ldots a_kb_1 = [10r]$$

Similarly, the next digit b_2 must appear according to the following formula:

$$a_1 \ldots a_kb_1b_2 = [100r]$$

And so on, with at step l, the needed digit b_l appearing according to:

$$a_1 \ldots a_kb_1b_2 \ldots b_l = [10^lr]$$

(2) Summarizing, we have our decimal writing for r, and the question is now, what this decimal writing exactly means. But this meaning comes from the above formula that we used, in order to define b_l, which corresponds to inequalities as follows:

$$a_1 \ldots a_kb_1b_2 \ldots b_l \leq 10^lr < a_1 \ldots a_kb_1b_2 \ldots b_l + 1$$

Indeed, by dividing all terms by 10^l, these inequalities become as follows:

$$\frac{a_1 \ldots a_kb_1 \ldots b_l}{10^l} \leq r < \frac{a_1 \ldots a_kb_1 \ldots b_l + 1}{10^l}$$

Equivalently, we can write the above inequalities in the following way:

$$a_1a_2\ldots a_k+\frac{b_1}{10}+\frac{b_2}{100}+\ldots+\frac{b_l}{10^l}\leq r<a_1a_2\ldots a_k+\frac{b_1}{10}+\frac{b_2}{100}+\ldots+\frac{b_l}{10^l}+\frac{1}{10^l}$$

We conclude that, alternatively, and without reference to be above algorithm, the decimal writing corresponds to the infinite sum formula in the statement, namely:

$$r=a_1a_2\ldots a_k+\frac{b_1}{10}+\frac{b_2}{100}+\frac{b_3}{1000}+\ldots$$

Summarizing, main assertion of the theorem proved, and with this coming either from the above explicit algorithm involving the integer part, or just from $1/10^l\to 0$, and some elementary analysis. All this is quite intuitive, and we will come back to this, with further details, and a generalization, in chapter 5 below, when discussing the real numbers.

(3) Getting now to bijectivity issues, one bug comes from the following equality:

$$0.999\ldots=1$$

Indeed, in case you don't believe me, let us compute the number on the left. According to our conventions for the decimal writing, this number is given by:

$$\begin{aligned}0.999\ldots&=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\ldots\\&=\frac{9}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\ldots\right)\\&=\frac{9}{10}\cdot\frac{1}{1-\frac{1}{10}}\\&=\frac{9}{10}\cdot\frac{10}{9}\\&=1\end{aligned}$$

More generally now, the same computation shows that we have:

$$\pm a_1a_2\ldots a_k.b_1b_2\ldots b_l999\ldots=\pm a_1a_2\ldots a_k.b_1b_2\ldots b_l1$$

Thus, we must make the convention in the statement, $999\ldots=1$, at the end.

(4) Still talking bijectivity, our claim now is that the decimal expressions that we can obtain in this way, from rational numbers, are precisely those which are periodic:

$$r=\pm a_1\ldots a_k.b_1\ldots b_l(c_1\ldots c_p)$$

Indeed, in one sense, this follows from the following computation, which shows that a number as in the statement is indeed rational:

$$\begin{aligned} r &= \pm\frac{1}{10^l}\, a_1 \ldots a_k b_1 \ldots b_l . c_1 \ldots c_p c_1 \ldots c_p \ldots \\ &= \pm\frac{1}{10^l}\left(a_1 \ldots a_k b_1 \ldots b_l + c_1 \ldots c_p \left(\frac{1}{10^p} + \frac{1}{10^{2p}} + \ldots\right)\right) \\ &= \pm\frac{1}{10^l}\left(a_1 \ldots a_k b_1 \ldots b_l + \frac{c_1 \ldots c_p}{10^p - 1}\right) \end{aligned}$$

As for the converse, given a rational number $r = k/l$, we can find its decimal writing by performing the usual division algorithm, k divided by l. But this algorithm will be surely periodic, after some time, so the decimal writing of r is indeed periodic, as claimed. □

Getting back now to percentages, and their mathematics, which are something very useful, we can use them for approximating rational numbers $r \in [0, 1]$, as follows:

PROPOSITION 4.3. *Any rational number $r \in [0, 1]$, written in decimal form as*

$$r = 0.a_1 a_2 a_3 \ldots$$

can be approximated by a percentage, as follows,

$$r \simeq a_1 a_2\%$$

with the error term being smaller than 1%.

PROOF. This is more of an empty statement, coming by comparing our conventions for percentages and decimals, from Definition 4.1 and afterwards, and Theorem 4.2. □

The above result is something quite intuitive, and very useful in practice, and we will heavily use it, in what follows. Of course, there are situations where our 1% error margin is too big, and in this case we can add decimals to our percentages, as follows:

THEOREM 4.4. *Any rational number $r \in [0, 1]$, written in decimal form as*

$$r = 0.a_1 a_2 a_3 \ldots$$

can be approximated by a percentage, as follows,

$$r \simeq a_1 a_2\%$$

$$r \simeq a_1 a_2 . a_3\%$$

$$r \simeq a_1 a_2 . a_3 a_4\%$$

$$\vdots$$

with the error term being smaller than 1%, 0.1%, 0.01% *and so on.*

PROOF. Again, this is more of an empty statement, coming by comparing our various conventions from Definition 4.1 and afterwards, and from Theorem 4.2. □

Finally, in relation with all this, as a question that you might have, shall the assumption $r \in [0,1]$ bother us, or no. In answer, not at all, because we have:

FACT 4.5. *The probability for anything in this world to happen is a number*

$$r \in [0,1]$$

and so, this probability can be approximated as a percentage, as above.

As a comment here, any probability $r \in [0,1]$ is in fact a percentage, even without approximating it, because in agreement with Theorem 4.4, we can say that we have:

$$r = 0.a_1a_2a_3\ldots \iff r = a_1a_2.a_3\ldots\%$$

Quite nice all this, philosophically speaking, hope you agree with me. In short, when you hear economists saying that everything in life is about percentages, they are in fact right, and this even in relation with complicated questions from theoretical physics.

4b. Cards, coins

But probably too much talking, we definitely know now what the percentages are. So, as an application to what we learned so far, let us do some probability. We first have the following theorem, solving a well-known problem, of key importance in real life:

THEOREM 4.6. *The probabilities at poker are as follows:*

(1) *One pair:* 0.533.
(2) *Two pairs:* 0.120.
(3) *Three of a kind:* 0.053.
(4) *Full house:* 0.006.
(5) *Straight:* 0.005.
(6) *Four of a kind:* 0.001.
(7) *Flush:* 0.000.
(8) *Straight flush:* 0.000.

PROOF. Let us consider indeed our deck of 32 cards:

$$\{7,8,9,10,J,Q,K,A\} \times \{\clubsuit,\diamondsuit,\heartsuit,\spadesuit\}$$

The total number of possibilities for a poker hand is:

$$\binom{32}{5} = \frac{32\cdot 31\cdot 30\cdot 29\cdot 28}{2\cdot 3\cdot 4\cdot 5} = 32\cdot 31\cdot 29\cdot 7$$

(1) For having a pair, the number of possibilities is:

$$N = \binom{8}{1}\binom{4}{2} \times \binom{7}{3}\binom{4}{1}^3 = 8\cdot 6\cdot 35\cdot 64$$

Thus, the probability of having a pair is:

$$P = \frac{8 \cdot 6 \cdot 35 \cdot 64}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6 \cdot 5 \cdot 16}{31 \cdot 29} = \frac{480}{899} = 0.533$$

(2) For having two pairs, the number of possibilities is:

$$N = \binom{8}{2}\binom{4}{2}^2 \times \binom{24}{1} = 28 \cdot 36 \cdot 24$$

Thus, the probability of having two pairs is:

$$P = \frac{28 \cdot 36 \cdot 24}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{36 \cdot 3}{31 \cdot 29} = \frac{108}{899} = 0.120$$

(3) For having three of a kind, the number of possibilities is:

$$N = \binom{8}{1}\binom{4}{3} \times \binom{7}{2}\binom{4}{1}^2 = 8 \cdot 4 \cdot 21 \cdot 16$$

Thus, the probability of having three of a kind is:

$$P = \frac{8 \cdot 4 \cdot 21 \cdot 16}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{3 \cdot 16}{31 \cdot 29} = \frac{48}{899} = 0.053$$

(4) For having full house, the number of possibilities is:

$$N = \binom{8}{1}\binom{4}{3} \times \binom{7}{1}\binom{4}{2} = 8 \cdot 4 \cdot 7 \cdot 6$$

Thus, the probability of having full house is:

$$P = \frac{8 \cdot 4 \cdot 7 \cdot 6}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6}{31 \cdot 29} = \frac{6}{899} = 0.006$$

(5) For having a straight, the number of possibilities is:

$$N = 4\left[\binom{4}{1}^4 - 4\right] = 16 \cdot 63$$

Thus, the probability of having a straight is:

$$P = \frac{16 \cdot 63}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{9}{2 \cdot 31 \cdot 29} = \frac{9}{1798} = 0.005$$

(6) For having four of a kind, the number of possibilities is:

$$N = \binom{8}{1}\binom{4}{4} \times \binom{7}{1}\binom{4}{1} = 8 \cdot 7 \cdot 4$$

Thus, the probability of having four of a kind is:

$$P = \frac{8 \cdot 7 \cdot 4}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{31 \cdot 29} = \frac{1}{899} = 0.001$$

(7) For having a flush, the number of possibilities is:

$$N = 4\left[\binom{8}{4} - 4\right] = 4 \cdot 66$$

Thus, the probability of having a flush is:

$$P = \frac{4 \cdot 66}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{33}{4 \cdot 31 \cdot 29 \cdot 7} = \frac{9}{25172} = 0.000$$

(8) For having a straight flush, the number of possibilities is:

$$N = 4 \cdot 4$$

Thus, the probability of having a straight flush is:

$$P = \frac{4 \cdot 4}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{2 \cdot 31 \cdot 29 \cdot 7} = \frac{1}{12586} = 0.000$$

Thus, we have obtained the numbers in the statement. □

So far, so good, but you might argue, what if we model our problem as for our poker hand to be ordered, do we still get the same answer? In answer, sure yes, but let us check this. The probability for having four of a kind, computed in this way, is then:

$$P(\text{four of a kind}) = \frac{8 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 28}{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28} = \frac{1}{31 \cdot 29} = \frac{1}{899}$$

To be more precise, here on the bottom $32 \cdot 31 \cdot 30 \cdot 29 \cdot 28$ stands for the total number of possibilities for an ordered poker hand, 5 out of 32, and on top, exercise for you to figure out what the above numbers 8, 5, then $4 \cdot 3 \cdot 2$, and 28, stand for.

At a more advanced level now, many things can be learned by flipping coins, and recording your findings. Let us start with something very basic, as follows:

FACT 4.7. *The probability of winning when flipping a coin is* $1/2$.

Obvious you would say, but there are some subtleties here, even in this simplest possible probability question. The first thing is that I said "winning", like everyone says when it comes to flipping coins, but winning against whom?

So, this is a first subtlety. Flipping a coin is best regarded as being a game, with you choosing between heads and tails, let us say heads, then flipping the coin, and winning if heads. But now, that we talked about a game, you need a partner for your game. That is, you are not playing a game alone, but with someone else, who wins when it's tails.

Which brings us into a second question, winning what? Many options here, like winning apples, or oranges, or luxury cars, assuming that both you and your partner have a considerable stock of those. Or why not, for making the game even more exciting, the right to slap your partner, or why not pulling a knife, and killing your partner.

So, what to choose? The answer here is money, this is what money is made for, for simplifying such things, transactions between humans. In the hope that we agree on this, and now with this discussion made, let us record our findings, as follows:

CONCLUSION 4.8. *Flipping a coin is best regarded as being a game, between you and a partner, the rules being:*

(1) *Every time it is heads, you win* \$1 *from your partner.*
(2) *Every time it is tails, your partner wins* \$1 *from you.*

With this conclusion recorded, we can see now more clearly what is behind coin flipping. Obviously, all sorts of interesting things that we can explore, and we will do that, and with the main question, which is surely on everyone's mind, being:

QUESTION 4.9. *Who wins?*

So, let us study now this question. Thus is a matter of understanding how the game axiomatized in Conclusion 4.8 evolves over the time, taking into account the 1/2 mathematics from Fact 4.7, and here are a few preliminary observations, about this:

PROPOSITION 4.10. *When flipping a coin k times, the following happen,*

(1) *The probability of you winning* $\$k$ *is* $1/2^k$.
(2) *The probability of you winning* $\$k-1$ *is* 0.
(3) *The probability of you winning* $\$k-2$ *is* $k/2^k$.
(4) *The probability of you winning* $\$k-3$ *is again* 0.
(5) *The probability of you winning* $\$k-4$ *is* $k(k-1)/2^{k+1}$.

and so on, with the probability increasing, up to the tie situation, and then decreasing.

PROOF. This follows indeed from some simple mathematics, as follows:

(1) You winning $\$k$ means you winning every time, over k attempts, so your probability here is $P = (1/2) \times \ldots \times (1/2)$, with k terms in the product, which reads $P = 1/2^k$.

(2) The point here is that you cannot win $\$k-1$, exactly. Indeed, you must lose at least once, and so you profit will be $\leq (k-1)-1 = k-2$.

(3) Here we have a similar computation as in (1). For winning $\$k-2$ you need to lose exactly once, and since there are k possibilities of losing exactly once, $P = k/2^k$.

(4) Here the situation is similar to that in (2). Indeed, for winning exactly $\$k-3$ you would certainly need to lose twice, so you profit will be $\leq (k-2)-2 = k-4$.

(5) With the same reasoning as in (3), here you need to lose exactly twice, and since there are $k(k-1)/2$ possibilities of losing exactly twice, $P = k(k-1)/2^{k+1}$.

(6) Finally, regarding the last assertion, which is a bit informal, we will leave the clarification here, both statement and proof, to you, as an instructive exercise. □

Obviously, some interesting mathematics is going on here, that needs to be better understood. We have the following result, generalizing Proposition 4.10:

THEOREM 4.11. *When flipping a coin k times what you can win are quantities of type $\$k-2s$, with $s=0,1,\ldots,k$, with the probability for this to happen being:*

$$P(k-2s)=\frac{1}{2^k}\binom{k}{s}$$

Geometrically, your winning curve starts with probability $1/2^k$ of winning $-\$k$, then increases up to the tie situation, and then decreases, up to probability $1/2^k$ of winning $\$k$.

PROOF. All this is quite clear, by fine-tuning our various observations from Proposition 4.10 and its proof, the point here being that, in order for you to win $k-s$ times and lose s times, over your k attempts, the number of possibilities is:

$$\binom{k}{s}=\frac{k!}{s!(k-s)!}$$

Thus, by dividing now by 2^k, which is the total number of possibilities, for the whole game, we are led to the probability in the statement, namely:

$$P(k-2s)=\frac{1}{2^k}\binom{k}{s}$$

Shall we doublecheck this? Sure yes, doublecheking is the first thing to be done, when you come across a theorem, in your mathematics. As a first check, the sum of probabilities that we found should be 1, which is intuitive, right, and 1 that is, as shown by:

$$\begin{aligned}\sum_{s=0}^{k}P(k-2s)&=\frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}\\&=\frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}1^s1^{k-s}\\&=\frac{1}{2^k}(1+1)^k\\&=\frac{1}{2^k}\times 2^k\\&=1\end{aligned}$$

But shall we really trust this. Imagine for instance that you play your game for \$1000 instead of \$1 as basic gain, your life is obviously at stake, so all this is worth a second doublecheck, before being used in practice. So, as second doublecheck, let us verify that, on average, what you win is exactly \$0, which is something very intuitive, the game itself

obviously not favoring you, nor your partner. But this can be checked as follows:

$$\begin{aligned}
\sum_{s=0}^{k} P(k-2s)\times(k-2s) &= \frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}(k-2s)\\
&= \frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}(k-s)-\frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}s\\
&= \frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}(k-s)-\frac{1}{2^k}\sum_{t=0}^{k}\binom{k}{k-t}(k-t)\\
&= \frac{1}{2^k}\sum_{s=0}^{k}\binom{k}{s}(k-s)-\frac{1}{2^k}\sum_{t=0}^{k}\binom{k}{t}(k-t)\\
&= 0
\end{aligned}$$

Summarizing, we have a good theorem here, proved, doublechecked and triplechecked, as per the highest scientific standards, ready to be used in practice. □

With Theorem 4.11 in hand, we are somehow done with math, and time now to turn to Question 4.9. Let us first examine a more concrete question, namely:

QUESTION 4.12. *What and how do you win, depending on your strategy?*

However, this appears to be a bit of a bad question, at least in the context of our very simple flipping game, because you have not so many options for developing a strategy. To be more precise, the only thing that you can do, as strategy, is that of pulling off the game, once you won enough money. And even this is something debatable, because you pulling off at the moment of your choice assumes that the rules are biased, favoring you. So, well, let us do this, and reformulate our strategy question as follows:

QUESTION 4.13. *Assuming that the rules are biased, favoring you, by allowing you to pull off at any moment of your choice, what is your best strategy?*

And with this, we are now straight into popular mathematics, because everyone in a casino, or buying lottery tickets, or doing stocks or crypto in front of a computer, thinks about such things, and at a highest possible seriousness level. You won't mess up things with your own money, hardly won via hard daily labor, won't you.

In relation with this, the legend has it that what you have to do is to play and play, until you reached a sum of money that you fixed as objective in advance, say $100. Then you pull off, with the money in your pocket. Simple like that.

So, let us see how this works. To start with, this can only work, I mean just play and play, as indicated above, and you will certainly end up with $100 in your pocket, no

question about it. However, this might take some precious time t, and the mathematics, based on our formula in Theorem 4.11, shows that this time t is as follows:

Time spent playing	Probability to win
$t = 100$	$1/2^{100} = 0.000$
$t = 102$	$102/2^{102} = 0.000$
$t = 104$	$\binom{104}{2}/2^{104} = 0.000$
$t = 106$	$\binom{106}{3}/2^{106} = 0.000$
$t = 108$	$\binom{108}{4}/2^{108} = 0.000$
$t = 110$	$\binom{110}{5}/2^{110} = 0.000$
$\vdots$	$\vdots$

Which does not loook very good, hope you agree with me. Obviously, we are here into some sort of very abstract math, not corresponding to anything in the real life. So, in order to reach to something more reasonable, good moment to remember that:

FACT 4.14. *Time is money.*

In view of this, let us downgrade our ambitions, and only wish to win a modest \$10. Here we reach to a more reasonable winning scheme, as follows:

Time spent playing	Probability to win
$t = 10$	$1/2^{10} = 0.001$
$t = 12$	$12/2^{12} = 0.003$
$t = 14$	$\binom{14}{2}/2^{14} = 0.006$
$t = 16$	$\binom{16}{3}/2^{16} = 0.009$
$t = 18$	$\binom{18}{4}/2^{18} = 0.012$
$t = 20$	$\binom{20}{5}/2^{20} = 0.015$
$\vdots$	$\vdots$

However, this is still not interesting, financially speaking, so in order to reach to something more viable, let us further downgrade our ambitions, and only wish to win a very modest \$5. And here, we reach to something more attractive, as follows:

Time spent playing	Probability to win
$t = 5$	$1/2^5 = 0.031$
$t = 7$	$7/2^7 = 0.055$
$t = 9$	$\binom{9}{2}/2^9 = 0.070$
$t = 11$	$\binom{11}{3}/2^{11} = 0.081$
$t = 13$	$\binom{13}{4}/2^{13} = 0.087$
$t = 15$	$\binom{15}{5}/2^{15} = 0.092$
$\vdots$	$\vdots$

But this still does not look very good, so going now for the true way of reason, let us simply wish to win a tiny \$3. And here, the situation becomes as follows:

Time spent playing	Probability to win
$t = 3$	$1/2^3 = 0.125$
$t = 5$	$5/2^5 = 0.156$
$t = 7$	$\binom{7}{2}/2^7 = 0.164$
$t = 9$	$\binom{9}{3}/2^9 = 0.164$
$t = 11$	$\binom{11}{4}/2^{11} = 0.163$
$t = 13$	$\binom{13}{5}/2^{13} = 0.157$
$\vdots$	$\vdots$

Which is sort of reasonable, but not really, observe for instance that the probabilities on the right start decreasing, and before putting this scheme into practice, we must probably do some more math, make sure that these probabilities won't start to decrease very sharply, which might complicate our business, and so on.

Moving ahead now, we talked in the above about "time is money", which is something that must be taken into account, but thinking well, what really matters in all this is the maximum amount of money that you can afford to lose. Which is something quite subtle, not included in our modeling above. So, let us further reformulate our strategy question, by making it more realistic, in touch with what happens in the real life, as follows:

QUESTION 4.15. *What is your best strategy, assuming that the game is asymmetric:*

(1) *With the rules being biased, favoring you, by allowing you to pull off from the game, at any moment of your choice.*
(2) *With the capital being unequal, favoring your partner, who has N money that he can afford to lose, compared to your $n < N$ money.*
(3) *And perhaps with a fee for playing the game too, again favoring your partner, to be paid by you, and this because N, n are normally secret.*

And good news, this is the good, final question, which perfectly makes sense, and is fully realistic. There is some math to be done here, and getting started with this, we can solve a simple case right away, namely that when your partner has endless money:

$$N = \infty$$

A player having this feature is called "the bank", and with this convention made, the answer to our various questions, and notably to Question 4.9 that we started with, is:

ANSWER 4.16. *The bank wins.*

To be more precise here, as already mentioned, we can certainly do some math here, and we will do this later. But, for our purposes now, the simplest is to argue that, in your situation, when you have \$$n$ and you lose \$$n$, say with $n = 1,000,000$ for having a

precise figure, you are dead, say with this coming from drug overdose, after reaching the street, after your bankruptcy. So, your strategy of pulling off once you won a precise sum of money, say \$100, is certainly flawed, because you can meet death on the way:

Time spent playing	Probability to win	Other outcomes
$t = 100$	small	losing
$t = 102$	small	losing
$t = 104$	small	losing
$\vdots$	$\vdots$	$\vdots$
$t = 1,000,000$	attractive	death
$t = 1,000,002$	attractive	death
$t = 1,000,004$	attractive	death
$\vdots$	$\vdots$	$\vdots$

As for the variations of this strategy, these can be certainly investigated too, but it is quite clear that all this will not lead to anything good, because originally you were there happy, looking for a strategy for winning the game, but all of the sudden, the rule (2) from Question 4.15 puts you in a very defensive situation, more caring about your life, than of winning the game. So, it is pretty much clear that we are led to Answer 4.16.

As a conclusion now to all this, and leaving aside the precise coin game that we were playing, "be the bank" is the winning strategy, in economics. Which is good to know.

4c. Rolling dice

At a more advanced level, we can roll dice. The difference with coins comes from the fact that the basic $1/2-1/2$ probabilities get now replaced by a better readable $1/6-5/6$. To be more precise, let us first convene for the following rules for the game:

RULES 4.17. *Rolling the die is played with the following rules:*

(1) *Every time it is* $1, 2, 3, 4, 5$*, your partner wins* \$1 *from you.*
(2) *And every time it is* 6*, you win* \$5 *from your partner.*

Of course, you might say that this is not very standard, but hey, we are just doing some math here, and we will complicate the rules later on, no worries for that. Now with these rules agreed on, we have the following analogue of Theorem 4.11:

THEOREM 4.18. *When rolling a die* k *times what you can win are quantities of type* $\$6w - k$*, with* $w = 0, 1, \ldots, k$*, with the probability for this to happen being:*

$$P(6w - k) = \frac{5^{k-w}}{6^k}\binom{k}{w}$$

Geometrically, your winning curve starts with probability $(5/6)^k$ *of losing* $\$k$*, then increases, up to some point, and then decreases, up to probability* $1/6^k$ *of winning* $\$5k$*.*

PROOF. There are several things going on here, the idea being as follows:

(1) When rolling the die k times, what will happen is that you will win w times and lose l times, with $k = w + l$. And in this situation, your profit will be, as stated:

$$\begin{aligned}\$ &= 5w - l\\ &= 5w - (k - w)\\ &= 6w - k\end{aligned}$$

(2) As for the probability for this to happen, this is the total number of possibilities for you to win w times, which is $5^{k-w}\binom{k}{w}$, because this amounts in choosing the w times when you will win, among k, then multiplying by 5^{k-w} possibilities, at places where your partner wins, and finally dividing by the total number of possibilities, which is 6^k:

$$P(6w - k) = \frac{5^{k-w}}{6^k}\binom{k}{w}$$

(3) As usual when doing complicated math, let us doublecheck all this, matter of being sure that we did not mess up our counting. First, the sum of all probabilities involved must be 1, and 1 that sum is, as shown by the following computation:

$$\begin{aligned}\sum_{w=0}^{k} P(6w - k) &= \sum_{w=0}^{k} \frac{5^{k-w}}{6^k}\binom{k}{w}\\ &= \frac{1}{6^k}\sum_{w=0}^{k}\binom{k}{w}1^w 5^{k-w}\\ &= \frac{1}{6^k}(1 + 5)^k\\ &= \frac{1}{6^k} \times 6^k\\ &= 1\end{aligned}$$

(4) Let us triplecheck this as well. Obviously, Rules 4.17 do not favor you, nor your partner, so on average, you should win 0. And this is the case indeed, because:

$$\begin{aligned}
\sum_{w=0}^{k} P(6w-k)\times(6w-k) &= \frac{1}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k}{w}(6w-k)\\
&= \frac{1}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k}{w}5w-\frac{1}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k}{w}(k-w)\\
&= \frac{5}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k}{w}w-\frac{1}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k}{w}(k-w)\\
&= \frac{5k}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k-1}{w-1}-\frac{k}{6^k}\sum_{w=0}^{k}5^{k-w}\binom{k-1}{w}\\
&= \frac{5k}{6^k}(1+5)^{k-1}-\frac{5k}{6^k}(1+5)^{k-1}\\
&= 0
\end{aligned}$$

This last computation was hot, wasn't it, but triplechecks are mandatory. In any case theorem proved, and the final conclusions in the statement are clear too. □

Quite interestingly, Theorem 4.18 is best seen, both at the level of the statement, and of the proof, from the viewpoint of your partner. Let us record this, as follows:

THEOREM 4.19. *When rolling a die k times what you can win are quantities of type $\$5k-6l$, with $l=0,1,\ldots,k$, with the probability for this to happen being:*

$$P(5k-6l)=\frac{5^l}{6^k}\binom{k}{l}$$

Geometrically, your winning curve starts with probability $(5/6)^k$ of losing $\$k$, then increases, up to some point, and then decreases, up to probability $1/6^k$ of winning $\$5k$.

PROOF. As before, when rolling the die k times, you will win w times and lose l times, with $k=w+l$. And in this situation, your profit will be, as stated:

$$\begin{aligned}
\$ &= 5w-l\\
&= 5(k-l)-l\\
&= 5k-6l
\end{aligned}$$

As for the rest, we already know all this from Theorem 4.18, but the point is that the proof of Theorem 4.18 becomes slightly simpler when using l instead of w. We will leave the details here, reworking the proof of Theorem 4.18 in this way, as an exercise. □

4d. Binomial laws

Now with Theorem 4.19 in hand, it is quite clear that the basic $1/6-5/6$ probabilities at dice can be repaced with something of type $p-(1-p)$, with $p\in[0,1]$ being arbitrary. We are led in this way to the following notions, which are quite general:

DEFINITION 4.20. *Given $p\in[0,1]$, the Bernoulli law of parameter p is given by:*

$$P(\text{win})=p\quad,\quad P(\text{lose})=1-p$$

More generally, the k-th binomial law of parameter p, with $k\in\mathbb{N}$, is given by

$$P(s)=p^s(1-p)^{k-s}\binom{k}{s}$$

with the Bernoulli law appearing at $k=1$, with $s=1,0$ here standing for win and lose.

To be more precise, what we call here "law" is something intuitive, based on what we did before with coins and dice, basically standing for "outcome of a game". As a first observation, the Bernoulli law generalizes indeed what we did before with coins and dice, which come respectively from the following choices of the parameter $p\in[0,1]$:

$$p_{coin}=1/2\quad,\quad p_{die}=1/6$$

Observe also that the last assertion holds indeed, because at $k=1$ the binomial law is as follows, coinciding indeed with the Bernoulli law of parameter p:

$$P(1)=p\quad,\quad P(0)=1-p$$

Finally, regarding the binomial law, observe that is indeed a "law", or what we can expect from a game, because the various probabilities sum up to 1, as they should:

$$\begin{aligned}\sum_{s=0}^{k}P(s)&=\sum_{s=0}^{k}p^s(1-p)^{k-s}\binom{k}{s}\\&=(p+(1-p))^k\\&=1\end{aligned}$$

Let us try now to better understand the relation between the Bernoulli and binomial laws. Indeed, we know from both coins and dice that the Bernoulli laws produce the binomial laws, simply by iterating the game, from 1 throw to $k\in\mathbb{N}$ throws.

The reasons behind this obviously come from the "independence" of our coin or dice throwings, when iterating. Let us record this finding, as follows:

CONCLUSION 4.21. *The Bernoulli laws produce the binomial laws, by iterating the game, via the independence of the throws.*

Of course, this finding is something quite intuitive, and temporary, and it remains to work out the precise mathematics of independence, producing the explicit formula of the binomial laws, out of the explicit formula of the Bernoulli laws. We will discuss this later, but coming a bit in advance, here is the answer to this question:

(1) The idea is to encapsulate the data from Definition 4.20 into so-called "probability measures" associated to the Bernoulli and binomial laws. For the Bernoulli law, the corresponding measure is as follows, with the δ symbols standing for Dirac masses:

$$\mu_{ber} = (1-p)\delta_0 + p\delta_1$$

As for the binomial law, here the measure is as follows, constructed in a similar way, you get the point I hope, again with the δ symbols standing for Dirac masses:

$$\mu_{bin} = \sum_{s=0}^{k} p^s(1-p)^{k-s}\binom{k}{s}\delta_s$$

(2) Getting now to independence, and to our finding from Conclusion 4.21, the mathematics there is that of the following formula, with $*$ standing for the convolution operation for probability measures, which on Dirac masses is given by $\delta_x * \delta_y = \delta_{x+y}$:

$$\mu_{bin} = \underbrace{\mu_{ber} * \ldots * \mu_{ber}}_{k\ terms}$$

(3) To be more precise, this latter formula does hold indeed, as a straightforward application of the binomial formula, the formal proof being as follows:

$$\begin{aligned}
\mu_{ber}^{*k} &= \big((1-p)\delta_0 + p\delta_1\big)^{*k}\\
&= \sum_{s=0}^{k} p^s(1-p)^{k-s}\binom{k}{s}\delta_0^{*(k-s)} * \delta_1^{*s}\\
&= \sum_{s=0}^{k} p^s(1-p)^{k-s}\binom{k}{s}\delta_s\\
&= \mu_{bin}
\end{aligned}$$

All this is very nice, and is perhaps worth a reformulation of Conclusion 4.21. We reach in this way to a quite drastic statement, as follows:

Conclusion 4.22. *Most of what we did with coins and dice reduces to the formula*

$$\mu_{ber}^{*k} = \mu_{bin}$$

relating the Bernoulli and binomial laws, via the convolution operation $*$.

And isn't this magic. We have proof here for the abstract power of mathematics. Or perhaps of physics, because the Dirac masses, involved in all this, come from Dirac.

Getting to formal mathematical work now, in relation with this, let us start with:

DEFINITION 4.23. *Given a set X, which can be finite, countable, or even uncountable, a discrete probability measure on it is a linear combination as follows,*

$$\mu = \sum_{x\in X} \lambda_x \delta_x$$

with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Given a function $f : X \to \mathbb{Q}$ we set

$$\int_X f(x) d\mu(x) = \sum_{x\in X} \lambda_x f(x)$$

with the convention that each Dirac mass integrates up to 1.

As a comment, we are using here rational numbers as scalars, because these are the numbers that we know about, at this point of this book. But the whole theory to be developed below works with real numbers too, and quite often, even with complex numbers. More on this later in this book, when talking real and complex numbers.

Observe that, with Definition 4.23, we are now into pure mathematics. However, and we insist on this, it is basic probability which is behind all this. Next, we have:

DEFINITION 4.24. *A random variable on a probability space X is a function*

$$f : X \to \mathbb{Q}$$

and the expectation of such a random variable is the quantity

$$E(f) = \sum_{x\in X} f(x) P(x)$$

which is best thought as being the average gain, when the game is played.

To be more precise here, in what regards the word "game" at the end, this comes from the following interpretation of what we have, which is something to be kept in mind:

(1) To start with, we can think of our abstract probability space X as corresponding to a generalized, biased die, with faces $x \in X$, having landing probabilities λ_x.

(2) Then, we can think of the random variable $f : X \to \mathbb{Q}$ as representing some sort of money, when the game is played, that is, when events $x \in X$ are picked.

(3) And the point now is that, with these conventions, the quantity $E(f)$ constructed above is what we win, on average, when playing the game.

As a basic illustration now, in case we are dealing with a usual die, what we win is what the die says, and on average, what we win is the following quantity:

$$E = \frac{1+2+3+4+5+6}{6} = 3.5$$

With this understood, what about coins? Here, before doing any computation, we have to assign some numbers to our events, and a standard choice here is as follows:

$$f : \{\text{heads, tails}\} \to \mathbb{R} \quad , \quad f(\text{heads}) = 1 \quad , \quad f(\text{tails}) = 0$$

With this choice made, what we can expect to win is the following quantity:

$$\begin{aligned} E(f) &= f(\text{heads}) \times P(\text{heads}) + f(\text{tails}) \times P(\text{tails}) \\ &= 1 \times \frac{1}{2} + 0 \times \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Moving on, with this discussed, time now to get into more delicate aspects. Imagine for instance that you want to set up some sort of business, with your random variable $f : X \to \mathbb{Q}$. You are of course mostly interested in the expectation $E(f) \in \mathbb{Q}$, but passed that, the way this expectation comes in matters too. For instance:

(1) When your variable is constant, $f = c$, you certainly have $E(f) = c$, and your business will run smoothly, with not so many surprises on the way.

(2) On the opposite, for a complicated variable satisfying $E(f) = c$, your business will be more bumpy, with wins or loses on the way, depending on your skills.

In short, and extrapolating now from business to mathematics, physics, chemistry and everything else, we must complement Definition 4.24 with something finer, regarding the "quality" of the expectation $E(f) \in \mathbb{Q}$ appearing there. And the first thought here, which is the correct one, goes to the following number, called variance of our variable:

$$\begin{aligned} V(f) &= E\left((f - E(f))^2\right) \\ &= E(f^2) - E(f)^2 \end{aligned}$$

However, let us not stop here. For a total control of your business, be that of financial, mathematical, physical or chemical type, you will certainly want to know more about your variable $f : X \to \mathbb{Q}$. Which leads us into general moments, constructed as follows:

DEFINITION 4.25. *The moments of a variable $f : X \to \mathbb{Q}$ are the numbers*

$$M_k = E(f^k)$$

which satisfy $M_0 = 1$, then $M_1 = E(f)$, and then $V(f) = M_2 - M_1^2$.

And with this, good news, we have now all the needed tools in our bag for doing some good business. To put things in a very compacted way, M_0 is about foundations, M_1 is about running some business, M_2 is about running that business well, and M_3 and higher are advanced level, about ruining all the competing businesses.

As a further piece of basic probability, coming this time as a theorem, we have:

THEOREM 4.26. *Given a random variable $f : X \to \mathbb{Q}$, if we define its law as being*

$$\mu = \sum_{x \in X} P(x)\delta_{f(x)}$$

regarded as probability measure on $\mathbb{Q}$, then the moments are given by the formula

$$E(f^k) = \int_{\mathbb{Q}} y^k d\mu(y)$$

with the usual convention that each Dirac mass integrates up to 1.

PROOF. There are several things going on here, the idea being as follows:

(1) To start with, given a random variable $f : X \to \mathbb{Q}$, we can certainly talk about its law μ, as being the formal linear combination of Dirac masses in the statement.

(2) Still talking basics, let us record as well the following alternative formula for the law, which is clear from definitions, and that we will often use, in what follows:

$$\mu = \sum_{y \in \mathbb{Q}} P(f = y)\delta_y$$

(3) Now let us compute the moments of f. With the usual convention that each Dirac mass integrates up to 1, as mentioned in the statement, we have:

$$\begin{aligned} E(f^k) &= \sum_{x \in X} P(x) f(x)^k \\ &= \sum_{y \in \mathbb{Q}} y^k \sum_{f(x) = y} P(x) \\ &= \int_{\mathbb{Q}} y^k d\mu(y) \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Next, we have the following definition, inspired by what happens for coins, dice and cards, as explored above, and which is the foundation for everything advanced:

DEFINITION 4.27. *We say that two variables $f, g : X \to \mathbb{Q}$ are independent when*

$$P(f = x, g = y) = P(f = x)P(g = y)$$

happens, for any $x, y \in \mathbb{Q}$.

As already mentioned, this is something very intuitive, inspired by what happens for coins, dice and cards. As a first result now regarding independence, we have:

THEOREM 4.28. *Assuming that $f, g : X \to \mathbb{Q}$ are independent, we have:*

$$E(fg) = E(f)E(g)$$

More generally, we have the following formula, for the mixed moments,

$$E(f^k g^l) = E(f^k)E(g^l)$$

and the converse holds, in the sense that this formula implies the independence of f, g.

PROOF. We have indeed the following computation, using the independence of f, g:

$$\begin{aligned} E(f^k g^l) &= \sum_{xy} x^k y^l P(f = x, g = y) \\ &= \sum_{xy} x^k y^l P(f = x)P(g = y) \\ &= \sum_{x} x^k P(f = x) \sum_{y} y^l P(g = y) \\ &= E(f^k)E(g^l) \end{aligned}$$

As for the last assertion, this is clear too, because having the above computation work, for any $k, l \in \mathbb{N}$, amounts in saying that the independence formula for f, g holds. □

Regarding now the convolution operation, motivated by what we found before, in Conclusion 4.22, let us start with the following abstract definition:

DEFINITION 4.29. *Given a space X with a sum operation $+$, we can define the convolution of any two discrete probability measures on it,*

$$\mu = \sum_i a_i \delta_{x_i} \quad , \quad \nu = \sum_j b_j \delta_{y_j}$$

as being the discrete probability measure given by the following formula:

$$\mu * \nu = \sum_{ij} a_i b_j \delta_{x_i + y_j}$$

That is, the convolution operation $*$ is defined by $\delta_x * \delta_y = \delta_{x+y}$, and linearity.

As a first observation, our operation is well-defined, with $\mu * \nu$ being indeed a discrete probability measure, because the weights are positive, $a_i b_j \geq 0$, and their sum is:

$$\sum_{ij} a_i b_j = \sum_i a_i \sum_j b_j = 1 \times 1 = 1$$

Also, the above definition agrees with what we did before with coins, and Bernoulli and binomial laws. We have in fact the following general result:

THEOREM 4.30. *Assuming that $f, g : X \to \mathbb{Q}$ are independent, we have*

$$\mu_{f+g} = \mu_f * \mu_g$$

where $*$ is the convolution of probability measures.

PROOF. We have indeed the following straightforward verification:

$$\begin{aligned}
\mu_{f+g} &= \sum_{x\in\mathbb{Q}} P(f+g=x)\delta_x \\
&= \sum_{y,z\in\mathbb{Q}} P(f=y,g=z)\delta_{y+z} \\
&= \sum_{y,z\in\mathbb{Q}} P(f=y)P(g=z)\delta_y * \delta_z \\
&= \left(\sum_{y\in\mathbb{Q}} P(f=y)\delta_y\right) * \left(\sum_{z\in\mathbb{Q}} P(g=z)\delta_z\right) \\
&= \mu_f * \mu_g
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Before going further, let us attempt as well to find a proof of Theorem 4.30, based on the moment characterization of independence, from Theorem 4.28. For this purpose, we will need the following standard fact, which is of certain theoretical interest:

THEOREM 4.31. *The sequence of moments*

$$M_k = \int_{\mathbb{Q}} x^k d\mu(x)$$

uniquely determines the law.

PROOF. Indeed, assume that the law of our variable is as follows:

$$\mu = \sum_i \lambda_i \delta_{x_i}$$

The sequence of moments is then given by the following formula:

$$M_k = \sum_i \lambda_i x_i^k$$

But it is then standard calculus to recover the numbers $\lambda_i, x_i \in \mathbb{Q}$, and so the measure μ, out of the sequence of numbers M_k. Indeed, assuming that the numbers x_i are $0 < x_1 < \ldots < x_n$ for simplifying, in the $k \to \infty$ limit we have the following formula:

$$M_k \sim \lambda_n x_n^k$$

Thus, we got the parameters $\lambda_n, x_n \in \mathbb{Q}$ of our measure μ, and then by substracting them and doing an obvious recurrence, we get the other parameters $\lambda_i, x_i \in \mathbb{Q}$ as well. □

Getting back now to our philosophical question above, namely recovering Theorem 4.30 via moment technology, we can now do this, the result being as follows:

THEOREM 4.32. *Assuming that $f, g : X \to \mathbb{Q}$ are independent, the measures*

$$\mu_{f+g} \quad , \quad \mu_f * \mu_g$$

have the same moments, and so, they coincide.

PROOF. We have the following computation, using the independence of f, g:

$$\begin{aligned} M_k(f+g) &= E((f+g)^k) \\ &= \sum_r \binom{k}{r} E(f^r g^{k-r}) \\ &= \sum_r \binom{k}{r} M_r(f) M_{k-r}(g) \end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned} \int_X x^k d(\mu_f * \mu_g)(x) &= \int_{X\times X} (x+y)^k d\mu_f(x) d\mu_g(y) \\ &= \sum_r \binom{k}{r} \int_X x^r d\mu_f(x) \int_X y^{k-r} d\mu_g(y) \\ &= \sum_r \binom{k}{r} M_r(f) M_{k-r}(g) \end{aligned}$$

Thus, job done, and theorem proved, or rather Theorem 4.30 reproved. □

Good news, with all this understood, we can now get back to what we found in Conclusion 4.22, and formulate a precise theorem about this, as follows:

THEOREM 4.33. *The following happen, in the context of the biased coin game:*

(1) *The Bernoulli laws μ_{ber} produce the binomial laws μ_{bin}, by iterating the game $k \in \mathbb{N}$ times, via the independence of the throws.*
(2) *We have in fact $\mu_{bin} = \mu_{ber}^{*k}$, with $*$ being the convolution operation for real probability measures, given by $\delta_x * \delta_y = \delta_{x+y}$, and linearity.*

PROOF. This is something that we knew from before, in a vague form, and which is now clear indeed, by using Theorem 4.30, or Theorem 4.32. □

Many other things can be said about the binomial laws. We will be back to this later in this book, once we will know about the real numbers, and in particular about the number e, which is needed in order to understand the asymptotics of these laws.

4e. Exercises

With percentages and probability, we are now into the real life, or rather into the real life of us modern humans, as we know it. Here are some exercises, on all this:

EXERCISE 4.34. *Review if needed the classical algorithm for dividing integers.*

EXERCISE 4.35. *Learn how computers deal, or not, with the* $999\ldots$ *issue.*

EXERCISE 4.36. *Does mathematics simplify with* $\mathbb{Q}$ *introduced via decimal form?*

EXERCISE 4.37. *Play some poker, matter of verifying the figures computed above.*

EXERCISE 4.38. *Learn more about the binomial laws, and their various properties.*

EXERCISE 4.39. *Mediate about the variance* V*, and its exact probabilistic meaning.*

EXERCISE 4.40. *Clarify the details, for the fact that the moments determine the law.*

EXERCISE 4.41. *Learn more about Dirac masses, from mathematicians and physicists.*

As bonus exercise, reiterated, start doing your food shopping at the market. This is how fractions, percentages and probability are best learned, the hard way.

Part II

Basic arithmetic

Do you remember
Before the rain came down
You were so full of life
So bring that right back around

CHAPTER 5

Real numbers

5a. Real numbers

We have certainly used a bit real numbers in the above, as everyone does, but time now to get more in detail into their definition, and philosophy. Among others, we will see how the knowledge of the real numbers tells us more about $\mathbb{Q}$, and even about $\mathbb{Z}$ or $\mathbb{N}$.

Let us start with something well-known to any mathematician or scientist, and to any computer too, which is quite concerning, and that you are surely aware of, namely:

FACT 5.1. *The real numbers $x \in \mathbb{R}$ can be introduced via their decimal form,*

$$x = \pm a_1 \ldots a_n . b_1 b_2 b_3 \ldots\ldots$$

with $a_i, b_i \in \{0, 1, \ldots, 9\}$, and $a_1 \neq 0$, with the convention at the end

$$\ldots b999 \ldots = \ldots (b+1)000 \ldots$$

but with this, the field structure of $\mathbb{R}$ remains something quite unclear.

To be more precise, no need of course to know about fields when stating this, it is all about the basic operations on the real numbers, namely addition and multiplication, that this is all about. Here are the problems, both being questions of common sense:

(1) With the addition, things are certainly complicated, because the addition of real numbers is given by a modified version of the algorithm that we know well for the integers, which is something quite complicated, and this even for the rational numbers.

(2) Things are similar with the multiplication of the real numbers, again appearing as a modified version of the algorithm that we know well for the integers, which is again something quite complicated, and this even for the rational numbers.

Well, it looks like we are a bit stuck, we certainly don't want to build our theory of real numbers on complicated algorithms, so we must find something clever, and new.

Fortunately, there is indeed a clever solution to this, due to Dedekind. His definition for the real numbers, please listen to me carefully, and do not fear, is as follows:

DEFINITION 5.2. *The real numbers $x \in \mathbb{R}$ are formal cuts in the set of rationals,*

$$\mathbb{Q} = A_x \sqcup B_x$$

with such a cut being by definition subject to the following conditions:

$$p \in A_x \ , \ q \in B_x \implies p < q \qquad , \qquad \inf B_x \notin B_x$$

These numbers add and multiply by adding and multiplying the corresponding cuts.

This might look quite original, but believe me, there is some genius behind this definition. As a first observation, we have an inclusion $\mathbb{Q} \subset \mathbb{R}$, obtained by identifying each rational number $r \in \mathbb{Q}$ with the obvious cut that it produces, namely:

$$A_r = \left\{ p \in \mathbb{Q} \middle| p \leq r \right\} \quad , \quad B_r = \left\{ q \in \mathbb{Q} \middle| q > r \right\}$$

As a second observation, the addition and multiplication of real numbers, obtained by adding and multiplying the corresponding cuts, in the obvious way, is something very simple. To be more precise, in what regards the addition, the formula is as follows:

$$A_{x+y} = A_x + A_y$$

As for the multiplication, the formula here is similar, namely $A_{xy} = A_x A_y$, up to some mess with positives and negatives, which is quite easy to untangle, and with this being a good exercise. We can also talk about order between real numbers, as follows:

$$x \leq y \iff A_x \subset A_y$$

But let us perhaps leave more abstractions for later, and go back to more concrete things. As a first success of our theory, we can formulate the following theorem:

THEOREM 5.3. *The equation $x^2 = 2$ has two solutions over the real numbers, namely the positive solution, denoted $\sqrt{2}$, and its negative counterpart, which is $-\sqrt{2}$.*

PROOF. By using $x \to -x$, it is enough to prove that $x^2 = 2$ has exactly one positive solution $\sqrt{2}$. But this is clear, because $\sqrt{2}$ can only come from the following cut:

$$A_{\sqrt{2}} = \mathbb{Q}_- \bigsqcup \left\{ p \in \mathbb{Q}_+ \middle| p^2 < 2 \right\} \quad , \quad B_{\sqrt{2}} = \left\{ q \in \mathbb{Q}_+ \middle| q^2 > 2 \right\}$$

Thus, we are led to the conclusion in the statement. □

More generally, the same method works in order to extract the square root $\sqrt{r}$ of any number $r \in \mathbb{Q}_+$, or even of any number $r \in \mathbb{R}_+$, and we have the following result:

THEOREM 5.4. *The solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provided that $b^2 - 4ac \geq 0$. In the case $b^2 - 4ac < 0$, there are no solutions.

PROOF. We can write indeed our equation in the following way:

$$\begin{aligned} ax^2+bx+c=0 &\iff x^2+\frac{b}{a}x+\frac{c}{a}=0\\ &\iff \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0\\ &\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\\ &\iff x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, we have a nice abstract definition for the real numbers, that we can certainly do some mathematics with. As a first general result now, which is something very useful, and puts us back into real life, and science and engineering, we have:

THEOREM 5.5. *The real numbers $x\in\mathbb{R}$ can be written in decimal form,*

$$x=\pm a_1\ldots a_n.b_1b_2b_3\ldots\ldots$$

with $a_i,b_i\in\{0,1,\ldots,9\}$, with the convention $\ldots b999\ldots=\ldots(b+1)000\ldots$

PROOF. This is something which requires a number of verifications, as follows:

(1) First of all, our precise claim is that any $x\in\mathbb{R}$ can be written in the form in the statement, with the integer $\pm a_1\ldots a_n$ and then each of the digits $b_1,b_2,b_3,\ldots$ providing the best approximation of x, at that stage of the approximation.

(2) Moreover, we have a second claim as well, namely that any expression of type $x=\pm a_1\ldots a_n.b_1b_2b_3\ldots\ldots$ corresponds to a real number $x\in\mathbb{R}$, and that with the convention $\ldots b999\ldots=\ldots(b+1)000\ldots$, the correspondence is bijective.

(3) In order to prove now these two assertions, our first claim is that we can restrict the attention to the case $x\in[0,1)$, and with this meaning of course $0\leq x<1$, with respect to the order relation for the reals discussed in the above.

(4) Getting started now, let $x\in\mathbb{R}$, coming from a cut $\mathbb{Q}=A_x\sqcup B_x$. Since the set $A_x\cap\mathbb{Z}$ consists of integers, and is bounded from above by any element $q\in B_x$ of your choice, this set has a maximal element, that we can denote $[x]$:

$$[x]=\max\left(A_x\cap\mathbb{Z}\right)$$

It follows from definitions that $[x]$ has the usual properties of the integer part, namely:

$$[x]\leq x<[x]+1$$

Thus we have $x = [x] + y$ with $[x] \in \mathbb{Z}$ and $y \in [0, 1)$, and getting back now to what we want to prove, namely (1,2) above, it is clear that it is enough to prove these assertions for the remainder $y \in [0, 1)$. Thus, we have proved (3), and we can assume $x \in [0, 1)$.

(5) So, assume $x \in [0, 1)$. We are first looking for a best approximation from below of type $0.b_1$, with $b_1 \in \{0, \ldots, 9\}$, and it is clear that such an approximation exists, simply by comparing x with the numbers $0.0, 0.1, \ldots, 0.9$. Thus, we have our first digit b_1, and then we can construct the second digit b_2 as well, by comparing x with the numbers $0.b_10, 0.b_11, \ldots, 0.b_19$. And so on, which finishes the proof of our claim (1).

(6) In order to prove now the remaining claim (2), let us restrict again the attention, as explained in (4), to the case $x \in [0, 1)$. First, it is clear that any expression of type $x = 0.b_1b_2b_3 \ldots$ defines a real number $x \in [0, 1]$, simply by declaring that the corresponding cut $\mathbb{Q} = A_x \sqcup B_x$ comes from the following set, and its complement:

$$A_x = \bigcup_{n \geq 1} \left\{ p \in \mathbb{Q} \middle| p \leq 0.b_1 \ldots b_n \right\}$$

(7) Thus, we have our correspondence between real numbers as cuts, and real numbers as decimal expressions, and we are left with the question of investigating the bijectivity of this correspondence. But here, the only bug that happens is that numbers of type $x = \ldots b999 \ldots$, which produce reals $x \in \mathbb{R}$ via (6), do not come from reals $x \in \mathbb{R}$ via (5). So, in order to finish our proof, we must investigate such numbers.

(8) So, consider an expression of type $\ldots b999 \ldots$ Going back to the construction in (6), we are led to the conclusion that we have the following equality:

$$A_{b999\ldots} = B_{(b+1)000\ldots}$$

Thus, at the level of the real numbers defined as cuts, we have:

$$\ldots b999 \ldots = \ldots (b+1)000 \ldots$$

But this solves our problem, because by identifying $\ldots b999 \ldots = \ldots (b+1)000 \ldots$ the bijectivity issue of our correspondence is fixed, and we are done. □

The above theorem was of course quite difficult, but this is how things are. Let us record as well the following result, coming as a useful complement to the above:

THEOREM 5.6. *A real number $r \in \mathbb{R}$ is rational precisely when*

$$r = \pm a_1 \ldots a_m.b_1 \ldots b_n(c_1 \ldots c_p)$$

that is, when its decimal writing is periodic.

PROOF. This is something that we know from chapter 4, coming from the following computation, which shows that a number as in the statement is indeed rational:

$$\begin{aligned} r &= \pm\frac{1}{10^n}\, a_1 \ldots a_m b_1 \ldots b_n . c_1 \ldots c_p c_1 \ldots c_p \ldots \\ &= \pm\frac{1}{10^n}\left(a_1 \ldots a_m b_1 \ldots b_n + c_1 \ldots c_p \left(\frac{1}{10^p} + \frac{1}{10^{2p}} + \ldots\right)\right) \\ &= \pm\frac{1}{10^n}\left(a_1 \ldots a_m b_1 \ldots b_n + \frac{c_1 \ldots c_p}{10^p - 1}\right) \end{aligned}$$

As for the converse, given a rational number $r = p/q$, we can find its decimal writing by performing the usual division algorithm, p divided by q. But this algorithm will be surely periodic, after some time, so the decimal writing of r is indeed periodic, as claimed. □

Getting back now to Theorem 5.5, that was definitely something quite difficult. Alternatively, we have the following definition for the real numbers:

THEOREM 5.7. *The field of real numbers $\mathbb{R}$ can be defined as well as the completion of $\mathbb{Q}$ with respect to the usual distance on the rationals, namely*

$$d\left(\frac{a}{b}, \frac{c}{d}\right) = \left|\frac{a}{b} - \frac{c}{d}\right|$$

and with the operations on $\mathbb{R}$ coming from those on $\mathbb{Q}$, via Cauchy sequences.

PROOF. There are several things going on here, the idea being as follows:

(1) Getting back to chapter 3, we know from there what the rational numbers are. But, as a continuation of the material there, we can talk about the distance between such rational numbers, as being given by the formula in the statement, namely:

$$d\left(\frac{a}{b}, \frac{c}{d}\right) = \left|\frac{a}{b} - \frac{c}{d}\right| = \frac{|ad - bc|}{|bd|}$$

(2) Very good, so let us get now into Cauchy sequences. We say that a sequence of rational numbers $\{r_n\} \subset \mathbb{Q}$ is Cauchy when the following condition is satisfied:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, m, n \geq N \implies d(r_m, r_n) < \varepsilon$$

Here of course $\varepsilon \in \mathbb{Q}$, because we do not know yet what the real numbers are.

(3) With this notion in hand, the idea will be to define the reals $x \in \mathbb{R}$ as being the limits of the Cauchy sequences $\{r_n\} \subset \mathbb{Q}$. But since these limits are not known yet to exist to us, precisely because they are real, we must employ a trick. So, let us define instead the reals $x \in \mathbb{R}$ as being the Cauchy sequences $\{r_n\} \subset \mathbb{Q}$ themselves.

(4) The question is now, will this work. As a first observation, we have an inclusion $\mathbb{Q} \subset \mathbb{R}$, obtained by identifying each rational $r \in \mathbb{Q}$ with the constant sequence $r_n = r$.

Also, we can sum and multiply our real numbers in the obvious way, namely:

$$(r_n) + (p_n) = (r_n + p_n) \quad , \quad (r_n)(p_n) = (r_n p_n)$$

We can also talk about the order between such reals, as follows:

$$(r_n) < (p_n) \iff \exists N, n \geq N \implies r_n < p_n$$

Finally, we can also solve equations of type $x^2 = 2$ over our real numbers, say by using our previous work on the decimal writing, which shows in particular that $\sqrt{2}$ can be approximated by rationals $r_n \in \mathbb{Q}$, by truncating the decimal writing.

(5) However, there is still a bug with our theory, because there are obviously more Cauchy sequences of rationals, than real numbers. In order to fix this, let us go back to the end of step (3) above, and make the following convention:

$$(r_n) = (p_n) \iff d(r_n, p_n) \to 0$$

(6) But, with this convention made, we have our theory. Indeed, the considerations in (4) apply again, with this change, and we obtain an ordered field $\mathbb{R}$, containing $\mathbb{Q}$. Moreover, the equivalence with the Dedekind cuts is something which is easy to establish, and we will leave this as an instructive exercise, and this gives all the results. □

Very nice all this, so have have two equivalent definitions for the real numbers. Finally, getting back to the decimal writing approach, that can be recycled too, with some analysis know-how, and we have a third possible definition for the real numbers, as follows:

THEOREM 5.8. *The real numbers $\mathbb{R}$ can be defined as well via the decimal form*

$$x = \pm a_1 \ldots a_n . a_{n+1} a_{n+2} a_{n+3} \ldots\ldots$$

with $a_i \in \{0, 1, \ldots, 9\}$, with the usual convention for such numbers, namely

$$\ldots a999 \ldots = \ldots (a+1)000 \ldots$$

and with the sum and multiplication coming by writing such numbers as

$$x = \pm \sum_{k \in \mathbb{Z}} a_k 10^{-k}$$

and then summing and multiplying, in the obvious way.

PROOF. Let us first forget about the precise decimal writing in the statement, and define the real numbers $x \in \mathbb{R}$ as being formal sums as follows, with the sum being over integers $k \in \mathbb{Z}$ assumed to be greater than a certain integer, $k \geq k_0$:

$$x = \pm \sum_{k \in \mathbb{Z}} a_k 10^{-k}$$

Now by truncating, we can see that what we have here are certain Cauchy sequences of rationals, and with a bit more work, we conclude that the $\mathbb{R}$ that we constructed is precisely the $\mathbb{R}$ that we constructed in Theorem 5.7. Thus, we get the result. □

5b. Limits, series

Time now to get into calculus, among others in order to better understand what was said in the above, regarding the reals. We already learned some good calculus for the rationals, in chapter 3, and the idea is that all that things extend to the reals, and that in addition, many more things can be said, in the real number case.

So, what we will be doing here will be for the most, at least to start with, a cheap remarke of what we learned in chapter 3. But, always good to talk about such things, which are important, so let us review this with full details. To start with, we have:

DEFINITION 5.9. *We say that a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *converges to* $x \in \mathbb{R}$ *when:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - x| < \varepsilon$$

In this case, we write $\lim_{n\to\infty} x_n = x$*, or simply* $x_n \to x$*.*

To be more precise, what we have here is the old convergence definition for the rationals, from chapter 3, extended now to the real numbers. We refer to chapter 3 for comments and basic examples of this, with a basic illustration being as follows:

PROPOSITION 5.10. *We have the following convergence,*

$$n^a \to \begin{cases} 0 & (a < 0) \\ 1 & (a = 0) \\ \infty & (a > 0) \end{cases}$$

with $n \to \infty$*, valid for any exponent* $a \in \mathbb{R}$*.*

PROOF. This is something that we know well from chapter 3 for the rational exponents, $a \in \mathbb{R}$, and the proof in the general case, where $a \in \mathbb{R}$, is similar. □

Coming next, we have some general results about limits, summarized as follows:

THEOREM 5.11. *The following happen:*

(1) *The limit* $\lim_{n\to\infty} x_n$*, if it exists, is unique.*
(2) *If* $x_n \to x$*, with* $x \in (-\infty, \infty)$*, then* x_n *is bounded.*
(3) *If* x_n *is increasing or descreasing, then it converges.*
(4) *Assuming* $x_n \to x$*, any subsequence of* x_n *converges to* x*.*

PROOF. Again, this is something elementary, coming from definitions:

(1) Assuming $x_n \to x$, $x_n \to y$ we have indeed, for any $\varepsilon > 0$, for n big enough:

$$|x - y| \leq |x - x_n| + |x_n - y| < 2\varepsilon$$

(2) Assuming $x_n \to x$, we have $|x_n - x| < 1$ for $n \geq N$, and so, for any $k \in \mathbb{N}$:

$$|x_k| < 1 + |x| + \sup\left(|x_1|, \ldots, |x_{n-1}|\right)$$

(3) By using $x \to -x$, it is enough to prove the result for increasing sequences. But here we can construct the limit $x \in (-\infty, \infty]$ in the following way:

$$\bigcup_{n\in\mathbb{N}} (-\infty, x_n) = (-\infty, x)$$

(4) This is clear from definitions. □

Here are as well some general rules for computing limits:

THEOREM 5.12. *The following happen, with the conventions $\infty+\infty = \infty$, $\infty\cdot\infty = \infty$, $1/\infty = 0$, and with the conventions that $\infty - \infty$ and $\infty \cdot 0$ are undefined:*

(1) $x_n \to x$ *implies* $\lambda x_n \to \lambda x$.
(2) $x_n \to x$, $y_n \to y$ *implies* $x_n + y_n \to x + y$.
(3) $x_n \to x$, $y_n \to y$ *implies* $x_n y_n \to xy$.
(4) $x_n \to x$ *with* $x \neq 0$ *implies* $1/x_n \to 1/x$.

PROOF. All this is again elementary, coming from definitions:

(1) This is something which is obvious from definitions.

(2) This follows indeed from the following estimate:

$$|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y|$$

(3) This follows indeed from the following estimate:

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)y_n + x(y_n - y)| \\ &\leq |x_n - x| \cdot |y_n| + |x| \cdot |y_n - y| \end{aligned}$$

(4) This is again clear, by estimating $1/x_n - 1/x$, in the obvious way. □

As an application of the above rules, we have the following useful result:

PROPOSITION 5.13. *The $n \to \infty$ limits of quotients of polynomials are given by*

$$\lim_{n\to\infty} \frac{a_p n^p + a_{p-1} n^{p-1} + \ldots + a_0}{b_q n^q + b_{q-1} n^{q-1} + \ldots + b_0} = \lim_{n\to\infty} \frac{a_p n^p}{b_q n^q}$$

with the limit on the right being $\pm\infty$, 0, a_p/b_q, depending on the values of p, q.

PROOF. The first assertion comes from the following computation:

$$\begin{aligned} \lim_{n\to\infty} \frac{a_p n^p + a_{p-1} n^{p-1} + \ldots + a_0}{b_q n^q + b_{q-1} n^{q-1} + \ldots + b_0} &= \lim_{n\to\infty} \frac{n^p}{n^q} \cdot \frac{a_p + a_{p-1} n^{-1} + \ldots + a_0 n^{-p}}{b_q + b_{q-1} n^{-1} + \ldots + b_0 n^{-q}} \\ &= \lim_{n\to\infty} \frac{a_p n^p}{b_q n^q} \end{aligned}$$

As for the second assertion, this comes from Proposition 5.10. □

Getting back now to theory, some sequences which obviously do not converge, like for instance $x_n = (-1)^n$, have however "2 limits instead of 1". So, let us formulate:

DEFINITION 5.14. *Given a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, we let*

$$\liminf_{n\to\infty} x_n \in [-\infty, \infty] \quad , \quad \limsup_{n\to\infty} x_n \in [-\infty, \infty]$$

to be the smallest and biggest limit of a subsequence of (x_n).

Observe that the above quantities are defined indeed for any sequence x_n. For instance, for $x_n = (-1)^n$ we obtain -1 and 1. Also, for $x_n = n$ we obtain ∞ and ∞. And so on. Of course, and generalizing the $x_n = n$ example, if $x_n \to x$ we obtain x and x.

Going ahead with more theory, here is a key result:

THEOREM 5.15. *A sequence x_n converges, with finite limit $x \in \mathbb{R}$, precisely when*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, |x_m - x_n| < \varepsilon$$

called Cauchy condition.

PROOF. In one sense, this is clear. In the other sense, we can say for instance that the Cauchy condition forces the decimal writings of our numbers x_n to coincide more and more, with $n \to \infty$, and so we can construct a limit $x = \lim_{n\to\infty} x_n$, as desired. □

The above result is quite interesting, and as an application, we can further clarify what was said before in this chapter, in relation with the reals. Indeed, we have:

THEOREM 5.16. *$\mathbb{R}$ is the completion of $\mathbb{Q}$, in the sense that it is the space of Cauchy sequences over $\mathbb{Q}$, identified when the virtual limit is the same, in the sense that:*

$$x_n \sim y_n \iff |x_n - y_n| \to 0$$

Moreover, $\mathbb{R}$ is complete, in the sense that it equals its own completion.

PROOF. Let us denote indeed the completion operation by $X \to \bar{X} = C_X/\sim$, where C_X is the space of Cauchy sequences over X, and $\sim$ is the above equivalence relation. Since by Theorem 5.15 any Cauchy sequence $(x_n) \in C_{\mathbb{Q}}$ has a limit $x \in \mathbb{R}$, we obtain $\bar{\mathbb{Q}} = \mathbb{R}$. As for the equality $\bar{\mathbb{R}} = \mathbb{R}$, this is clear again by using Theorem 5.15. □

With this discussed, let us get now into series. Let us start with:

DEFINITION 5.17. *Given numbers $x_0, x_1, x_2, \ldots \in \mathbb{R}$, we write*

$$\sum_{n=0}^{\infty} x_n = x$$

with $x \in [-\infty, \infty]$ when $\lim_{k\to\infty} \sum_{n=0}^{k} x_n = x$.

As before with the sequences, there is some general theory that can be developed for the series, and more on this in a moment. As a first, basic example, we have:

THEOREM 5.18. *We have the "geometric series" formula*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

valid for any $|x| < 1$. For $|x| \geq 1$, the series diverges.

PROOF. Our first claim, which comes by multiplying and simplifying, is that:

$$\sum_{n=0}^{k} x^n = \frac{1-x^{k+1}}{1-x}$$

But this proves the first assertion, because with $k \to \infty$ we get:

$$\sum_{n=0}^{k} x^n \to \frac{1}{1-x}$$

As for the second assertion, this is clear as well from our formula above. □

Less trivial now is the following result, due to Riemann:

THEOREM 5.19. *We have the following formula:*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty$$

In fact, $\sum_n 1/n^a$ converges for $a > 1$, and diverges for $a \leq 1$.

PROOF. We have to prove several things, the idea being as follows:

(1) The first assertion comes from the following computation:

$$\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \\
&= \infty
\end{aligned}$$

(2) Regarding now the second assertion, we have that at $a = 1$, and so at any $a \leq 1$. Thus, it remains to prove that at $a > 1$ the series converges. Let us first discuss the case

$a = 2$, which will prove the convergence at any $a \geq 2$. The trick here is as follows:

$$\begin{aligned}
1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots &\leq 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots \\
&= 2\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots\right) \\
&= 2\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)\ldots\right] \\
&= 2
\end{aligned}$$

(3) It remains to prove that the series converges at $a \in (1, 2)$, and here it is enough to deal with the case of the exponents $a = 1 + 1/p$ with $p \in \mathbb{N}$. We already know how to do this at $p = 1$, and the proof at $p \in \mathbb{N}$ will be based on a similar trick. We have:

$$\sum_{n=0}^{\infty} \frac{1}{n^{1/p}} - \frac{1}{(n+1)^{1/p}} = 1$$

Let us compute, or rather estimate, the generic term of this series. By using the formula $a^p - b^p = (a-b)(a^{p-1} + a^{p-2}b + \ldots + ab^{p-2} + b^{p-1})$, we have:

$$\begin{aligned}
\frac{1}{n^{1/p}} - \frac{1}{(n+1)^{1/p}} &= \frac{(n+1)^{1/p} - n^{1/p}}{n^{1/p}(n+1)^{1/p}} \\
&= \frac{1}{n^{1/p}(n+1)^{1/p}[(n+1)^{1-1/p} + \ldots + n^{1-1/p}]} \\
&\geq \frac{1}{n^{1/p}(n+1)^{1/p} \cdot p(n+1)^{1-1/p}} \\
&= \frac{1}{pn^{1/p}(n+1)} \\
&\geq \frac{1}{p(n+1)^{1+1/p}}
\end{aligned}$$

We therefore obtain the following estimate for the Riemann sum:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n^{1+1/p}} &= 1 + \sum_{n=0}^{\infty} \frac{1}{(n+1)^{1+1/p}} \\
&\leq 1 + p\sum_{n=0}^{\infty} \left(\frac{1}{n^{1/p}} - \frac{1}{(n+1)^{1/p}}\right) \\
&= 1 + p
\end{aligned}$$

Thus, we are done with the case $a = 1 + 1/p$, which finishes the proof. □

Here is another tricky result, this time about alternating sums:

THEOREM 5.20. *We have the following convergence result:*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots<\infty$$

However, when rearranging terms, we can obtain any $x\in[-\infty,\infty]$ as limit.

PROOF. Both the assertions follow from Theorem 5.19, as follows:

(1) We have the following computation, using the Riemann criterion at $a=2$:

$$\begin{aligned}1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots&=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\\&=\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\ldots\\&<\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\\&<\infty\end{aligned}$$

(2) We have the following formulae, coming from the Riemann criterion at $a=1$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right)=\infty$$

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\geq\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots=\infty$$

Thus, both these series diverge. The point now is that, by using this, when rearranging terms in the alternating series in the statement, we can arrange for the partial sums to go arbitrarily high, or arbitrarily low, and we can obtain any $x\in[-\infty,\infty]$ as limit. □

Back now to the general case, we first have the following statement:

THEOREM 5.21. *The following hold, with the converses of* (1) *and* (2) *being wrong, and with* (3) *not holding when the assumption $x_n\geq 0$ is removed:*

(1) *If $\sum_n x_n$ converges then $x_n\to 0$.*
(2) *If $\sum_n|x_n|$ converges then $\sum_n x_n$ converges.*
(3) *If $\sum_n x_n$ converges, $x_n\geq 0$ and $x_n/y_n\to 1$ then $\sum_n y_n$ converges.*

PROOF. This is a mixture of trivial and non-trivial results, as follows:

(1) We know that $\sum_n x_n$ converges when $S_k=\sum_{n=0}^k x_n$ converges. Thus by Cauchy we have $x_k=S_k-S_{k-1}\to 0$, and this gives the result. As for the simplest counterexample for the converse, this is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\infty$, coming from Theorem 5.19.

(2) This follows again from the Cauchy criterion, by using:

$$|x_n+x_{n+1}+\ldots+x_{n+k}|\leq|x_n|+|x_{n+1}|+\ldots+|x_{n+k}|$$

As for the simplest counterexample for the converse, this is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots<\infty$, coming from Theorem 5.20, coupled with $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\infty$ from (1).

(3) Again, the main assertion here is clear, coming from, for n big:

$$(1-\varepsilon)x_n \leq y_n \leq (1+\varepsilon)x_n$$

In what regards now the failure of the result, when the assumption $x_n \geq 0$ is removed, this is something quite tricky, the simplest counterexample being as follows:

$$x_n = \frac{(-1)^n}{\sqrt{n}} \quad , \quad y_n = \frac{1}{n} + \frac{(-1)^n}{\sqrt{n}}$$

To be more precise, we have $y_n/x_n \to 1$, so $x_n/y_n \to 1$ too, but according to the above-mentioned results from (1,2), modified a bit, $\sum_n x_n$ converges, while $\sum_n y_n$ diverges. □

Summarizing, we have some useful positive results about series, which are however quite trivial, along with various counterexamples to their possible modifications, which are non-trivial. Staying positive, here are some more positive results:

THEOREM 5.22. *The following happen, and in all cases, the situtation where $c = 1$ is indeterminate, in the sense that the series can converge or diverge:*

(1) *If $|x_{n+1}/x_n| \to c$, the series $\sum_n x_n$ converges if $c < 1$, and diverges if $c > 1$.*
(2) *If $\sqrt[n]{|x_n|} \to c$, the series $\sum_n x_n$ converges if $c < 1$, and diverges if $c > 1$.*
(3) *With $c = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$, $\sum_n x_n$ converges if $c < 1$, and diverges if $c > 1$.*

PROOF. Again, this is a mixture of trivial and non-trivial results, as follows:

(1) Here the main assertions, regarding the cases $c < 1$ and $c > 1$, are both clear by comparing with the geometric series $\sum_n c^n$. As for the case $c = 1$, this is what happens for the Riemann series $\sum_n 1/n^a$, so we can have both convergent and divergent series.

(2) Again, the main assertions, where $c < 1$ or $c > 1$, are clear by comparing with the geometric series $\sum_n c^n$, and the $c = 1$ examples come from the Riemann series.

(3) Here the case $c < 1$ is dealt with as in (2), and the same goes for the examples at $c = 1$. As for the case $c > 1$, this is clear too, because here $x_n \to 0$ fails. □

Finally, generalizing the first assertion in Theorem 5.20, we have:

THEOREM 5.23. *If $x_n \searrow 0$ then $\sum_n (-1)^n x_n$ converges.*

PROOF. We have the $\sum_n (-1)^n x_n = \sum_k y_k$, where:

$$y_k = x_{2k} - x_{2k+1}$$

But, by drawing for instance the numbers x_i on the real line, we see that y_k are positive numbers, and that $\sum_k y_k$ is the sum of lengths of certain disjoint intervals, included in the interval $[0, x_0]$. Thus we have $\sum_k y_k \leq x_0$, and this gives the result. □

Very nice all this, good math that we learned. Let us mention, however, that the above is not the end of the story, because most of the above results extend, and even look better in extended form, to the complex numbers. More on this later in this book.

5c. The number e

All the above was a bit theoretical, and as something more concrete now, which will turn to be essential for arithmetic, and for mathematics in general, we have:

THEOREM 5.24. *We have the following convergence*

$$\left(1+\frac{1}{n}\right)^n\to e$$

where $e=2.71828\ldots$ *is a certain number.*

PROOF. This is something quite tricky, as follows:

(1) Our first claim is that the following sequence is increasing:

$$x_n=\left(1+\frac{1}{n}\right)^n$$

In order to prove this, we use the following arithmetic-geometric inequality:

$$\frac{1+\sum_{i=1}^n\left(1+\frac{1}{n}\right)}{n+1}\geq\sqrt[n+1]{1\cdot\prod_{i=1}^n\left(1+\frac{1}{n}\right)}$$

In practice, after arranging terms, this gives the following inequality:

$$1+\frac{1}{n+1}\geq\left(1+\frac{1}{n}\right)^{n/(n+1)}$$

Now by raising to the power $n+1$ we obtain, as desired:

$$\left(1+\frac{1}{n+1}\right)^{n+1}\geq\left(1+\frac{1}{n}\right)^n$$

(2) Normally we are left with proving that x_n is bounded from above, but this is non-trivial, and we have to use a trick. Consider the following sequence:

$$y_n=\left(1+\frac{1}{n}\right)^{n+1}$$

We will prove that this sequence y_n is decreasing, and together with the fact that we have $x_n/y_n\to 1$, this will give the result. So, this will be our plan.

(3) In order to prove now that y_n is decreasing, we use, a bit as before:

$$\frac{1+\sum_{i=1}^n\left(1-\frac{1}{n}\right)}{n+1}\geq\sqrt[n+1]{1\cdot\prod_{i=1}^n\left(1-\frac{1}{n}\right)}$$

In practice, this gives the following inequality:

$$1-\frac{1}{n+1}\geq\left(1-\frac{1}{n}\right)^{n/(n+1)}$$

Now by raising to the power $n+1$ we obtain from this:

$$\left(1-\frac{1}{n+1}\right)^{n+1}\geq\left(1-\frac{1}{n}\right)^{n}$$

The point now is that we have the following inversion formulae:

$$\left(1-\frac{1}{n+1}\right)^{-1}=\left(\frac{n}{n+1}\right)^{-1}=\frac{n+1}{n}=1+\frac{1}{n}$$

$$\left(1-\frac{1}{n}\right)^{-1}=\left(\frac{n-1}{n}\right)^{-1}=\frac{n}{n-1}=1+\frac{1}{n-1}$$

Thus by inverting the inequality that we found, we obtain, as desired:

$$\left(1+\frac{1}{n}\right)^{n+1}\leq\left(1+\frac{1}{n-1}\right)^{n}$$

(4) But with this, we can now finish. Indeed, the sequence x_n is increasing, the sequence y_n is decreasing, and we have $x_n<y_n$, as well as:

$$\frac{y_n}{x_n}=1+\frac{1}{n}\to 1$$

Thus, both sequences x_n,y_n converge to a certain number e, as desired.

(5) Finally, regarding the numerics for our limiting number e, we know from the above that we have $x_n<e<y_n$ for any $n\in\mathbb{N}$, which reads:

$$\left(1+\frac{1}{n}\right)^{n}<e<\left(1+\frac{1}{n}\right)^{n+1}$$

Thus $e\in[2,3]$, and with a bit of patience, or a computer, we obtain $e=2.71828\ldots$ We will actually come back to this question later, with better methods. □

More generally now, along the same lines, we have the following result:

THEOREM 5.25. *We have the following formula,*

$$\left(1+\frac{x}{n}\right)^{n}\to e^{x}$$

valid for any $x\in\mathbb{R}$.

PROOF. We already know from Theorem 5.24 that the result holds at $x = 1$, and this because the number e was by definition given by the following formula:

$$\left(1 + \frac{1}{n}\right)^n \to e$$

By taking inverses, we obtain as well the result at $x = -1$, namely:

$$\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}$$

In general now, when $\in \mathbb{R}$ is arbitrary, the best is to proceed as follows:

$$\left(1 + \frac{x}{n}\right)^n = \left[\left(1 + \frac{x}{n}\right)^{n/x}\right]^x \to e^x$$

Thus, we are led to the conclusion in the statement. □

Next, we have the following result, which is something quite far-reaching:

THEOREM 5.26. *We have the formula*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

valid for any $x \in \mathbb{R}$.

PROOF. This can be done in several steps, as follows:

(1) At $x = 1$, which is the key step, we want to prove that we have the following equality, between the sum of a series, and a limit of a sequence:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

(2) For this purpose, the first observation is that we have the following estimate:

$$2 < \sum_{k=0}^{\infty} \frac{1}{k!} < \sum_{k=0}^{\infty} \frac{1}{2^{k-1}} = 3$$

Thus, the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges indeed, towards a limit in $(2, 3)$.

(3) In order to prove now that this limit is e, observe that we have:

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\cdot\frac{1}{n^k}\\ &= \sum_{k=0}^{n}\frac{n(n-1)\ldots(n-k+1)}{k!}\cdot\frac{1}{n^k}\\ &\leq \sum_{k=0}^{n}\frac{1}{k!}\end{aligned}$$

Thus, with $n\to\infty$, we get that the limit of the series $\sum_{k=0}^{\infty}\frac{1}{k!}$ belongs to $[e,3)$.

(4) For the reverse inequality, we use the following computation:

$$\begin{aligned}\sum_{k=0}^{n}\frac{1}{k!}-\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\frac{1}{k!}-\sum_{k=0}^{n}\frac{n(n-1)\ldots(n-k+1)}{k!}\cdot\frac{1}{n^k}\\ &= \sum_{k=2}^{n}\frac{1}{k!}-\sum_{k=2}^{n}\frac{n(n-1)\ldots(n-k+1)}{k!}\cdot\frac{1}{n^k}\\ &= \sum_{k=2}^{n}\frac{n^k-n(n-1)\ldots(n-k+1)}{n^k k!}\\ &\leq \sum_{k=2}^{n}\frac{n^k-(n-k)^k}{n^k k!}\\ &= \sum_{k=2}^{n}\frac{1-\left(1-\frac{k}{n}\right)^k}{k!}\end{aligned}$$

(5) In order to estimate the above expression that we found, we can use the following trivial inequality, valid for any number $x\in(0,1)$:

$$1-x^k=(1-x)(1+x+x^2+\ldots+x^{k-1})\leq(1-x)k$$

Indeed, we can use this with $x = 1 - k/n$, and we obtain in this way:

$$\begin{aligned}
\sum_{k=0}^{n}\frac{1}{k!}-\left(1+\frac{1}{n}\right)^n &\leq \sum_{k=2}^{n}\frac{\frac{k}{n}\cdot k}{k!}\\
&= \frac{1}{n}\sum_{k=2}^{n}\frac{k}{(k-1)!}\\
&= \frac{1}{n}\sum_{k=2}^{n}\frac{k}{k-1}\cdot\frac{1}{(k-2)!}\\
&\leq \frac{1}{n}\sum_{k=2}^{n}\frac{2}{2^{k-2}}\\
&< \frac{4}{n}
\end{aligned}$$

Now since with $n \to \infty$ this goes to 0, we obtain that the limit of the series $\sum_{k=0}^{\infty}\frac{1}{k!}$ is the same as the limit of the sequence $\left(1+\frac{1}{n}\right)^n$, manely e. Thus, getting back now to what we wanted to prove, our theorem, we are done in this way with the case $x = 1$.

(6) In order to deal now with the general case, consider the following function:

$$f(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Observe that, by using our various results above, this function is indeed well-defined. Moreover, again by using our various results above, f is continuous.

(7) Our next claim, which is the key one, is that we have:

$$f(x+y) = f(x)f(y)$$

Indeed, by using the binomial formula, we have the following computation:

$$\begin{aligned}
f(x+y) &= \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!}\\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\binom{k}{s}\cdot\frac{x^s y^{k-s}}{k!}\\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{x^s y^{k-s}}{s!(k-s)!}\\
&= f(x)f(y)
\end{aligned}$$

(8) In order to finish now, we know that our function f is continuous, that it satisfies $f(x+y)=f(x)f(y)$, and that we have:

$$f(0)=1\quad,\quad f(1)=e$$

But it is easy to prove that such a function is necessarily unique, and since e^x obviously has all these properties too, we must have $f(x)=e^x$, as desired. □

All the above is quite nice, and as a first application, in relation with the counting for permutations from chapter 2, we can now formulate the following result:

THEOREM 5.27. *The number of fixed points of the permutations $\sigma\in S_N$, regarded as random variable $\chi:S_N\to\mathbb{N}$, follows with $N\to\infty$ the law*

$$p_1=\frac{1}{e}\sum_{k\geq 0}\frac{\delta_k}{k!}$$

called Poisson law of parameter 1.

PROOF. Observe first that p_1 has indeed mass 1, as it should, due to $e=\sum_k 1/k!$. Regarding now the proof, as explained in chapter 2, by inclusion-exclusion we have:

$$\chi(0)=\sum_{r=0}^{N}\frac{(-1)^r}{r!}$$

Thus we have $\chi(0)\simeq 1/e$, and a variation of this computation, that we will leave here as an instructive exercise, gives $\chi(k)\simeq 1/ek!$ for any $k\in\mathbb{N}$, as desired. □

Many other things can be said about this, and about the Poisson laws too, the general idea being that these measures appear a bit everywhere, in the discrete probability context, the reasons for this coming from the Poisson Limit Theorem (PLT), which is closely related to our previous investigations from chapter 4, regarding the Bernoulli and binomial laws. For more on all this, have a look at any standard probability book.

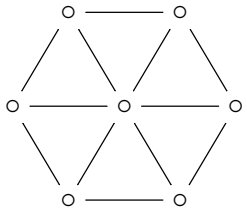
5d. The number pi

As the saying goes, there is no e without π, so let us get now into π and trigonometry. Many things can be said here, and to start with, sort of axiomatically, we have:

THEOREM 5.28. *The following two definitions of π are equivalent:*

(1) *The length of the unit circle is $L=2\pi$.*
(2) *The area of the unit disk is $A=\pi$.*

PROOF. In order to prove this theorem let us cut the unit disk as a pizza, into N slices, and forgetting about gastronomy, leave aside the rounded parts:



The area to be eaten can be then computed as follows, where H is the height of the slices, S is the length of their sides, and $P = NS$ is the total length of the sides:

$$A = N \times \frac{HS}{2} = \frac{HP}{2} \simeq \frac{1 \times L}{2}$$

Thus, with $N \to \infty$ we obtain that we have $A = L/2$, as desired. □

In what regards now the precise value of π, the above picture at $N = 6$ shows that we have $\pi > 3$, but not by much. The precise figure is $\pi = 3.14159\ldots$, but we will come back to this later, once we will have appropriate tools for dealing with such questions.

Now that we know about π, let us discuss trigonometry. We first have:

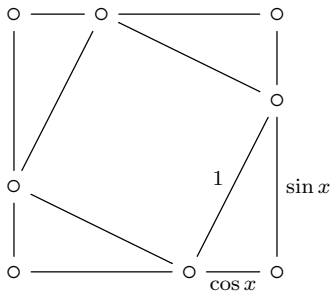
THEOREM 5.29. *The following happen:*

(1) *We can talk about angles $x \in \mathbb{R}$, by using the unit circle, in the usual way, and in this correspondence, the right angle has a value of $\pi/2$.*
(2) *Associated to any $x \in \mathbb{R}$ are numbers $\sin x, \cos x \in \mathbb{R}$, constructed in the usual way, by using a triangle. These numbers satisfy $\sin^2 x + \cos^2 x = 1$.*

PROOF. There are certainly things that you know, the idea being as follows:

(1) The formula $L = 2\pi$ from Theorem 5.28 shows that the length of a quarter of the unit circle is $l = \pi/2$, and so the right angle has indeed this value, $\pi/2$.

(2) As for $\sin^2 x + \cos^2 x = 1$, called Pythagoras' theorem, this comes from the following picture, consisting of two squares and four identical triangles, as indicated:



Indeed, when computing the area of the outer square, we obtain:

$$(\sin x+\cos x)^2=1+4\times\frac{\sin x\cos x}{2}$$

Now when expanding we obtain $\sin^2 x+\cos^2 x=1$, as claimed. □

Still at the level of the basics, we have the following result:

THEOREM 5.30. *The sines and cosines of sums are given by*

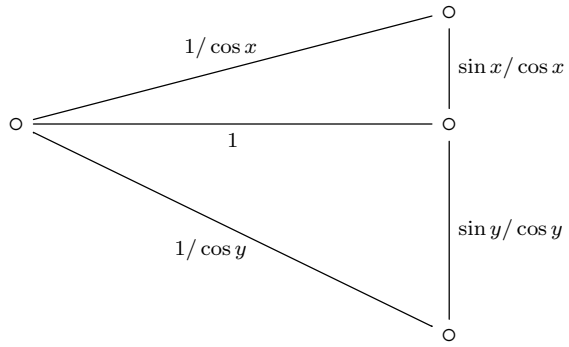
$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

and these formulae give a formula for $\tan(x+y)$ *too.*

PROOF. This is something quite tricky, using the same idea as in the proof of Pythagoras' theorem, that is, computing certain areas, the idea being as follows:

(1) Consider the following picture, consisting of a length 1 line segment, with angles x,y drawn on each side, and with the lengths being computed, as indicated:



Now let us compute the area of the big triangle, or rather the double of that area. We can do this in two ways, either directly, with a formula involving $\sin(x+y)$, or by using the two small triangles, involving functions of x,y. We obtain in this way:

$$\frac{1}{\cos x}\cdot\frac{1}{\cos y}\cdot\sin(x+y)=\frac{\sin x}{\cos x}\cdot 1+\frac{\sin y}{\cos y}\cdot 1$$

But this gives the formula for $\sin(x+y)$ from the statement.

(2) By using $\sin(x+y)$ we can deduce a formula for $\cos(x+y)$, as follows:

$$\begin{aligned}\cos(x+y)&=\sin\left(\frac{\pi}{2}-x-y\right)\\&=\sin\left[\left(\frac{\pi}{2}-x\right)+(-y)\right]\\&=\sin\left(\frac{\pi}{2}-x\right)\cos(-y)+\cos\left(\frac{\pi}{2}-x\right)\sin(-y)\\&=\cos x\cos y-\sin x\sin y\end{aligned}$$

(3) Finally, in what regards the tangents, we have, according to the above:

$$\tan(x+y)=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}$$

Thus, we are led to the conclusions in the statement. □

And with this, good news, we have all tools in our bag for doing some truly tough calculus, with the real numbers. Hang on, difficult material to come, right next.

5e. Exercises

This was a very basic analysis chapter, and as exercises here, we have:

EXERCISE 5.31. *Quickly rewrite the theory of $\mathbb{R}$, introduced as completion of $\mathbb{Q}$.*

EXERCISE 5.32. *Then, rewrite the theory of $\mathbb{R}$, introduced via the decimal form.*

EXERCISE 5.33. *Then, rewrite again, with $\mathbb{R}$ introduced via binary numbers.*

EXERCISE 5.34. *Clarify what we said about rearranging terms of $\sum_k(-1)^k/k$.*

EXERCISE 5.35. *Review, if needed, the proof of the arithmetic-geometric inequality.*

EXERCISE 5.36. *Fill in all the details, regarding the properties of $f(x)=\sum_k x^k/k!$.*

EXERCISE 5.37. *Learn more about the Poisson laws, and their various properties.*

EXERCISE 5.38. *Further meditate on the two definitions of π, and their equivalence.*

As bonus exercise, find 100 exercises on sequences and series, and solve them.

CHAPTER 6

Some calculus

6a. Derivatives, rules

The idea of calculus is very simple. We are interested in the functions $f : \mathbb{R} \to \mathbb{R}$, and when f is continuous at a point x, this means by definition that we can write an approximation formula as follows, for the values of f around that point x:

$$f(x + t) \simeq f(x)$$

The problem is now, how to improve this? And a bit of thinking at all this suggests to look at the slope of f at the point x. Which leads us into the following notion:

DEFINITION 6.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called differentiable at x when*

$$f'(x) = \lim_{t \to 0} \frac{f(x + t) - f(x)}{t}$$

called derivative of f at that point x, exists.

As a first remark, in order for f to be differentiable at x, that is to say, in order for the above limit to converge, the numerator must go to 0, as the denominator t does:

$$\lim_{t \to 0} [f(x + t) - f(x)] = 0$$

Thus, f must be continuous at x. However, the converse is not true, a basic counterexample being $f(x) = |x|$ at $x = 0$. Let us summarize these findings as follows:

PROPOSITION 6.2. *If f is differentiable at x, then f must be continuous at x. However, the converse is not true, a basic counterexample being $f(x) = |x|$, at $x = 0$.*

PROOF. The first assertion is something that we already know, from the above. As for the second assertion, regarding $f(x) = |x|$, this is something quite clear on the picture of f, but let us prove this mathematically, based on Definition 6.1. We have:

$$\lim_{t \searrow 0} \frac{|0 + t| - |0|}{t} = \lim_{t \searrow 0} \frac{t - 0}{t} = 1$$

On the other hand, we have as well the following computation:

$$\lim_{t \nearrow 0} \frac{|0 + t| - |0|}{t} = \lim_{t \nearrow 0} \frac{-t - 0}{t} = -1$$

Thus, the limit in Definition 6.1 does not converge, as desired. □

Generally speaking, the last assertion in Proposition 6.2 should not bother us much, because most of the basic continuous functions are differentiable, and we will see examples in a moment. Before that, however, let us recall why we are here, namely improving the basic estimate $f(x+t)\simeq f(x)$. We can now do this, using the derivative, as follows:

THEOREM 6.3. *Assuming that f is differentiable at x, we have:*

$$f(x+t)\simeq f(x)+f'(x)t$$

In other words, f is, approximately, locally affine at x.

PROOF. Assume indeed that f is differentiable at x, and let us set, as before:

$$f'(x)=\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$$

By multiplying by t, we obtain that we have, once again in the $t\to 0$ limit:

$$f(x+t)-f(x)\simeq f'(x)t$$

Thus, we are led to the conclusion in the statement. □

All this is very nice, and before developing more theory, let us work out some examples. As a first illustration, the derivatives of the power functions are as follows:

PROPOSITION 6.4. *We have the differentiation formula*

$$(x^p)'=px^{p-1}$$

valid for any exponent $p\in\mathbb{R}$.

PROOF. We can do this in three steps, as follows:

(1) In the case $p\in\mathbb{N}$ we can use the binomial formula, which gives, as desired:

$$\begin{aligned}(x+t)^p&=\sum_{k=0}^{n}\binom{p}{k}x^{p-k}t^k\\&=x^p+px^{p-1}t+\ldots+t^p\\&\simeq x^p+px^{p-1}t\end{aligned}$$

(2) Let us discuss now the general case $p\in\mathbb{Q}$. We write $p=m/n$, with $m\in\mathbb{Z}$ and $n\in\mathbb{N}$. In order to do the computation, we use the following formula:

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\ldots+b^{n-1})$$

We set in this formula $a = (x+t)^{m/n}$ and $b = x^{m/n}$. We obtain, as desired:

$$\begin{aligned}(x+t)^{m/n} - x^{m/n} &= \frac{(x+t)^m - x^m}{(x+t)^{m(n-1)/n} + \ldots + x^{m(n-1)/n}}\\ &\simeq \frac{(x+t)^m - x^m}{nx^{m(n-1)/n}}\\ &\simeq \frac{mx^{m-1}t}{nx^{m(n-1)/n}}\\ &= \frac{m}{n} \cdot x^{m-1-m+m/n} \cdot t\\ &= \frac{m}{n} \cdot x^{m/n-1} \cdot t\end{aligned}$$

(3) In the general case now, where $p \in \mathbb{R}$ is real, we can use a similar argument. Indeed, given any integer $n \in \mathbb{N}$, we have the following computation:

$$\begin{aligned}(x+t)^p - x^p &= \frac{(x+t)^{pn} - x^{pn}}{(x+t)^{p(n-1)} + \ldots + x^{p(n-1)}}\\ &\simeq \frac{(x+t)^{pn} - x^{pn}}{nx^{p(n-1)}}\end{aligned}$$

Now observe that we have the following estimate, with $[.]$ being the integer part:

$$(x+t)^{[pn]} \leq (x+t)^{pn} \leq (x+t)^{[pn]+1}$$

By using the binomial formula on both sides, for the integer exponents $[pn]$ and $[pn]+1$ there, we deduce that with $n >> 0$ we have the following estimate:

$$(x+t)^{pn} \simeq x^{pn} + pnx^{pn-1}t$$

Thus, we can finish our computation started above as follows:

$$(x+t)^p - x^p \simeq \frac{pnx^{pn-1}t}{nx^{pn-p}} = px^{p-1}t$$

But this gives $(x^p)' = px^{p-1}$, which finishes the proof. □

Here are some further computations, for other basic functions that we know, with the logarithm being by definition the inverse of the exponential function:

PROPOSITION 6.5. *We have the following results:*

(1) $(\sin x)' = \cos x$.
(2) $(\cos x)' = -\sin x$.
(3) $(e^x)' = e^x$.
(4) $(\log x)' = x^{-1}$.

PROOF. This is quite tricky, as always when computing derivatives, as follows:

(1) Regarding sin, the computation here goes as follows:

$$\begin{aligned}(\sin x)' &= \lim_{t\to 0}\frac{\sin(x+t)-\sin x}{t}\\ &= \lim_{t\to 0}\frac{\sin x\cos t+\cos x\sin t-\sin x}{t}\\ &= \lim_{t\to 0}\sin x\cdot\frac{\cos t-1}{t}+\cos x\cdot\frac{\sin t}{t}\\ &= \cos x\end{aligned}$$

(2) The computation for cos is similar, as follows:

$$\begin{aligned}(\cos x)' &= \lim_{t\to 0}\frac{\cos(x+t)-\cos x}{t}\\ &= \lim_{t\to 0}\frac{\cos x\cos t-\sin x\sin t-\cos x}{t}\\ &= \lim_{t\to 0}\cos x\cdot\frac{\cos t-1}{t}-\sin x\cdot\frac{\sin t}{t}\\ &= -\sin x\end{aligned}$$

(3) For the exponential, the derivative can be computed as follows:

$$\begin{aligned}(e^x)' &= \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)'\\ &= \sum_{k=0}^{\infty}\frac{kx^{k-1}}{k!}\\ &= e^x\end{aligned}$$

(4) As for the logarithm, the computation here is as follows, using $\log(1+y)\simeq y$ for $y\simeq 0$, which follows from $e^y\simeq 1+y$ that we found in (3), by taking the logarithm:

$$\begin{aligned}(\log x)' &= \lim_{t\to 0}\frac{\log(x+t)-\log x}{t}\\ &= \lim_{t\to 0}\frac{\log(1+t/x)}{t}\\ &= \frac{1}{x}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

Speaking exponentials, we can now formulate a nice result about them:

THEOREM 6.6. *The exponential function, namely*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is the unique power series satisfying $f' = f$ and $f(0) = 1$.

PROOF. Consider indeed a power series satisfying $f' = f$ and $f(0) = 1$. Due to $f(0) = 1$, the first term must be 1, and so our function must look as follows:

$$f(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$$

According to our differentiation rules, the derivative of this series is given by:

$$f(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

Thus, the equation $f' = f$ is equivalent to the following equalities:

$$a_1 = 1 \quad , \quad 2a_2 = a_1 \quad , \quad 3a_3 = a_2 \quad , \quad 4a_4 = a_3 \quad , \quad \ldots$$

But this system of equations can be solved by recurrence, as follows:

$$a_1 = 1 \quad , \quad a_2 = \frac{1}{2} \quad , \quad a_3 = \frac{1}{2 \times 3} \quad , \quad a_4 = \frac{1}{2 \times 3 \times 4} \quad , \quad \ldots$$

Thus we have $a_k = 1/k!$, leading to the conclusion in the statement. □

Observe that the above result leads to a more conceptual explanation for the number e itself. To be more precise, $e \in \mathbb{R}$ is the unique number satisfying:

$$(e^x)' = e^x$$

Let us work out now some general results. We have here the following statement:

THEOREM 6.7. *We have the following formulae:*

(1) $(f + g)' = f' + g'$.
(2) $(fg)' = f'g + fg'$.
(3) $(f \circ g)' = (f' \circ g) \cdot g'$.

PROOF. All these formulae are elementary, the idea being as follows:

(1) This follows indeed from definitions, the computation being as follows:

$$\begin{aligned}(f+g)'(x) &= \lim_{t\to 0}\frac{(f+g)(x+t)-(f+g)(x)}{t}\\ &= \lim_{t\to 0}\left(\frac{f(x+t)-f(x)}{t}+\frac{g(x+t)-g(x)}{t}\right)\\ &= \lim_{t\to 0}\frac{f(x+t)-f(x)}{t}+\lim_{t\to 0}\frac{g(x+t)-g(x)}{t}\\ &= f'(x)+g'(x)\end{aligned}$$

(2) This follows from definitions too, the computation, by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative, being as follows:

$$\begin{aligned}(fg)(x+t) &= f(x+t)g(x+t)\\ &\simeq (f(x)+f'(x)t)(g(x)+g'(x)t)\\ &\simeq f(x)g(x)+(f'(x)g(x)+f(x)g'(x))t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of t, namely:

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x)$$

(3) Regarding compositions, the computation here is as follows, again by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative:

$$\begin{aligned}(f\circ g)(x+t) &= f(g(x+t))\\ &\simeq f(g(x)+g'(x)t)\\ &\simeq f(g(x))+f'(g(x))g'(x)t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of t, namely:

$$(f\circ g)'(x)=f'(g(x))g'(x)$$

Thus, we are led to the conclusions in the statement. □

We can of course combine the above formulae, and we obtain for instance:

PROPOSITION 6.8. *The derivatives of fractions are given by:*

$$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

In particular, we have the following formula, for the derivative of inverses:

$$\left(\frac{1}{f}\right)'=-\frac{f'}{f^2}$$

In fact, we have $(f^p)'=pf^{p-1}$, for any exponent $p\in\mathbb{R}$.

PROOF. This statement is written a bit upside down, and for the proof it is better to proceed backwards. To be more precise, by using $(x^p)' = px^{p-1}$ and Theorem 6.7 (3), we obtain the third formula. Then, with $p = -1$, we obtain from this the second formula. And finally, by using this second formula and Theorem 6.7 (2), we obtain:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' \\ &= f' \cdot \frac{1}{g} + f\left(\frac{1}{g}\right)' \\ &= \frac{f'}{g} - \frac{fg'}{g^2} \\ &= \frac{f'g - fg'}{g^2}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

With the above formulae in hand, we can do all sorts of computations for other basic functions that we know, including $\tan x$, or $\arctan x$:

PROPOSITION 6.9. *We have the following formulae,*

$$(\tan x)' = \frac{1}{\cos^2 x} \quad , \quad (\arctan x)' = \frac{1}{1+x^2}$$

and the derivatives of the remaining trigonometric functions can be computed as well.

PROOF. For tan, we have the following computation:

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' \\ &= \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x}\end{aligned}$$

As for arctan, we can use here the following computation:

$$\begin{aligned}(\tan \circ \arctan)'(x) &= \tan'(\arctan x)\arctan'(x) \\ &= \frac{1}{\cos^2(\arctan x)}\arctan'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x' = 1$, we obtain from this:

$$\arctan'(x) = \cos^2(\arctan x)$$

On the other hand, with $t = \arctan x$ we know that we have $\tan t = x$, and so:

$$\cos^2(\arctan x) = \cos^2 t = \frac{1}{1 + \tan^2 t} = \frac{1}{1 + x^2}$$

Thus, we are led to the formula in the statement, namely:

$$(\arctan x)' = \frac{1}{1 + x^2}$$

As for the last assertion, we will leave this as an exercise. □

At the theoretical level now, further building on Theorem 6.3, we have:

THEOREM 6.10. *The local minima and maxima of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ appear at the points $x \in \mathbb{R}$ where:*

$$f'(x) = 0$$

However, the converse of this fact is not true in general.

PROOF. The first assertion follows from the formula in Theorem 6.3, namely:

$$f(x + t) \simeq f(x) + f'(x)t$$

Indeed, let us rewrite this formula, more conveniently, in the following way:

$$f(x + t) - f(x) \simeq f'(x)t$$

Now saying that our function f has a local maximum at $x \in \mathbb{R}$ means that there exists a number $\varepsilon > 0$ such that the following happens:

$$f(x + t) \geq f(x) \quad , \quad \forall t \in [-\varepsilon, \varepsilon]$$

We conclude that we must have $f'(x)t \geq 0$ for sufficiently small t, and since this small t can be both positive or negative, this gives, as desired:

$$f'(x) = 0$$

Similarly, saying that our function f has a local minimum at $x \in \mathbb{R}$ means that there exists a number $\varepsilon > 0$ such that the following happens:

$$f(x + t) \leq f(x) \quad , \quad \forall t \in [-\varepsilon, \varepsilon]$$

Thus $f'(x)t \leq 0$ for small t, and this gives, as before, $f'(x) = 0$. Finally, in what regards the converse, the simplest counterexample here is the following function:

$$f(x) = x^3$$

Indeed, we have $f'(x) = 3x^2$, and in particular $f'(0) = 0$. But our function being clearly increasing, $x = 0$ is not a local maximum, nor a local minimum. □

As an important theoretical consequence of Theorem 6.10, let us record:

THEOREM 6.11. *Assuming that $f : [a, b] \to \mathbb{R}$ is differentiable, we have*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$, called mean value property of f.

PROOF. In the case $f(a) = f(b)$, the result, called Rolle theorem, states that we have $f'(c) = 0$ for some $c \in (a, b)$, and follows from Theorem 6.10. Now in what regards our statement, due to Lagrange, this follows from Rolle, applied to the following function:

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a} \cdot x$$

Indeed, we have $g(a) = g(b)$, due to our choice of the constant on the right, so we get $g'(c) = 0$ for some $c \in (a, b)$, which translates into the formula in the statement. □

In practice, Theorem 6.10 can be used in order to find the maximum and minimum of any differentiable function, and this method is best recalled as follows:

ALGORITHM 6.12. *In order to find the minimum and maximum of $f : [a, b] \to \mathbb{R}$:*

(1) *Compute the derivative f'.*
(2) *Solve the equation $f'(x) = 0$.*
(3) *Add a, b to your set of solutions.*
(4) *Compute $f(x)$, for all your solutions.*
(5) *Compute the min/max of all these $f(x)$ values.*
(6) *Then this is the min/max of your function.*

Needless to say, all this is very interesting, and powerful. The general problem in any type of applied mathematics is that of finding the minimum or maximum of some function, and we have now an algorithm for dealing with such questions. Very nice.

6b. Second derivatives

The derivative theory that we have is already quite powerful, and can be used in order to solve all sorts of interesting questions, but with a bit more effort, we can do better. Indeed, at a more advanced level, we can come up with the following notion:

DEFINITION 6.13. *We say that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable if it is differentiable, and its derivative $f' : \mathbb{R} \to \mathbb{R}$ is differentiable too. The derivative of f' is denoted*

$$f'' : \mathbb{R} \to \mathbb{R}$$

and is called second derivative of f.

You might probably wonder why coming with this definition, which looks a bit abstract and complicated, instead of further developing the theory of the first derivative, which looks like something very reasonable and useful. Good point, and answer to this coming in a moment. But before that, let us get a bit familiar with f''. We first have:

INTERPRETATION 6.14. *The second derivative $f''(x) \in \mathbb{R}$ is the number which:*

(1) *Expresses the growth rate of the slope $f'(z)$ at the point x.*
(2) *Gives us the acceleration of the function f at the point x.*
(3) *Computes how much different is $f(x)$, compared to $f(z)$ with $z \simeq x$.*
(4) *Tells us how much convex or concave is f, around the point x.*

So, this is the truth about the second derivative, making it clear that what we have here is a very interesting notion. In practice now, the situation is as follows:

(1) This is something very intuitive, which follows from the usual interpretation of the derivative, both as a growth rate, and a slope.

(2) This is some sort of reformulation of (1), using the intuitive meaning of the word "acceleration", with the relevant physics equations, due to Newton, being as follows:

$$v = \dot{x} \quad , \quad a = \dot{v}$$

To be more precise, here x, v, a are the position, speed and acceleration, and the dot denotes the time derivative, and according to these equations, we have $a = \ddot{x}$, second derivative. We will be back to these equations later, later in this book.

(3) This is something more subtle, and very useful too, which is of statistical nature, and that we will clarify with some mathematics, in a moment.

(4) This is something quite subtle too, and again very useful in practice, that we will again clarify with some mathematics, later in this chapter.

All in all, what we have above is a mixture of trivial and non-trivial facts, and do not worry, we will get familiar with all this, in the next few pages.

In practice now, let us first compute the second derivatives of the functions that we are familiar with, see what we get. The result here, which is perhaps not very enlightening at this stage of things, but which certainly looks technically useful, is as follows:

PROPOSITION 6.15. *The second derivatives of the basic functions are as follows:*

(1) $(x^p)'' = p(p-1)x^{p-2}$.
(2) $\sin'' = -\sin$.
(3) $\cos'' = -\cos$.
(4) $\exp' = \exp$.
(5) $\log'(x) = -1/x^2$.

Also, there are functions which are differentiable, but not twice differentiable.

PROOF. We have several assertions here, the idea being as follows:

(1) Regarding the various formulae in the statement, these all follow from the various formulae for the derivatives established before, as follows:

$$(x^p)'' = (px^{p-1})' = p(p-1)x^{p-2}$$

$$(\sin x)'' = (\cos x)' = -\sin x$$

$$(\cos x)'' = (-\sin x)' = -\cos x$$

$$(e^x)'' = (e^x)' = e^x$$

$$(\log x)'' = (-1/x)' = -1/x^2$$

Of course, this is not the end of the story, because these formulae remain quite opaque, and must be examined in view of Interpretation 6.14, in order to see what exactly is going on. Also, we have tan and the inverse trigonometric functions too. In short, plenty of good exercises here, for you, and the more you solve, the better your calculus will be.

(2) Regarding now the counterexample, recall first that the simplest example of a function which is continuous, but not differentiable, was $f(x) = |x|$, the idea behind this being to use a "piecewise linear function whose branches do not fit well". In connection now with our question, piecewise linear will not do, but we can use a similar idea, namely "piecewise quadratic function whose branches do not fit well". So, let us set:

$$f(x) = \begin{cases} ax^2 & (x \leq 0) \\ bx^2 & (x \geq 0) \end{cases}$$

This function is then differentiable, with its derivative being:

$$f'(x) = \begin{cases} 2ax & (x \leq 0) \\ 2bx & (x \geq 0) \end{cases}$$

Now for getting our counterexample, we can set $a = -1, b = 1$, so that f is:

$$f(x) = \begin{cases} -x^2 & (x \leq 0) \\ x^2 & (x \geq 0) \end{cases}$$

Indeed, the derivative is $f'(x) = 2|x|$, which is not differentiable, as desired. □

Getting now to theory, our main purpose will be that of improving, with the help of the second derivative, the basic approximation formula for functions, namely:

$$f(x+t) \simeq f(x) + f'(x)t$$

In order to do so, things will be quite tricky, and a bit more geometric, and perhaps less intuitive, than before. We will be in need of the following standard result:

THEOREM 6.16. *The* $0/0$ *type limits can be computed according to the formula*

$$\frac{f(x)}{g(x)} \simeq \frac{f'(x)}{g'(x)}$$

called L'Hôpital's rule.

PROOF. The above formula holds indeed, as an application of the derivative theory from the beginning of this chapter, which gives, in the situation from the statement:

$$\begin{aligned}\frac{f(x+t)}{g(x+t)} &\simeq \frac{f(x)+f'(x)t}{g(x)+g'(x)t}\\ &= \frac{f'(x)t}{g'(x)t}\\ &= \frac{f'(x)}{g'(x)}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We can now formulate the following key result:

THEOREM 6.17. *Any twice differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *is locally quadratic,*

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

with $f''(x)$ *being as usual the derivative of the function* $f' : \mathbb{R} \to \mathbb{R}$ *at the point* x*.*

PROOF. Assume indeed that f is twice differentiable at x, and let us try to construct an approximation of f around x by a quadratic function, as follows:

$$f(x+t) \simeq a + bt + ct^2$$

We must have $a = f(x)$, and we also know from before that $b = f'(x)$ is the correct choice for the coefficient of t. Thus, our approximation must be as follows:

$$f(x+t) \simeq f(x) + f'(x)t + ct^2$$

In order to find the correct choice for $c \in \mathbb{R}$, observe that the function $t \to f(x+t)$ matches with $t \to f(x) + f'(x)t + ct^2$ in what regards the value at $t = 0$, and also in what regards the value of the derivative at $t = 0$. Thus, the correct choice of $c \in \mathbb{R}$ should be the one making match the second derivatives at $t = 0$, and this gives:

$$f''(x) = 2c$$

We are therefore led to the approximation formula in the statement, namely:

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

In order to prove now that this formula holds indeed, we can use L'Hôpital's rule. Indeed, by using this, if we denote by $\varphi(t) \simeq P(t)$ the formula to be proved, we have:

$$\begin{aligned}\frac{\varphi(t)-P(t)}{t^2} &\simeq \frac{\varphi'(t)-P'(t)}{2t}\\ &\simeq \frac{\varphi''(t)-P''(t)}{2}\\ &= \frac{f''(x)-f''(x)}{2}\\ &= 0\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result substantially improves Theorem 6.3, and there are many applications of it. As a first such application, justifying Interpretation 6.14 (3), we have the following statement, which is a bit heuristic, but we will call it however Proposition:

PROPOSITION 6.18. *Intuitively speaking, the second derivative $f''(x) \in \mathbb{R}$ computes how much different is $f(x)$, compared to the average of $f(z)$, with $z \simeq x$.*

PROOF. As already mentioned, this is something a bit heuristic, but which is good to know. Let us write the formula in Theorem 6.17, as such, and with $t \to -t$ too:

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

$$f(x-t) \simeq f(x) - f'(x)t + \frac{f''(x)}{2}\,t^2$$

By making the average, we obtain the following formula:

$$\frac{f(x+t)+f(x-t)}{2} \simeq f(x) + \frac{f''(x)}{2}\,t^2$$

But this is what our statement says, save for some uncertainties regarding the averaging method, and for the precise value of $I(t^2/2)$. We will leave this for later. □

Back to rigorous mathematics, we can improve as well Theorem 6.10, as follows:

THEOREM 6.19. *The local minima and local maxima of a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ appear at the points $x \in \mathbb{R}$ where*

$$f'(x) = 0$$

with the local minima corresponding to the case $f'(x) \geq 0$, and with the local maxima corresponding to the case $f''(x) \leq 0$.

PROOF. The first assertion is something that we already know. As for the second assertion, we can use the formula in Theorem 6.17, which in the case $f'(x) = 0$ reads:

$$f(x+t) \simeq f(x) + \frac{f''(x)}{2}\,t^2$$

Indeed, assuming $f''(x) \neq 0$, it is clear that the condition $f''(x) > 0$ will produce a local minimum, and that the condition $f''(x) < 0$ will produce a local maximum. □

As before with Theorem 6.10, the above result is not the end of the story with the mathematics of the local minima and maxima, because things are undetermined when:

$$f'(x) = f''(x) = 0$$

For instance the functions $\pm x^n$ with $n \in \mathbb{N}$ all satisfy this condition at $x = 0$, which is a minimum for the functions of type x^{2m}, a maximum for the functions of type $-x^{2m}$, and not a local minimum or local maximum for the functions of type $\pm x^{2m+1}$.

There are some comments to be made in relation with Algorithm 6.12 as well. Normally that algorithm stays strong, because Theorem 6.19 can only help in relation with the final steps, and is it worth it to compute the second derivative f'', just for getting rid of roughly $1/2$ of the $f(x)$ values to be compared. However, in certain cases, this method proves to be useful, so Theorem 6.19 is good to know, when applying that algorithm.

Again, we will be back to such questions later in this chapter, with a full, more advanced discussion about all this, by using higher derivatives.

6c. Convex functions

As a main concrete application now of the second derivative, which is something very useful in practice, and related to Interpretation 6.14 (4), we have the following result:

THEOREM 6.20. *Given a convex function $f : \mathbb{R} \to \mathbb{R}$, we have the following Jensen inequality, for any $x_1, \ldots, x_N \in \mathbb{R}$, and any $\lambda_1, \ldots, \lambda_N > 0$ summing up to 1,*

$$f(\lambda_1 x_1 + \ldots + \lambda_N x_N) \leq \lambda_1 f(x_1) + \ldots + \lambda_N x_N$$

with equality when $x_1 = \ldots = x_N$. In particular, by taking the weights λ_i to be all equal, we obtain the following Jensen inequality, valid for any $x_1, \ldots, x_N \in \mathbb{R}$,

$$f\left(\frac{x_1 + \ldots + x_N}{N}\right) \leq \frac{f(x_1) + \ldots + f(x_N)}{N}$$

and once again with equality when $x_1 = \ldots = x_N$. A similar statement holds for the concave functions, with all the inequalities being reversed.

PROOF. This is indeed something quite routine, the idea being as follows:

(1) First, we can talk about convex functions in a usual, intuitive way, with this meaning by definition that the following inequality must be satisfied:

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}$$

(2) But this means, via a simple argument, by approximating numbers $t\in[0,1]$ by sums of powers 2^{-k}, that for any $t\in[0,1]$ we must have:

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

Alternatively, via yet another simple argument, this time by doing some geometry with triangles, this means that we must have:

$$f\left(\frac{x_1+\ldots+x_N}{N}\right)\leq\frac{f(x_1)+\ldots+f(x_N)}{N}$$

But then, again alternatively, by combining the above two simple arguments, the following must happen, for any $\lambda_1,\ldots,\lambda_N>0$ summing up to 1:

$$f(\lambda_1x_1+\ldots+\lambda_Nx_N)\leq\lambda_1f(x_1)+\ldots+\lambda_Nx_N$$

(3) Summarizing, all our Jensen inequalities, at $N=2$ and at $N\in\mathbb{N}$ arbitrary, are equivalent. The point now is that, if we look at what the first Jensen inequality, that we took as definition for the convexity, exactly means, this is simply equivalent to:

$$f''(x)\geq 0$$

(4) Thus, we are led to the conclusions in the statement, regarding the convex functions. As for the concave functions, the proof here is similar. Alternatively, we can say that f is concave precisely when $-f$ is convex, and get the results from what we have. □

As a basic application of the Jensen inequality, which is very classical, we have:

THEOREM 6.21. *For any $p\in(1,\infty)$ we have the following inequality,*

$$\left|\frac{x_1+\ldots+x_N}{N}\right|^p\leq\frac{|x_1|^p+\ldots+|x_N|^p}{N}$$

and for any $p\in(0,1)$ we have the following inequality,

$$\left|\frac{x_1+\ldots+x_N}{N}\right|^p\geq\frac{|x_1|^p+\ldots+|x_N|^p}{N}$$

with in both cases equality precisely when $|x_1|=\ldots=|x_N|$.

PROOF. This follows indeed from Theorem 6.20, because we have:

$$(x^p)''=p(p-1)x^{p-2}$$

Thus x^p is convex for $p>1$ and concave for $p<1$, which gives the results. □

Observe that at $p = 2$ we obtain as particular case of the above inequality the Cauchy-Schwarz inequality, or rather something equivalent to it, namely:

$$\left(\frac{x_1 + \ldots + x_N}{N}\right)^2 \leq \frac{x_1^2 + \ldots + x_N^2}{N}$$

Finally, as yet another important application of the Jensen inequality, we have:

THEOREM 6.22. *We have the Young inequality,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

valid for any $a, b \geq 0$, and any exponents $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. We use the logarithm function, which is concave on $(0, \infty)$, due to:

$$(\log x)'' = \left(-\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Thus we can apply the Jensen inequality, and we obtain in this way:

$$\begin{aligned} \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) &\geq \frac{\log(a^p)}{p} + \frac{\log(b^q)}{q} \\ &= \log(a) + \log(b) \\ &= \log(ab) \end{aligned}$$

Now by exponentiating, we obtain the Young inequality. □

Observe that for the simplest exponents, namely $p = q = 2$, the Young inequality gives something which is trivial, but is very useful and basic, namely:

$$ab \leq \frac{a^2 + b^2}{2}$$

Moving forward now, as a consequence of the Young inequality, we have:

THEOREM 6.23 (Hölder). *Assuming that $p, q \geq 1$ are conjugate, in the sense that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

we have the following inequality, valid for any two vectors $x, y \in \mathbb{R}^N$,

$$\sum_i |x_i y_i| \leq \left(\sum_i |x_i|^p\right)^{1/p} \left(\sum_i |y_i|^q\right)^{1/q}$$

with the convention that an ∞ exponent produces a $\max |x_i|$ quantity.

PROOF. This is something very standard, the idea being as follows:

(1) Assume first that we are dealing with finite exponents, $p, q \in (1, \infty)$. By linearity we can assume that x, y are normalized, in the following way:

$$\sum_i |x_i|^p = \sum_i |y_i|^q = 1$$

By using now the Young inequality we obtain, as desired:

$$\sum_i |x_i y_i| \leq \sum_i \frac{|x_i|^p}{p} + \sum_i \frac{|y_i|^q}{q} = 1$$

(2) In the case $p = 1$ and $q = \infty$, or vice versa, the inequality holds too, trivially, with the convention that an ∞ exponent produces a max quantity, according to:

$$\lim_{p \to \infty} \left(\sum_i |x_i|^p \right)^{1/p} = \max |x_i|$$

Thus, we are led to the conclusion in the statement. □

As a consequence now of the Hölder inequality, we have:

THEOREM 6.24 (Minkowski). *Assuming $p \in [1, \infty]$, we have the inequality*

$$\left(\sum_i |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_i |x_i|^p \right)^{1/p} + \left(\sum_i |y_i|^p \right)^{1/p}$$

for any two vectors $x, y \in \mathbb{R}^N$, with our usual conventions at $p = \infty$.

PROOF. We have indeed the following estimate, using the Hölder inequality, and the conjugate exponent $q \in [1, \infty]$, given by $1/p + 1/q = 1$:

$$\begin{aligned}
\sum_i |x_i + y_i|^p &= \sum_i |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&\leq \sum_i |x_i| \cdot |x_i + y_i|^{p-1} + \sum_i |y_i| \cdot |x_i + y_i|^{p-1} \\
&\leq \left(\sum_i |x_i|^p \right)^{1/p} \left(\sum_i |x_i + y_i|^{(p-1)q} \right)^{1/q} \\
&+ \left(\sum_i |y_i|^p \right)^{1/p} \left(\sum_i |x_i + y_i|^{(p-1)q} \right)^{1/q} \\
&= \left[\left(\sum_i |x_i|^p \right)^{1/p} + \left(\sum_i |y_i|^p \right)^{1/p} \right] \left(\sum_i |x_i + y_i|^p \right)^{1-1/p}
\end{aligned}$$

Here we have used the following fact, at the end:

$$\frac{1}{p}+\frac{1}{q}=1\implies\frac{1}{q}=\frac{p-1}{p}\implies(p-1)q=p$$

Thus, we are led to the conclusion in the statement. □

Good news, done with inequalities, and as a consequence of the above results, and more specifically of the Minkowski inequality, that we just proved, we can formulate:

THEOREM 6.25. *Given an exponent $p\in[1,\infty]$, the formula*

$$||x||_p=\left(\sum_i|x_i|^p\right)^{1/p}$$

with usual conventions at $p=\infty$, defines a norm on $\mathbb{R}^N$.

PROOF. This follows indeed from the Minkowski inequality, which proves the triangle inequality for our norm, with the other two norm axioms being trivial, in our case. □

And with this discussed, we are now experts in functional analysis. And of course do not worry, all this knowledge is useful when doing arithmetic. More later.

6d. Taylor formula

Back now to the general theory of the derivatives, and their theoretical applications, we can further develop our basic approximation method, at order 3, at order 4, and so on. Let us start with something nice and intuitive, as follows:

FACT 6.26. *The third derivative f''' can be thought of as being the jerk of f.*

Here the terminology comes from real life and classical mechanics, where the jerk is by definition the derivative of the acceleration, and so is the second derivative of the speed, and so is the third derivative of the position, according to the following formulae:

$$j=\dot{a}=\ddot{v}=\dddot{x}$$

As before with second derivatives, many other things can be said. Let us also record the formulae of the third derivatives of the basic functions, which are as follows:

PROPOSITION 6.27. *The third derivatives of the basic functions are as follows:*

(1) $(x^p)'''=p(p-1)(p-2)x^{p-3}$.
(2) $\sin'''=-\cos$.
(3) $\cos'''=\sin$.
(4) $\exp'''=\exp$.
(5) $\log'''(x)=2/x^3$.

PROOF. The various formulae in the statement all follow from the various formulae for the second derivatives established before, as follows:

$$(x^p)''' = (p(p-1)x^{p-2})' = p(p-1)(p-2)x^{p-3}$$

$$(\sin x)''' = (-\sin x)' = -\cos x$$

$$(\cos x)''' = (-\cos x)' = \sin x$$

$$(e^x)''' = (e^x)' = e^x$$

$$(\log x)''' = (-1/x^2)' = 2/x^3$$

Thus, we are led to the formulae in the statement. □

Getting now to the fourth derivatives, things are less intuitive here, in what regards the interpretation, but we can nevertheless do some computations, as follows:

THEOREM 6.28. *The fourth derivatives of the basic functions are as follows:*

(1) $(x^p)'''' = p(p-1)(p-2)(p-3)x^{p-4}$.
(2) $\sin'''' = \sin$.
(3) $\cos'''' = \cos$.
(4) $\exp'''' = \exp$.
(5) $\log''''(x) = -6/x^4$.

PROOF. The various formulae in the statement all follow from the various formulae for the third derivatives established before, as follows:

$$(x^p)'''' = (p(p-1)(p-2)x^{p-3})' = p(p-1)(p-2)(p-3)x^{p-4}$$

$$(\sin x)'''' = (-\cos x)' = \sin x$$

$$(\cos x)'''' = (\sin x)' = \cos x$$

$$(e^x)'''' = (e^x)' = e^x$$

$$(\log x)'''' = (2/x^3)' = -6/x^4$$

Thus, we are led to the formulae in the statement. □

Observe the magic brought by the fourth derivative at the level of basic trigonometric functions. This is perhaps something worth recording, as follows:

OBSERVATION 6.29. *The fourth derivative brings some periodicity magic at the level of basic trigonometric functions.*

With this discussed, and getting back now to our usual approximation business, the ultimate result on the subject, called Taylor formula, is as follows:

THEOREM 6.30. *Any function $f : \mathbb{R} \to \mathbb{R}$ can be locally approximated as*

$$f(x+t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}\, t^k$$

where $f^{(k)}(x)$ are the higher derivatives of f at the point x.

PROOF. Consider the function to be approximated, namely:

$$\varphi(t) = f(x+t)$$

Let us try to best approximate this function at a given order $n \in \mathbb{N}$. We are therefore looking for a certain polynomial in t, of the following type:

$$P(t) = a_0 + a_1 t + \ldots + a_n t^n$$

The natural conditions to be imposed are those stating that P and φ should match at $t = 0$, at the level of the actual value, of the derivative, second derivative, and so on up the n-th derivative. Thus, we are led to the approximation in the statement:

$$f(x+t) \simeq \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}\, t^k$$

In order to prove now that this approximation holds indeed, we can use L'Hôpital's rule, applied several times, as in the proof of Theorem 6.17. To be more precise, if we denote by $\varphi(t) \simeq P(t)$ the approximation to be proved, we have:

$$\begin{aligned}
\frac{\varphi(t) - P(t)}{t^n} &\simeq \frac{\varphi'(t) - P'(t)}{nt^{n-1}} \\
&\simeq \frac{\varphi''(t) - P''(t)}{n(n-1)t^{n-2}} \\
&\vdots \\
&\simeq \frac{\varphi^{(n)}(t) - P^{(n)}(t)}{n!} \\
&= \frac{f^{(n)}(x) - f^{(n)}(x)}{n!} \\
&= 0
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Here is a related interesting statement, inspired from the above proof:

PROPOSITION 6.31. *For a polynomial of degree n, the Taylor approximation*

$$f(x+t) \simeq \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}\, t^k$$

is an equality. The converse of this statement holds too.

PROOF. By linearity, it is enough to check the equality in question for the monomials $f(x) = x^p$, with $p \leq n$. But here, the formula to be proved is as follows:

$$(x+t)^p \simeq \sum_{k=0}^{p} \frac{p(p-1)\ldots(p-k+1)}{k!}\, x^{p-k}t^k$$

We recognize the binomial formula, so our result holds indeed. As for the converse, this is clear, because the Taylor approximation is a polynomial of degree n. □

There are many other things that can be said about the Taylor formula, at the theoretical level, notably with a study of the remainder, when truncating this formula at a given order $n \in \mathbb{N}$. We will be back to this, later in this book.

As a first application of the Taylor formula, we can now improve the binomial formula, which was actually our main tool so far, in the following way:

THEOREM 6.32. *We have the following generalized binomial formula, with $p \in \mathbb{R}$,*

$$(x+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^{p-k}t^k$$

with the generalized binomial coefficients being given by the formula

$$\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}$$

valid for any $|t| < |x|$. With $p \in \mathbb{N}$, we recover the usual binomial formula.

PROOF. It is customary to divide everything by x, which is the same as assuming $x = 1$. The formula to be proved is then as follows, under the assumption $|t| < 1$:

$$(1+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} t^k$$

Let us discuss now the validity of this formula, depending on $p \in \mathbb{R}$:

(1) Case $p \in \mathbb{N}$. According to our definition of the generalized binomial coefficients, we have $\binom{p}{k} = 0$ for $k > p$, so the series is stationary, and the formula to be proved is:

$$(1+t)^p = \sum_{k=0}^{p} \binom{p}{k} t^k$$

But this is the usual binomial formula, which holds for any $t \in \mathbb{R}$.

(2) Case $p = -1$. Here we can use the following formula, valid for $|t| < 1$:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$$

But this is exactly our generalized binomial formula at $p = -1$, because:

$$\binom{-1}{k} = \frac{(-1)(-2)\ldots(-k)}{k!} = (-1)^k$$

(3) Case $p \in -\mathbb{N}$. This is a continuation of our study at $p = -1$, which will finish the study at $p \in \mathbb{Z}$. With $p = -m$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{-m}{k} &= \frac{(-m)(-m-1)\ldots(-m-k+1)}{k!}\\ &= (-1)^k\frac{m(m+1)\ldots(m+k-1)}{k!}\\ &= (-1)^k\frac{(m+k-1)!}{(m-1)!k!}\\ &= (-1)^k\binom{m+k-1}{m-1}\end{aligned}$$

Thus, our generalized binomial formula at $p = -m$ reads:

$$\frac{1}{(1+t)^m} = \sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{m-1}t^k$$

But this is something which holds indeed, as we know from chapter 2.

(4) General case, $p \in \mathbb{R}$. As we can see, things escalate quickly, so we will skip the next step, $p \in \mathbb{Q}$, and discuss directly the case $p \in \mathbb{R}$. Consider the following function:

$$f(x) = x^p$$

The derivatives at $x = 1$ are then given by the following formula:

$$f^{(k)}(1) = p(p-1)\ldots(p-k+1)$$

Thus, the Taylor approximation at $x = 1$ is as follows:

$$f(1+t) = \sum_{k=0}^{\infty}\frac{p(p-1)\ldots(p-k+1)}{k!}\,t^k$$

But this is exactly our generalized binomial formula, so we are done with the case where t is small. With a bit more care, we obtain that this holds for any $|t| < 1$, and we will leave this as an instructive exercise, and come back to it, later in this book. □

We can see from the above the power of the Taylor formula, saving us from quite complicated combinatorics. Remember indeed the mess from chapter 2, when trying to directly establish particular cases of the generalized binomial formula. Gone all that.

As a main application now of our generalized binomial formula, which is something very useful in practice, we can extract square roots, as follows:

THEOREM 6.33. *We have the following formula,*

$$\sqrt{1+t} = 1 - 2\sum_{k=1}^{\infty} C_{k-1}\left(\frac{-t}{4}\right)^k$$

with $C_k = \frac{1}{k+1}\binom{2k}{k}$ *being the Catalan numbers. Also, we have*

$$\frac{1}{\sqrt{1+t}} = \sum_{k=0}^{\infty} D_k\left(\frac{-t}{4}\right)^k$$

with $D_k = \binom{2k}{k}$ *being the central binomial coefficients.*

PROOF. This is something that we already know from chapter 2, but time now to review all this. At $p = 1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{1/2}{k} &= \frac{1/2(-1/2)\ldots(3/2-k)}{k!}\\ &= (-1)^{k-1}\frac{(2k-2)!}{2^{k-1}(k-1)!2^k k!}\\ &= -2\left(\frac{-1}{4}\right)^k C_{k-1}\end{aligned}$$

Also, at $p = -1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{-1/2}{k} &= \frac{-1/2(-3/2)\ldots(1/2-k)}{k!}\\ &= (-1)^k\frac{(2k)!}{2^k k!2^k k!}\\ &= \left(\frac{-1}{4}\right)^k D_k\end{aligned}$$

Thus, Theorem 6.32 at $p = \pm 1/2$ gives the formulae in the statement. □

As another basic application of the Taylor series, we have:

THEOREM 6.34. *We have the following formulae,*

$$\sin x = \sum_{l=0}^{\infty}(-1)^l\frac{x^{2l+1}}{(2l+1)!} \quad , \quad \cos x = \sum_{l=0}^{\infty}(-1)^l\frac{x^{2l}}{(2l)!}$$

as well as the following formulae,

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad , \quad \log(1+x) = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^k}{k}$$

as Taylor series, and in general as well, with $|x| < 1$ *needed for* log.

PROOF. There are several statements here, the proofs being as follows:

(1) Regarding sin and cos, we can use here the following formulae:

$$(\sin x)' = \cos x \quad , \quad (\cos x)' = -\sin x$$

Thus, we can differentiate sin and cos as many times as we want to, and compute the corresponding Taylor series, and we obtain the formulae in the statement.

(2) Regarding exp and log, here the needed formulae, which lead to the formulae in the statement for the corresponding Taylor series, are as follows:

$$(e^x)' = e^x$$

$$(\log x)' = x^{-1}$$

$$(x^p)' = px^{p-1}$$

(3) Finally, the fact that the formulae in the statement extend beyond the small t setting, coming from Taylor series, is something standard too. We will be back to this later in this book, once we will have more tools for dealing with such questions. □

6e. Exercises

Welcome to calculus, such a joy to have you here, and as exercises, we have:

EXERCISE 6.35. *Clarify all the details, in the proof of* $(x^p)' = px^{p-1}$.

EXERCISE 6.36. *Compute the derivatives of all trigonometric functions.*

EXERCISE 6.37. *Learn more about Rolle, Lagrange and the mean value property.*

EXERCISE 6.38. *Compute the second derivatives of all trigonometric functions.*

EXERCISE 6.39. *Further work on the various interpretations of the second derivative.*

EXERCISE 6.40. *Fill in all the details, for what we said on the convex functions.*

EXERCISE 6.41. *Learn the standard proof of the Cauchy-Schwarz inequality.*

EXERCISE 6.42. *Learn more about higher derivatives, and the Taylor formula.*

As bonus exercise, compute the Taylor series of all trigonometric functions. Enjoy.

CHAPTER 7

Prime numbers

7a. Euler formula

Good news, we can go back now to arithmetic, and more specifically to the prime numbers. We already know a bit about primes from chapter 1, and there are many other things can be said, both of algebraic and analytic nature. In this chapter we present a preliminary exploration of this, to be continued later, as this book further develops.

To start with, many things can be said about the primes, of analytic nature. At the beginning of everything here, we have the following famous formula, due to Euler:

THEOREM 7.1. *We have the following formula, with P being the set of primes,*

$$\sum_{p\in P}\frac{1}{p}=\infty$$

implying $|P|=\infty$.

PROOF. Here is the original proof, due to Euler:

(1) The idea is to use the factorization theorem for the integers, stating that we must have $n=p_1^{a_1}\ldots p_k^{a_k}$, but written upside down, as follows:

$$\frac{1}{n}=\frac{1}{p_1^{a_1}}\cdots\frac{1}{p_k^{a_k}}$$

Indeed, summing now over $n\geq 1$ gives the following beautiful formula:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}&=\prod_{p\in P}\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots\right)\\&=\prod_{p\in P}\left(1-\frac{1}{p}\right)^{-1}\end{aligned}$$

(2) In what concerns the sum on the left, we know from chapter 3 that this equals ∞. As for the product on the right, this can be estimated by using log, as follows:

$$\begin{aligned}
\log\left[\prod_{p\in P}\left(1-\frac{1}{p}\right)^{-1}\right] &= -\sum_{p\in P}\log\left(1-\frac{1}{p}\right)\\
&= \sum_{p\in P}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\\
&< \sum_{p\in P}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{2p^3}+\frac{1}{2p^4}+\ldots\\
&= \sum_{p\in P}\frac{1}{p}+\frac{1}{2}\sum_{p\in P}\frac{1}{p^2}\cdot\frac{1}{1-1/p}\\
&= \sum_{p\in P}\frac{1}{p}+\frac{1}{2}\sum_{p\in P}\frac{1}{p(p-1)}\\
&< \sum_{p\in P}\frac{1}{p}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\\
&= \sum_{p\in P}\frac{1}{p}+\frac{1}{2}
\end{aligned}$$

We therefore obtain the following estimate, which gives the first assertion:

$$\sum_{p\in P}\frac{1}{p}+\frac{1}{2}>\log\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)=\infty$$

(3) Regarding the second assertion, $|P|=\infty$, this is clear from what we have, because the sum $\sum_{p\in P}1/p$ being infinite, so must be the indexing set P. □

The Euler formula, making a link with calculus, is something quite interesting, so let us try now to improve it. But first, what does this formula teach us? In view of Theorem 7.1, this formula tells us that there are "many primes", with this coming from:

$$\sum_{p\in P}\frac{1}{p}>\infty$$

With this interpretation in mind, and looking for improvements, we are led to:

QUESTION 7.2. *How to find estimates as follows, for the partial Euler sums,*

$$\sum_{p<N}\frac{1}{p}>C_N$$

with $C_N\to\infty$ of course, with the sum being as usual over the primes?

Indeed, assuming that we have found a good estimate of this type, we can declare afterwards that "there are many primes $p < N$, with their number controlled by C_N", with this being a valuable theorem, regarding the intimate structure of the primes.

Which sounds quite good, so let us try now to solve the above question. Looking at the proof of Theorem 7.1, normally there should be no problem for truncating the computations there, but we will certainly get into a difficulty at the end, as follows:

QUESTION 7.3. *How to find estimates as follows, needed for improving Euler,*

$$\sum_{n<N}\frac{1}{n} > D_N$$

with $D_N \to \infty$ of course, chosen as sharp as possible?

In answer now, calculus you would say, but the problem is, browsing through the material from chapter 6, many interesting things there, but none helping with this. Well, mea cupla, this is because I forgot to tell you in chapter 6 about the fundamental theorem of calculus, we all sometimes forget fundamentals, right. Here that theorem is:

THEOREM 7.4. *Given a function $F : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_a^b F'(x)dx = F(b) - F(a)$$

for any interval $[a, b]$.

PROOF. This is something very standard, the idea being as follows:

(1) To start with, we can talk about the integral of a continuous function $f : [a, b] \to \mathbb{R}$, as being the signed area below its graph. By drawing rectangles, we have the following formula for the integral, which can stand as a formal definition for this integral:

$$\int_a^b f(x)dx = \lim_{N\to\infty}\sum_{k=1}^{N}\frac{b-a}{N}\cdot f\left(a+\frac{b-a}{N}\cdot k\right)$$

(2) As an illustration for this, by using the known formulae for $1^s + 2^s + \ldots + N^s$ at $s = 1, 2$, we can compute the integral of the function $f(x) = x^2$, as follows:

$$\begin{aligned}
\int_a^b x^2dx &= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{b-a}{N}\left(a+\frac{b-a}{N}\cdot k\right)^2\\
&= \lim_{N\to\infty}a^2(b-a)+a(b-a)^2\frac{N+1}{N}+(b-a)^3\frac{(N+1)(2N+1)}{6N^2}\\
&= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{3}\\
&= \frac{b^3-a^3}{3}
\end{aligned}$$

(3) As a first general result now, we can always find $c \in [a, b]$ such that:

$$\int_a^b f(x)dx = (b-a)f(c)$$

Indeed, by integrating $\min(f) \leq f \leq \max(f)$, we obtain the following estimate:

$$(b-a)\min(f) \leq \int_a^b f(x)dx \leq (b-a)\max(f)$$

Now since f must takes all values on $[\min(f), \max(f)]$, we obtain the result.

(4) Next, our claim is that with $F(x) = \int_a^x f(s)ds$, the following happens:

$$F' = f$$

In order to prove this, we must compute F'. Observe first that we have:

$$\frac{F(x+t) - F(x)}{t} = \frac{1}{t}\int_x^{x+t} f(x)dx$$

Now by using (3), we can find a number $c \in [x, x+t]$ such that:

$$\frac{1}{t}\int_x^{x+t} f(x)dx = f(c)$$

Thus, putting our formulae together, we conclude that we have:

$$\frac{F(x+t) - F(x)}{t} = f(c)$$

Now with $t \to 0$, no matter how the number $c \in [x, x+t]$ varies, one thing that we can be sure about is that we have $c \to x$. Thus, by continuity of f, we obtain:

$$\lim_{t \to 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

But this means exactly that we have $F' = f$, and we are done.

(5) We are now ready to prove our theorem. Consider the following function:

$$G(s) = \int_a^s F'(x)dx$$

By using (4) we have $G' = F'$, so our functions F, G differ by a constant. But with $s = a$ we have $G(a) = 0$, and so the constant is $F(a)$, and we get:

$$F(s) = G(s) + F(a)$$

Now with $s = b$ this gives $F(b) = G(b) + F(a)$, which reads:

$$F(b) = \int_a^b F'(x)dx + F(a)$$

Thus, we are led to the conclusion in the statement. □

As an illustration now for this, solving Question 7.3, we have:

PROPOSITION 7.5. *We have the following estimate,*

$$\sum_{n=1}^{N-1}\frac{1}{n} > \int_1^N \frac{1}{x}\,dx = \log N$$

coming from the fundamental theorem of calculus.

PROOF. In what regards the inequality at left, this comes from the definition of the integral of $1/x$, which is decreasing, by drawing rectangles, which in practice gives:

$$\begin{aligned}\int_1^N \frac{1}{x}\,dx &= \sum_{n=1}^{N-1}\int_n^{n+1}\frac{1}{x}\,dx\\ &< \sum_{n=1}^{N-1}\int_n^{n+1}\frac{1}{n}\,dx\\ &= \sum_{n=1}^{N-1}\frac{1}{n}\end{aligned}$$

As for the equality at right, this comes from Theorem 7.4 applied to the logarithm:

$$\begin{aligned}\int_1^N \frac{1}{x}\,dx &= \int_1^N (\log x)'dx\\ &= \log N - \log 1\\ &= \log N\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Good news, we can now solve Question 7.2, as follows:

THEOREM 7.6. *We have the following formula, with sum over primes,*

$$\sum_{p<N}\frac{1}{p} > \log\log N - \frac{1}{2}$$

for the partial Euler sums.

PROOF. This is something quite straightforward, as follows:

(1) By using $n = p_1^{a_1} \ldots p_k^{a_k}$ as before, and Proposition 7.5, we have:

$$\begin{aligned}\prod_{p<N}\left(1-\frac{1}{p}\right)^{-1} &= \prod_{p<N}\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots\right)\\ &> \sum_{n=1}^{N-1}\frac{1}{n}\\ &> \log N\end{aligned}$$

(2) But the product on the left can be estimated by using log, as follows:

$$\begin{aligned}\log\left[\prod_{p<N}\left(1-\frac{1}{p}\right)^{-1}\right] &= -\sum_{p<N}\log\left(1-\frac{1}{p}\right)\\ &= \sum_{p<N}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\\ &< \sum_{p<N}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{2p^3}+\frac{1}{2p^4}+\ldots\\ &= \sum_{p<N}\frac{1}{p}+\frac{1}{2}\sum_{p<N}\frac{1}{p^2}\cdot\frac{1}{1-1/p}\\ &= \sum_{p<N}\frac{1}{p}+\frac{1}{2}\sum_{p<N}\frac{1}{p(p-1)}\\ &< \sum_{p<N}\frac{1}{p}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\\ &= \sum_{p<N}\frac{1}{p}+\frac{1}{2}\end{aligned}$$

(3) Now by putting everything together, we obtain:

$$\sum_{p<N}\frac{1}{p}+\frac{1}{2} > \log\left[\prod_{p<N}\left(1-\frac{1}{p}\right)^{-1}\right] > \log\log N$$

Thus, we are led to the conclusion in the statement. □

7b. Further estimates

Ready for some more analysis? As explained after Question 7.2, further improving the Euler formula is a key problem, which is definitely worth more study. And with a bit of work, we can further improve what we have, in the following way:

THEOREM 7.7. *We have the following formula, with sum over primes,*

$$\sum_{p<N}\frac{1}{p}>\log\log N-\frac{1}{2}$$

and the $1/2$ *constant on the right can be improved to* $\log(\pi^2/6)=0.49770..$

PROOF. This is something a bit more tricky, as follows:

(1) The first assertion is something that we know, from Theorem 7.6, and a quick look at what we did in the proof there reveals four $>$ signs, which appear as follows:

$$\begin{aligned}\sum_{p<N}\frac{1}{p}+\frac{1}{2}&>\sum_{p<N}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{2p^3}+\frac{1}{2p^4}+\ldots\\&>\log\left[\prod_{p<N}\left(1-\frac{1}{p}\right)^{-1}\right]\\&>\log\left(\sum_{n=1}^{N-1}\frac{1}{n}\right)\\&>\log\log N\end{aligned}$$

In principle, we can improve all these four estimates, if we want to. However, we would like to use instead a different technique, which is quite instructive.

(2) The point indeed is that we have a rival method, based by using the factorization $n=p_1\ldots p_km^2$, with p_i distinct primes. This factorization gives:

$$\begin{aligned}\sum_{n=1}^{N-1}\frac{1}{n}&<\prod_{p<N}\left(1+\frac{1}{p}\right)\sum_{m=1}^{N}\frac{1}{m^2}\\&<\prod_{p<N}\exp\left(\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{1}{(m-1/2)(m+1/2)}\\&=\exp\left(\sum_{p<N}\frac{1}{p}\right)\sum_{m=1}^{\infty}\frac{1}{m-1/2}-\frac{1}{m+1/2}\\&=2\exp\left(\sum_{p<N}\frac{1}{p}\right)\end{aligned}$$

Now by taking the logarithm, this gives the following formula:

$$\sum_{p<N}\frac{1}{p}+\log 2>\log\left(\sum_{n=1}^{N-1}\frac{1}{n}\right)$$

We therefore obtain the following estimate, for our sum:

$$\sum_{p<N}\frac{1}{p}>\log\log N-\log 2$$

(3) However, $\log 2=0.69314..$ does not improve our $1/2$ constant, and we have to be more careful with our telescoping in (2). By separating the first term, we get closer:

$$\sum_{m=1}^{\infty}\frac{1}{m^2}<1+\frac{2}{3}=\frac{5}{3}\quad,\quad\log\left(\frac{5}{3}\right)=0.51082..$$

By separating the first two terms, we get even closer, but still not there:

$$\sum_{m=1}^{\infty}\frac{1}{m^2}<1+\frac{1}{4}+\frac{2}{5}=\frac{33}{20}\quad,\quad\log\left(\frac{33}{20}\right)=0.50077..$$

However, with the first three terms separated, what we get is a win:

$$\sum_{m=1}^{\infty}\frac{1}{m^2}<1+\frac{1}{4}+\frac{1}{9}+\frac{2}{7}=\frac{415}{252}\quad,\quad\log\left(\frac{415}{252}\right)=0.49884..$$

(4) In practice now, in order to finish this discussion, in a neat way, we can invoke the Basel formula, due to Euler, which is however something quite complicated:

$$\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}$$

To be more precise, assuming this, and putting everything together, which is probably worth it, we conclude that the factorization method $n=p_1\ldots p_km^2$ gives:

$$\begin{aligned}\exp\left(\sum_{p<N}\frac{1}{p}\right)\times\frac{\pi^2}{6}&>\prod_{p<N}\left(1+\frac{1}{p}\right)\times\frac{\pi^2}{6}\\&=\prod_{p<N}\left(1+\frac{1}{p}\right)\sum_{m=1}^{N}\frac{1}{m^2}\\&>\sum_{n=1}^{N-1}\frac{1}{n}\\&>\int_1^N\frac{1}{x}\,dx\\&=\log N\end{aligned}$$

Thus, by applying log, we are led to the conclusion in the statement. □

As already mentioned, the above estimates are to be later improved, by certain theorems of Mertens. Let us discuss however the formula that we used at the end:

THEOREM 7.8. *We have the following formula, due to Euler too,*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

answering the Basel problem, asking for the computation of this sum.

PROOF. This is something quite tricky, as follows:

(1) The original proof of Euler is as follows, making some clever manipulations on the Taylor series expansion of the function $\sin x/x$, based on the fact that the zeroes of this function appear precisely at the points $x=k\pi$, with $k\in\mathbb{Z}$:

$$\begin{aligned}\frac{\sin x}{x}&=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\\&=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\\&=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots\\&=1-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,x^2+\ldots\end{aligned}$$

(2) However, in practice, all this needs a bit more justification, which can be obtained by taking the logarithm, or passing to complex numbers, or to Fourier analysis, and getting the result from the Parseval formula. Exercise for you, to learn about this. □

As a conclusion now, quite exciting all this, and by the end of the day, we are left with more questions that those that we originally started with, as follows:

QUESTIONS 7.9. *Do we have a formula of the following type,*

$$\sum_{p<N}\frac{1}{p}\simeq\log\log N+M$$

and what the constant M is? Can we better understand the related function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

at $s=2$, and in general? And, what does all this tell us, concretely, about primes?

Which sounds quite good, as a plan. However, I don't know about you, but in what concerns me, I'm a bit worried, this book which was intended to be a clean-cut text is turning into some sort of mess, now that we are into analytic number theory. Probably a good idea to intercept the cat, see what she thinks about this. And cat declares:

CAT 7.10. *Advanced number theory needs a good knowledge of e and π. By the way, you should learn more calculus, and some complex analysis too.*

Okay, so take it easy, and as a plan for the remainder of this book, in the rest of the present Part II we will do some algebra, as a matter of having something concrete on the dinner table, about the primes, then in Part III we will learn about complex numbers, and about more calculus too, among others with some applications to e and π, and finally in Part IV we will go back, courageously, to analytic number theory.

7c. Groups and fields

Getting now to algebraic aspects, we have briefly seen in chapter 3 that the integers modulo N form a field when $N = p$ is prime, and form something weaker, called ring, otherwise. Our purpose here will be to talk in detail about this, rings, fields and other algebraic beasts, which are all intimately related to the prime numbers.

Let us first talk about groups. We have here the following definition:

DEFINITION 7.11. *A group is a set G endowed with a multiplication operation*

$$(g, h) \to gh$$

which must satisfy the following conditions:

(1) *Associativity: we have, $(gh)k = g(hk)$, for any $g, h, k \in G$.*
(2) *Unit: there is an element $1 \in G$ such that $g1 = 1g = g$, for any $g \in G$.*
(3) *Inverses: for any $g \in G$ there is $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.*

The multiplication law is not necessarily commutative. In the case where it is, in the sense that $gh = hg$, for any $g, h \in G$, we call G abelian, en hommage to Abel, and we usually denote its multiplication, unit and inverse operation as follows:

$$(g, h) \to g + h \quad , \quad 0 \in G \quad , \quad g \to -g$$

However, this is not a general rule, and rather the converse is true, in the sense that if a group is denoted as above, this means that the group must be abelian.

The simplest possible finite group is the cyclic group $\mathbb{Z}_N$, constructed as follows:

DEFINITION 7.12. *The cyclic group $\mathbb{Z}_N$ is defined as follows:*

(1) *As the additive group of remainders modulo N.*
(2) *As the multiplicative group of the N-th roots of unity.*

The two definitions are equivalent, because if we set $w = e^{2\pi i/N}$, then any remainder modulo N defines a N-th root of unity, according to the following formula:

$$k \to w^k$$

We obtain in this way all the N-roots of unity, and so our correspondence is bijective. Moreover, our correspondence transforms the sum of remainders modulo N into the multiplication of the N-th roots of unity, due to the following formula:

$$w^k w^l = w^{k+l}$$

Thus, the groups defined in (1,2) above are isomorphic, via $k \to w^k$, and we agree to denote by $\mathbb{Z}_N$ the corresponding group. Observe that this group $\mathbb{Z}_N$ is abelian. We will be back to the finite abelian groups later, on several occasions.

As a second basic example of a finite group, we have the symmetric group S_N. This is again something very familiar, appearing as follows:

DEFINITION 7.13. *A permutation of $\{1, \ldots, N\}$ is a bijection, as follows:*

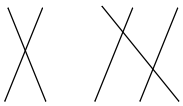
$$\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$$

The permutations form a group, denoted S_N.

There are many possible notations for the permutations, the basic one consisting in writing the numbers $1, \ldots, N$, and below them, their permuted versions:

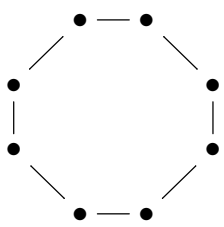
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

Another method, which is faster, and that I personally prefer, remember that time is money, is by denoting the permutations as diagrams, acting from top to bottom:

$\sigma =$ 

As a third interesting example now of a finite group, which is something more advanced, we have the dihedral group D_N, which appears as follows:

DEFINITION 7.14. *The dihedral group D_N is the symmetry group of*



that is, of the regular polygon having N vertices.

Many interesting things can be said about D_N. To start with, we have $D_2 = \mathbb{Z}_2$, and $D_3 = S_3$. In general, the dihedral group D_N has $2N$ elements, as follows:

(1) First we have N rotations $R_1, \ldots, R_N$, with R_k being the rotation of angle $2k\pi/N$. When labeling the vertices of the N-gon $1, \ldots, N$ we have $R_k : i \to k + i$.

(2) Then we have N symmetries $S_1, \ldots, S_N$, with S_k being the symmetry with respect to the Ox axis rotated by $k\pi/N$. The symmetry formula is $S_k : i \to k - i$.

Now let us see how these rotations and symmetries multiply. We have:

$$\begin{aligned} R_kR_l &: i \to l + i \to k + l + i \\ R_kS_l &: i \to l - i \to k + l - i \\ S_kR_l &: i \to l + i \to k - l - i \\ S_kS_l &: i \to l - i \to k - l + i \end{aligned}$$

We conclude that we can talk about D_N abstractly, as being the group having $2N$ elements, $R_1, \ldots, R_N$ and $S_1, \ldots, S_N$, which multiply as follows:

$$R_kR_l = R_{k+l} \quad , \quad R_kS_l = S_{k+l}$$

$$S_kR_l = S_{k-l} \quad , \quad S_kS_l = R_{k-l}$$

But this shows that D_N appears as some sort of "twisted version" of $\mathbb{Z}_N \times \mathbb{Z}_2$, and with a bit of algebraic know-how, we can even write a formula as follows:

$$D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$$

So long for the basic examples of finite groups. Getting now to some general theory, for the finite groups, we first have here the following result:

THEOREM 7.15. *Given a finite group G and a subgroup $H \subset G$, the sets*

$$G/H = \{gH \Big| g \in G\} \quad , \quad H\backslash G = \{Hg \Big| g \in G\}$$

consist of partitions of G into subsets of size H, and we have the following formula:

$$|G/H| = |H\backslash G| = \frac{|G|}{|H|}$$

In particular, the order of the subgroup divides the order of the group, $|H| \Big| |G|$.

PROOF. The partition claim for the set G/H constructed in the statement can be deduced as follows, and the proof for $H\backslash G$ is similar:

$$gH \cap kH \neq \emptyset \iff g^{-1}k \in H \iff gH = kH$$

But with this in hand, the cardinality formulae are all clear. □

As a continuation of the above, which is something fundamental, we have:

THEOREM 7.16. *Given a subgroup $H \subset G$ which is normal, in the sense that*

$$gH = Hg \quad , \quad \forall g \in G$$

the space $G/H = H\backslash G$ is a group, with multiplication $(gH)(sH) = gsH$.

PROOF. Assume indeed that $H \subset G$ is normal, and that g, k, s, t are such that:

$$gH = kH \quad , \quad sH = tH$$

We have then the following computation, by using the normality condition:

$$gsH = gtH = gHt = kHt = ktH$$

Thus $G/H = H\backslash G$ is a indeed group, with multiplication $(gH)(sH) = gsH$. □

As another continuation of Theorem 7.15, which is something fundamental too, we can talk about the order of group elements, inside any finite group, as follows:

THEOREM 7.17. *Given a finite group G, any $g \in G$ generates a cyclic subgroup*

$$< g >= \{1, g, g^2, \ldots, g^{k-1}\}$$

with $k = ord(g)$ being the smallest number $k \in \mathbb{N}$ satisfying $g^k = 1$. Also, we have

$$ord(g) \ \Big| \ |G|$$

that is, the order of any group element divides the order of the group.

PROOF. In order to prove the first assertion, let $g \in G$, and consider the semigroup $< g > \subset G$ formed by the sequence of powers of g:

$$< g >= \{1, g, g^2, g^3, \ldots\} \subset G$$

Since G was assumed to be finite, the sequence of powers must cycle, $g^n = g^m$ for some $n < m$. But this shows that $g^k = 1$, with $k = m - n$, which gives:

$$< g >= \{1, g, g^2, \ldots, g^{k-1}\}$$

Moreover, we can choose the number $k \in \mathbb{N}$ to be minimal with this property, and with this choice, we have a set without repetitions. Thus $< g > \subset G$ is indeed a group, and more specifically a cyclic group, whose order is as follows:

$$| < g > | = k = ord(g)$$

Thus, we proved the first assertion, and with this in hand, the second assertion, namely $ord(g) \,|\, |G|$, follows from Theorem 7.15, applied to the subgroup $< g > \subset G$. □

As a main result now about groups, dealing with the abelian case, we have:

THEOREM 7.18. *The finite abelian groups are the following groups:*

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

That is, the finite abelian groups are the products of cyclic groups.

PROOF. This is something quite tricky, the idea being as follows:

(1) In order to prove our result, assume that G is finite and abelian. For any prime number $p \in \mathbb{N}$, let us define $G_p \subset G$ to be the subset of elements having as order a power of p. Equivalently, this subset $G_p \subset G$ can be defined as follows:

$$G_p = \left\{ g \in G \middle| \exists k \in \mathbb{N}, g^{p^k} = 1 \right\}$$

(2) It is then routine to check, based on definitions, that each G_p is a subgroup. Our claim now is that we have a direct product decomposition as follows:

$$G = \prod_p G_p$$

(3) Indeed, by using the fact that our group G is abelian, we have a morphism as follows, with the order of the factors when computing $\prod_p g_p$ being irrelevant:

$$\prod_p G_p \to G \quad , \quad (g_p) \to \prod_p g_p$$

Moreover, it is routine to check that this morphism is both injective and surjective, via some simple manipulations, so we have our group decomposition, as in (2).

(4) Thus, we are left with proving that each component G_p decomposes as a product of cyclic groups, having as orders powers of p, as follows:

$$G_p = \mathbb{Z}_{p^{r_1}} \times \ldots \times \mathbb{Z}_{p^{r_s}}$$

But this is something that can be checked by recurrence on $|G_p|$, via some routine computations, and so we are led to the conclusion in the statement. □

Switching topics now, but still in relation with primes, we would like to talk about fields. Let us start with the following key theorem of Fermat, for the usual integers:

THEOREM 7.19. *We have the following congruence, for any prime p,*

$$a^p = a(p)$$

called Fermat's little theorem.

PROOF. The simplest way is to do this by recurrence on $a \in \mathbb{N}$, as follows:

$$\begin{aligned}(a+1)^p &= \sum_{k=0}^{p} \binom{p}{k} a^k \\ &= a^p + 1(p) \\ &= a + 1(p)\end{aligned}$$

Here we have used the fact that all non-trivial binomial coefficients $\binom{p}{k}$ are multiples of p, as shown by a close inspection of these binomial coeffients, given by:

$$\binom{p}{k}=\frac{p(p-1)\ldots(p-k+1)}{k!}$$

Thus, we have the result for any $a\in\mathbb{N}$, and with the case $p=2$ being trivial, we can assume $p\geq 3$, and here by using $a\to -a$ we get it for any $a\in\mathbb{Z}$, as desired. □

The Fermat theorem is particularly interesting when extended from the integers to the arbitrary field case. In order to discuss this question, let us start with:

THEOREM 7.20. *Given a field F, define its characteristic $p=char(F)$ as being the smallest $p\in\mathbb{N}$ such that the following happens, and as $p=0$, if this never happens:*

$$\underbrace{1+\ldots+1}_{p\ times}=0$$

Then, assuming $p>0$, this characteristic p must be a prime number, we have a field embedding $\mathbb{F}_p\subset F$, and $q=|F|$ must be of the form $q=p^k$, with $k\in\mathbb{N}$.

PROOF. Very crowded statement that we have here, the idea being as follows:

(1) The fact that $p>0$ must be prime comes by contradiction, by using:

$$(\underbrace{1+\ldots+1}_{a\ times})\times(\underbrace{1+\ldots+1}_{b\ times})=\underbrace{1+\ldots+1}_{ab\ times}$$

Indeed, assuming that we have $p=ab$ with $a,b>1$, the above formula corresponds to an equality of type $AB=0$ with $A,B\neq 0$ inside F, which is impossible.

(2) Back to the general case, F has a smallest subfield $E\subset F$, called prime field, consisting of the various sums $1+\ldots+1$, and their quotients. In the case $p=0$ we obviously have $E=\mathbb{Q}$. In the case $p>0$ now, the multiplication formula in (1) shows that the set $S=\{1+\ldots+1\}$ is stable under taking quotients, and so $E=S$.

(3) Now with $E=S$ in hand, we obviously have $(E,+)=\mathbb{Z}_p$, and since the multiplication is given by the formula in (1), we conclude that we have $E=\mathbb{F}_p$, as a field. Thus, in the case $p>0$, we have constructed an embedding $\mathbb{F}_p\subset F$, as claimed.

(4) In the context of the above embedding $\mathbb{F}_p\subset F$, we can say that F is a vector space over $\mathbb{F}_p$, and so we have $|F|=p^k$, with $k\in\mathbb{N}$ being the dimension of this space. □

In relation with Fermat, we can extend the trick in the proof there, as follows:

PROPOSITION 7.21. *In a field F of characteristic $p>0$ we have*

$$(a+b)^p=a^p+b^p$$

for any two elements $a,b\in F$.

PROOF. We have indeed the computation, exactly as in the proof of Fermat, by using the fact that the non-trivial binomial coefficients are all multiples of p:

$$(a+b)^p=\sum_{k=0}^{p}\binom{p}{k}a^kb^{p-k}=a^p+b^p$$

Thus, we are led to the conclusion in the statement. □

Observe that we can iterate the Fermat formula, and we obtain $(a+b)^r=a^r+b^r$ for any power $r=p^s$. In particular we have, with $q=|F|$, the following formula:

$$(a+b)^q=a^q+b^q$$

But this is something quite interesting, showing that the following subset of F, which is closed under multiplication, is closed under addition too, and so is a subfield:

$$E=\left\{a\in F\Big|a^q=a\right\}$$

So, what is this subfield $E\subset F$? In the lack of examples, or general theory for subfields $E\subset F$, we are a bit in the dark here, but it seems quite reasonable to conjecture that we have $E=F$. Thus, our conjecture would be that we have the following formula, for any $a\in F$, and with this being the field extension of the Fermat theorem itself:

$$a^q=a$$

Now that we have our conjecture, let us think at a potential proof. And here, by looking at the proof of the Fermat theorem, the recurrence method from there, based on $a\to a+1$, cannot work as such, and must be suitably fine-tuned.

Thinking a bit, the recurrence from the proof of Fermat somehow rests on the fact that the additive group $\mathbb{Z}$ is singly generated, by $1\in\mathbb{Z}$. Thus, we need some sort of field extension of this single generation result, and in the lack of something additive here, the following theorem, which is something multiplicative, comes to the rescue:

THEOREM 7.22. *Given a field F, any finite subgroup of its multiplicative group*

$$G\subset F-\{0\}$$

must be cyclic.

PROOF. This can be done via some standard arithmetic, as follows:

(1) Let us pick an element $g\in G$ of highest order, $n=ord(g)$. Our claim, which will easily prove the result, is that the order $m=ord(h)$ of any $h\in G$ satisfies $m|n$.

(2) In order to prove this claim, let $d=(m,n)$, write $d=am+bn$ with $a,b\in\mathbb{Z}$, and set $k=g^ah^b$. We have then the following computations:

$$k^m=g^{am}h^{bm}=g^{am}=g^{d-bn}=g^d$$
$$k^n=g^{an}h^{bn}=h^{bn}=h^{d-am}=h^d$$

By using either of these formulae, say the first one, we obtain:

$$k^{[m,n]} = k^{mn/d} = (k^m)^{n/d} = (g^d)^{n/d} = g^n = 1$$

Thus $ord(k)|[m,n]$, and our claim is that we have in fact $ord(k) = [m,n]$.

(3) In order to prove this latter claim, assume first that we are in the case $d = 1$. But here the result is clear, because the formulae in (2) read $g = k^m, h = g^n$, and since $n = ord(g), m = ord(g)$ are prime to each other, we conclude that we have $ord(k) = mn$, as desired. As for the general case, where d is arbitrary, this follows from this.

(4) Summarizing, we have proved our claim in (2). Now since the order $n = ord(g)$ was assumed to be maximal, we must have $[m,n]|n$, and so $m|n$. Thus, we have proved our claim in (1), namely that the order $m = ord(h)$ of any $h \in G$ satisfies $m|n$.

(5) But with this claim in hand, the result follows. Indeed, since the polynomial $x^n - 1$ has all the elements $h \in G$ as roots, its degree must satisfy $n \geq |G|$. On the other hand, from $n = ord(g)$ with $g \in G$, we have $n||G|$. We therefore conclude that we have $n = |G|$, which shows that G is indeed cyclic, generated by the element $g \in G$. □

We can now extend the Fermat theorem to the finite fields, as follows:

THEOREM 7.23. *Given a finite field F, with $q = |F|$ we have*

$$a^q = a$$

for any $a \in F$.

PROOF. According to Theorem 7.22 the multiplicative group $F - \{0\}$ is cyclic, of order $q - 1$. Thus, the following formula is satisfied, for any $a \in F - \{0\}$:

$$a^{q-1} = 1$$

Now by multiplying by a, we are led to the conclusion in the statement, with of course the remark that the formula there trivially holds for $a = 0$. □

The Fermat polynomial $X^p - X$ is something very useful, and its field generalization $X^q - X$, with $q = p^k$ prime power, can be used in order to elucidate the structure of finite fields. In order to discuss this question, let us start with a basic fact, as follows:

PROPOSITION 7.24. *Given a finite field F, we have*

$$X^q - X = \prod_{a \in F} (X - a)$$

with $q = |F|$.

PROOF. We know from the Fermat theorem above that we have $a^q = a$, for any $a \in F$. We conclude from this that all the elements $a \in F$ are roots of the polynomial $X^q - X$, and so this polynomial must factorize as in the statement. □

The continuation of the story is more complicated, as follows:

THEOREM 7.25. *For any prime power $q = p^k$ there is a unique field $\mathbb{F}_q$ having q elements. At $k = 1$ this is the usual $\mathbb{F}_p$, and in general, this is the field making*

$$X^q - X = \prod_{a \in F} (X - a)$$

happen, in some abstract algebraic sense.

PROOF. We are punching here a bit above our weight, the idea being as follows:

(1) At $k = 1$ there is nothing much to be said, because the prime field embedding $\mathbb{F}_p \subset F$ found in Theorem 7.20 must be an isomorphism. Thus, done with this.

(2) At $k \geq 2$ however, both the construction and uniqueness of $\mathbb{F}_q$ are non-trivial. However, the idea is not that complicated. Indeed, instead of struggling first with finding a model for $\mathbb{F}_q$, and then struggling some more with proving the uniqueness, the point is that we can solve both these problems, at the same time, by looking at $X^q - X$.

(3) To be more precise, this polynomial $X^q - X$ must have some sort of abstract, minimal "splitting field", and this is how $\mathbb{F}_q$ comes, both existence and uniqueness. We will be back to this, which is something non-trivial, later in this book, with details. □

7d. p-adic numbers

We would like to discuss now some arithmetic tricks, for dealing with equations over the rationals, and with the rational numbers themselves, based on the notion of p-adic number. The idea will be very simple, namely that of completing $\mathbb{Q}$ with respect to a different norm, which privileges the prime number p that we have chosen in advance.

Before that, some motivational talk. The dream in arithmetic, usually concerned with solving equations $f = 0$ over the rationals, is something very simple, namely:

DREAM 7.26. *I checked that my equation $f = 0$ has solutions modulo p, for any prime p, so my equation must have solutions over $\mathbb{Q}$.*

As a first observation, the dream holds when f is constant, $f = c$. Indeed, ignoring a bit the differences between integers and rationals, $c = 0(p)$ for any prime p means $c = 0$, so our equation is $c = 0$, having any rational number $x \in \mathbb{Q}$ as solution.

Along the same lines, there are some other examples of very simple equations $f = 0$ for which the dream holds. However, such equations are usually so simple, that we can solve them right away, and so our dream for them is not useful. In general, for more complicated equations, our dream remains wrong, and must be fine-tuned.

As a second piece of motivation, let us talk some analysis too. As explained before, everything in analytic number theory comes from the Euler formula, namely:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\prod_{p\in P}\left(1-\frac{1}{p}\right)^{-1}$$

But this is again something of "local-global" type, with on the left the global quantity, that is, a usual number, which actually happens to be ∞, in our case, and on the right the "local" versions of this number, with respect to the various primes p.

Summarizing, our dream is something important, both from the algebraic and analytic perspective, and is definitely worth a second look, with the aim of fixing it. We are led in this way to the following update to it, which is a bit more modest:

HOPE 7.27. *I checked that my equation $f=0$ has solutions with respect to any prime p, in a suitable sense, so my equation must have solutions over $\mathbb{Q}$.*

So, this will be our plan for what follows, doing some mathematics, as for this hope come true. We will see that this can indeed be done, with our vague wording above "with respect to any prime p, in a suitable sense" being replaced by something very precise and mathematical, namely "over the p-adics, for any prime p", and with the statement itself being a deep principle in number theory, called Hasse local-global principle.

Getting to work now, let us further reformulate our dreams and hopes, as follows:

QUESTION 7.28. *What are the p-adic numbers, defined with respect to a chosen prime number p, making the local-global principle work?*

In answer, let us temporarily forget about equations, and the local-global principle, and simply pick a prime number p, and look at the world from the perspective of p. So, imagining that we are p, both me and you, what we see is something as follows:

(1) First, we see all sorts of integers $a\in\mathbb{Z}$. Some appear friendly, namely those of the form $a\in p\mathbb{Z}$, while the others, of the form $a\notin p\mathbb{Z}$, appear bizarre and distant.

(2) Moreover, between friends $a\in p\mathbb{Z}$, those of the form $a\in p^2\mathbb{Z}$ appear particularly close. And among them, $a\in p^3\mathbb{Z}$ are truly very close friends. And so on.

(3) Then, we see all sorts of rationals, $r=a/b$, and again, some are close, some are distant, depending on the exact p^k factor, with $k\in\mathbb{Z}$, appearing inside r.

(4) In particular, the rationals of the form $r=1/p^k$ with $k>>0$ appear really frightening. Fortunately they are very far away from us, we can barely see them.

(5) And finally, we can see some irrationals $x\notin\mathbb{Q}$ too, but these being uncountable, it is quite hard to figure out how they look like, and are distributed in space.

Very good, so getting back to Earth now, let us write down a definition, based on what we saw in our Prime Number Experience. By focusing on the integers, and more generally the rationals, and leaving the irrationals for later, we have:

DEFINITION 7.29. *Given p prime, we define the p-adic norm of $r \in \mathbb{Q}$ as being:*

$$|r| = p^{-k} \quad , \quad r = p^k \frac{a}{b} \quad , \quad a, b \neq 0(p)$$

Also, we call the integer $k \in \mathbb{Z}$ the p-adic valuation of r, and denote it $k = v(r)$.

As a comment here, $|r| = p^{-k}$ is the natural choice, because according to our Prime Number Experience, the bigger $k \in \mathbb{Z}$ is, the smaller $|r| > 0$ must be, and so we are looking for a formula of type $|r| = \beta^{-k}$ with $\beta > 1$, as for this to happen. Of course, there is still a question left, in regards with the value of $\beta > 1$. But, again coming from our Prime Number Experience, if I am for instance $p = 11$, why shall I use $\beta = 17$.

Of course you might argue here that there might be some mighty universal number, such as $e = 2.7182\ldots$ or $\pi = 3.1415\ldots$ or $1/\alpha = 137.0359\ldots$ doing the job for all prime numbers p. But this cannot work, as we will see next, with some simple math.

Going ahead now with math, the question is, is our Definition 7.29 correct? That is, is $|r|$ indeed a norm? And here, is depends a bit on your background, with mathematicians being a bit dissapointed, to the point of even choosing to stop calling $|r|$ a norm, but physicists and others being fully happy with it, the result being as follows:

THEOREM 7.30. *The p-adic norm $|r| = p^{-k}$ is not exactly a norm, but satisfies the following conditions, which are even better:*

(1) *First axiom: $|x| \geq 0$, with $|x| = 0$ when $x = 0$.*
(2) *Modified second axiom: $|xy| = |x| \cdot |y|$.*
(3) *Strong triangle inequality: $|x + y| \leq \max(|x|, |y|)$.*

PROOF. All this follows indeed from some simple arithmetic modulo p:

(1) That axiom clearly holds, with the remark that we forgot to say in Definition 7.29 that $v(0) = \infty$, by definition, because any p^k, no matter how big $k \in \mathbb{N}$ is, divides 0.

(2) As a first observation, the usual second norm axiom, namely $|\lambda x| = ||\lambda|| \cdot |x|$, with $||.||$ standing here for the usual absolute value of the numbers, definitely fails, and this because all the p-adic norms $|r|$ are by definition integer powers of p, and an arbitrary $\lambda \in \mathbb{Q}$ will mess up this. However, we have instead $|xy| = |x| \cdot |y|$, coming from:

$$v(xy) = v(x)v(y)$$

And is this good news or not. After some thinking, this modified second axiom is just as good as the failed usual second axiom, because who cares about arbitrary numbers $\lambda \in \mathbb{Q}$, not viewed from the perspective of p, I mean. More on this in a moment.

(3) Finally, let us look at sums $x+y$. Over the integers $p^k|x,y$ implies $p^k|x+y$, and with a bit of fractions arithmetic, that we will leave here as an easy exercise, the same holds for rationals, in the sense that we have, in terms of the p-adic valuation:

$$v(x+y)\geq\min(v(x),v(y))$$

Thus the p-adic norm itself, $|r|=p^{-v(r)}$, satisfies the following inequality:

$$|x+y|\leq\max(|x|,|y|)$$

Now, what does this inequality mean, geometrically? Good question, and as a first remark, since this is obviously something stronger than the usual triangle inequality satisfied by the norms, $|x+y|\leq|x|+|y|$, we will call it strong triangle inequality. □

Before going ahead, let us further examine the strong triangle inequality found in the above. This is something new to us, and as a further result on it, we have:

PROPOSITION 7.31. *The strong triangle inequality implies*

$$|x|\neq|y|\implies|x+y|=\max(|x|,|y|)$$

and with this being valid for any modified norm, in the sense of Theorem 7.30.

PROOF. This is again something elementary, the idea being as follows:

(1) In what regards the p-adic norm, going back to (3) in the proof of Theorem 7.30, we can add there the observation that, trivially over the integers, and then over the rationals too, with a bit of fraction work, the p-adic valuation satisfies:

$$v(x)\neq v(y)\implies v(x+y)=\min(v(x),v(y))$$

Thus the p-adic norm itself satisfies the condition in the statement.

(2) More generally now, and with this being something quite interesting, our claim is that this phenomenon is valid for any generalized norm in the sense of Theorem 7.30. Indeed, assume that $|x|\geq 0$, with $|x|=0$ when $x=0$, as usual, and that:

$$|xy|=|x|\cdot|y|\quad,\quad|x+y|\leq\max(|x|,|y|)$$

In order to prove our result, assume $|x|>|y|$. We then have, trivially:

$$|x+y|\leq\max(|x|,|y|)=|x|$$

(3) In the other sense now, we have to work a bit. We have the following computation, with at the end the observation that the max cannot be $|y|$, because if that would be the case, the inequality that we would obtain would be $|x|\leq|y|$, contradicting $|x|>|y|$:

$$\begin{aligned}|x|&=|(x+y)-y|\\&\leq\max(|x+y|,|y|)\\&=|x+y|\end{aligned}$$

Thus, we have equality in the estimate in (2), as desired. □

Very nice all this, and getting back now to what we have in Theorem 7.30, namely the modified norm axioms there, we can formulate, as a simple consequence:

PROPOSITION 7.32. *The p-adic norm $|r| = p^{-k}$ is not exactly a norm, but*

$$d(x, y) = |x - y|$$

is a distance. Thus, the rationals $\mathbb{Q}$ become in this way a metric space.

PROOF. With the conditions satisfied by the p-norm $|r|$ in hand, it follows, trivially, that $d(x, y) = |x - y|$ is indeed a distance, making $\mathbb{Q}$ a metric space. □

Now let us turn to irrationals. The quite blurry picture that we saw during our Prime Number Experience, and with the blame at that time being on the uncountability of these beasts, in the lack of something better, can be now explained. Indeed, what we saw were not the "usual" irrationals $x \in \mathbb{R} - \mathbb{Q}$, but rather some irrationals $x \in \mathbb{Q}_p - \mathbb{Q}$ viewed from the perspective of p, constructed according to the following result:

THEOREM 7.33. *By completing $\mathbb{Q}$ with respect to the p-adic distance*

$$d(x, y) = |x - y|$$

we obtain a certain field $\mathbb{Q}_p$, called field of p-adic numbers.

PROOF. This is something very standard, with the passage $\mathbb{Q} \to \mathbb{Q}_p$ being very similar to the passage $\mathbb{Q} \to \mathbb{R}$, that we are very familiar with. In fact, some things get even simpler for p-adics, due to the strong triangle inequality satisfied by the norm. □

What is next? Many things, especially in relation with understanding what the p-adic irrationals $x \in \mathbb{Q}_p - \mathbb{Q}$ really are, concretely speaking. But before that, inspired by the theory of usual numbers, $\mathbb{Z} \subset \mathbb{Q}$, we can introduce the p-adic integers, as follows:

THEOREM 7.34. *We can introduce the p-adic integers $\mathbf{Z}_p \subset \mathbb{Q}_p$ as being*

$$\mathbf{Z}_p = \left\{ x \in \mathbb{Q}_p \,\middle|\, |x| \leq 1 \right\}$$

not to be confused with $\mathbb{Z}_p$, and this is a ring, appearing as completion of $\mathbb{Z} \subset \mathbf{Z}_p$.

PROOF. There are several things going on here, the idea being as follows:

(1) We can certainly introduce a set $\mathbf{Z}_p \subset \mathbb{Q}_p$ by the condition in the statement, and the ring axioms are all clear from the modified norm conditions, from Theorem 7.30, the verifications of the fact that $\mathbf{Z}_p$ is stable under sums and products being as follows:

$$|x|, |y| \leq 1 \implies |x + y| \leq \max(|x|, |y|) \leq 1$$

$$|x|, |y| \leq 1 \implies |xy| = |x| \cdot |y| \leq 1$$

(2) Next, since the valuation of a usual integer $x \in \mathbb{Z}$ satisfies $v(x) \geq 0$, the norm satisfies $|x| \leq 1$, and so we have an inclusion $\mathbb{Z} \subset \mathbf{Z}_p$, as in the statement.

(3) With a bit more work, we can see that $\mathbf{Z}_p$ is closed with respect to the p-adic norm, and also, that is appears as the completion of its subring $\mathbb{Z} \subset \mathbf{Z}_p$.

(4) Finally, and getting now into hot stories and other funny facts, the ring of p-adic integers $\mathbf{Z}_p$ is obviously not to be confused with the cyclic group $\mathbb{Z}_p$. There are actually two schools of thought here, with the other school denoting the p-adic integers by $\mathbb{Z}_p$, and using for the cyclic group all sorts of bizarre notations, such as C_p.

(5) In what regards our philosophy, that is very simple. If you need some sort of integers with respect to p, for your mathematics, this is a no-brainer, go with the remainders modulo p, or even better, with the p-th roots of unity, and that will solve your mathematical question, in 99% of the cases. And in the remaining 1% cases, what you need are probably the p-adic integers. So, assuming at least a little bit of modesty and common sense, the simplest notation, $\mathbb{Z}_p$, should be attributed to the cyclic group. □

With this understood, let us get now to the irrationals, and non-integers, and the p-adic numbers in general, viewed as a whole. Obviously, in order to understand them, we must understand well the Cauchy sequences and convergence in $\mathbb{Q}_p$. But here, many surprises are waiting for us, as for instance the following notorious formula:

THEOREM 7.35. *We have the following formula,*

$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$$

with respect to the p-adic norm.

PROOF. By using $p^n \to 0$, with respect to the p-adic norm, we have:

$$\begin{aligned}
\sum_{k=0}^{n-1} p^k &= \frac{1-p^n}{1-p} \\
&= \frac{1}{1-p} - \frac{p^n}{1-p} \\
&\simeq \frac{1}{1-p} - \frac{0}{1-p} \\
&= \frac{1}{1-p}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Quite cool the above formula, we are learning new things here, aren't we, and even more spectacular is its $p = 2$ particular case, which reads:

$$\sum_{k=0}^{\infty} 2^k = -1$$

As a matter of doublecheking, this latter formula can be proved as follows:

$$\begin{aligned}\sum_{k=0}^{n-1}2^k &= 2^n-1\\ &\simeq 0-1\\ &= -1\end{aligned}$$

Moving ahead, the continuation of the story is more complicated, involving some non-trivial algebra, with the main result here, called Hasse local-global principle, stating more or less that for certain questions in arithmetic, for instance in relation with the elliptic curves, our dreams and hopes expressed in the beginning of this section come true. For more on this, have a look at any advanced number theory book, such as Serre [79].

7e. Exercises

This was our first truly advanced arithmetic chapter, and as exercises, we have:

EXERCISE 7.36. *Learn more about the Euler formula and its story, and about zeta too.*

EXERCISE 7.37. *Further improve the Euler formula, even a little bit would be nice.*

EXERCISE 7.38. *How to write the dihedral group D_N as some sort of product $\mathbb{Z}_N\rtimes\mathbb{Z}_2$?*

EXERCISE 7.39. *Learn more about normal subgroups, and their various properties.*

EXERCISE 7.40. *Learn also about Sylow subgroups, which are related to primes too.*

EXERCISE 7.41. *Fill in the details, for the classification of finite abelian groups.*

EXERCISE 7.42. *Learn more about the classification and modeling of finite fields.*

EXERCISE 7.43. *Learn more about the p-adic numbers, and what they can do.*

As bonus exercise, read some systematic group theory. All good learning.

CHAPTER 8

Squares, residues

8a. Squares, residues

Let us go back now to what we did in Part I with congruences. Our aim here will be that of further building on some of the theorems there. To be more precise, we will be interested in solving the following ubiquitous equation, over the integers:

$$a^2 = b(c)$$

Many things can be said here, of various levels of difficulty. At the simplest level, we have the following result, which is something very useful, in practice:

THEOREM 8.1. *The squares modulo* 4 *can be of the form*

$$a^2 = 0, 1(4)$$

that is, with the cases $a^2 = 2, 3(4)$ *being excluded.*

PROOF. When our number to be squared is even, $a = 2k$, we have:

$$a^2 = (2k)^2 = 4k^2 = 0(4)$$

As for the case where our number is odd, $a = 2k + 1$, here we have:

$$a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 1(4)$$

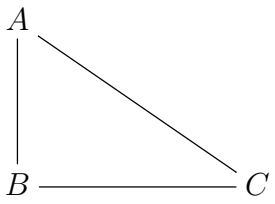
Thus, we are led to the conclusion in the statement. □

As a comment here, in the odd case, $a = 2k + 1$, since $k(k + 1)$ is even, we obtain in fact $a^2 = 1(8)$. Thus, we have in fact $a^2 = 0, 1, 4(8)$, improving what Theorem 8.1 says. However, for most applications, Theorem 8.1 as stated will be just fine.

The above statement is quite interesting, and it is quite clear that this is just the tip of the iceberg, with the equation $a^2 = b(c)$, with $c = p^k$ being a prime power, being something of interest, waiting to be systematically investigated. We will be back to this question, with several answers to it, later in this chapter.

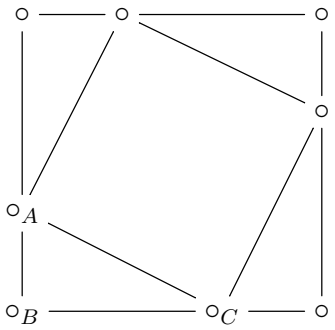
Before that, however, let us discuss some older mathematics in relation with the squares. We have indeed the following key result, due to Pythagoras:

THEOREM 8.2 (Pythagoras). *In a right triangle ABC,*



we have $AB^2 + BC^2 = AC^2$.

PROOF. This is something that we already know from before, but always good to talk about this, again. Consider indeed from the following picture, consisting of two squares, and four triangles which are identical to our triangle ABC, as indicated:



Now let us compute the area S of the outer square. This can be done in two ways. First, since the side of this square is $AB + BC$, we obtain:

$$\begin{aligned} S &= (AB + BC)^2 \\ &= AB^2 + BC^2 + 2 \times AB \times BC \end{aligned}$$

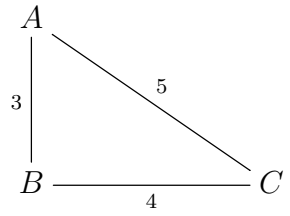
On the other hand, the outer square is made of the smaller square, having side AC, and of four identical right triangles, having sizes AB, BC. Thus:

$$\begin{aligned} S &= AC^2 + 4 \times \frac{AB \times BC}{2} \\ &= AC^2 + 2 \times AB \times BC \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a basic application of the Pythagoras theorem, which is something widely useful in practice, and this since the ancient times, we have:

THEOREM 8.3. *A triangle having sides* $3, 4, 5$ *is a right triangle:*



Thus, for drawing right angles, you only need a loop, with 12 *knots on it.*

PROOF. Here the first assertion comes from the following equality, and from the obvious converse of the Pythagoras theorem, and up to you to check the details here:

$$16 + 9 = 25$$

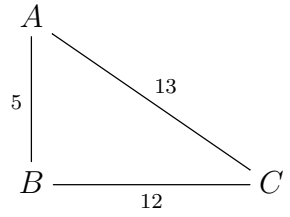
As for the second assertion, what does that exactly mean, and how can that be used in practice, we will leave this as an interesting engineering exercise. □

Still speaking engineering, having 12 knots equally spaced on a loop is certainly possible, and reliable for most tasks, but if we want to improve our tool, it would be desirable to have more knots on our loop. So, here we are, looking for integer solutions of:

$$a^2 + b^2 = c^2$$

Which is not exactly obvious, but with a bit of patience, we are led to:

THEOREM 8.4. *A triangle having sides* $5, 12, 13$ *is a right triangle:*



Thus, for drawing right angles, you only need a loop, with 30 *knots on it.*

PROOF. Here the first assertion comes from the following equality, and with the comment that this is the simplest possible one, passed $16 + 9 = 25$:

$$144 + 25 = 169$$

As for the second assertion, we will leave this again as an engineering exercise. As a bonus exercise, try further improving this, say with a solution using 90 knots. □

Along the same lines, at a more advanced level, we have the following result, which fully closes the discussion, regarding the Pythagoras equation over the integers:

THEOREM 8.5. *The Pythagoras equation, namely*

$$a^2+b^2=c^2$$

can be fully solved over the integers, the solutions being

$$a=d(m^2-n^2)\quad,\quad b=2dmn\quad,\quad c=d(m^2+n^2)$$

with $(m,n)=1$*, up to exchanging* a,b*.*

PROOF. This is something standard, due to Euclid, the idea being as follows:

(1) Let us try to solve $a^2+b^2=c^2$. If we divide a,b,c by their greatest common divisor $d=(a,b,c)$, the equation is still satisfied. Thus, we can assume $(a,b,c)=1$, and we want to prove that the solutions are as follows, up to exchanging a,b:

$$a=m^2-n^2\quad,\quad b=2mn\quad,\quad c=m^2+n^2$$

(2) To start with, in one sense our result is clear, because given any two numbers m,n, the above formulae produce a solution to our equation, as shown by:

$$\begin{aligned}(m^2-n^2)^2+(2mn)^2&=m^4+n^4-2m^2n^2+4m^2n^2\\&=m^4+n^4+2m^2n^2\\&=(m^2+n^2)^2\end{aligned}$$

(3) So, we must prove now the converse, stating that if a,b,c satisfying $(a,b,c)=1$ are solutions of $a^2+b^2=c^2$, then we can write them as in (1). For this purpose, the first observation is that, due to $a^2+b^2=c^2$, our assumption $(a,b,c)=1$ implies:

$$(a,b)=(a,c)=(b,c)=1$$

(4) Let us study now the parity of a,b,c. Since $(a,b)=1$, one of these two numbers, say a, is odd. Now assuming that b is odd too, we would get $a^2+b^2=2(4)$, which is impossible, due to $a^2+b^2=c^2$. Thus b must be even, and as a conclusion to this study, up to exchanging a,b, we can assume that the parity of our numbers is as follows:

$$a=\text{odd}\quad,\quad b=\text{even}\quad,\quad c=\text{odd}$$

(5) Now comes the trick. We can rewrite our equation in the following way:

$$\begin{aligned}a^2+b^2=c^2&\iff b^2=c^2-a^2\\&\iff b^2-(c-a)(c+a)\\&\iff\frac{c+a}{b}=\frac{b}{c-a}\end{aligned}$$

(6) With this done, let us look at the fraction on the left. This is a rational number, so we can write it in reduced form, as follows, with $(m,n)=1$:

$$\frac{c+a}{b}=\frac{m}{n}$$

Now observe that our equation, as reformulated in (5), takes the following form:

$$\frac{c+a}{b}=\frac{m}{n}\quad,\quad\frac{c-a}{b}=\frac{n}{m}$$

Equivalently, our equation, as reformulated in (5), takes the following form:

$$\frac{c}{b}+\frac{a}{b}=\frac{m}{n}\quad,\quad\frac{c}{b}-\frac{a}{b}=\frac{n}{m}$$

But this latter system is equivalent to the following two formulae:

$$\frac{a}{b}=\frac{1}{2}\left(\frac{m}{n}-\frac{m}{n}\right)=\frac{m^2-n^2}{2mn}$$

$$\frac{c}{b}=\frac{1}{2}\left(\frac{m}{n}+\frac{m}{n}\right)=\frac{m^2+n^2}{2mn}$$

(7) Good work that we did, and time to breathe, and see what we have. We have proved so far that if a,b,c satisfying $(a,b,c)=1$ are solutions of $a^2+b^2=c^2$, then up to exchanging a,b, we can find numbers m,n satisfying $(m,n)=1$, such that:

$$\frac{a}{b}=\frac{m^2-n^2}{2mn}\quad,\quad\frac{c}{b}=\frac{m^2+n^2}{2mn}$$

Which sounds nice, because due to $(a,b)=(b,c)=1$, as noted in (3), the two fractions on the left are in reduced form. So, if we manage to prove that the two fractions on the right are in reduced form too, this would finish the proof, because we would get:

$$a=m^2-n^2\quad,\quad b=2mn\quad,\quad c=m^2+n^2$$

(8) So, let us look now at the two fractions on the right, appearing above. As a first observation, due to $(m,n)=1$, the following two fractions are in reduced form:

$$\frac{m^2-n^2}{mn}\quad,\quad\frac{m^2+n^2}{mn}$$

The problem, however, is that the fractions in (7) are the halves of these quantities. So, all we need is a study modulo 2, and with this, normally done.

(9) Getting now to the endgame, from $(m,n)=1$, the case where both m,n are even is excluded. But the case where both m,n are odd is excluded too, due to:

$$\frac{a}{b}=\frac{m^2-n^2}{2mn}$$

Indeed, if m,n were both to be odd, we would have $m^2-n^2=0(4)$ and $2mn=2(4)$, so the fraction on the right, when reduced, would have an even denominator. But this would tell us that b must be even, which contradicts our b odd choice from (4).

(10) Summarizing, one of the numbers m,n must be even, and the other must be odd. But this does the job, because it shows that m^2-n^2 and m^2+n^2 are both odd, so when

dividing the reduced fractions from (7) by 2, these fractions remain still reduced. Thus, as a conclusion to our study, the following two fractions are reduced:

$$\frac{m^2-n^2}{2mn} \quad , \quad \frac{m^2+n^2}{2mn}$$

(11) So, theorem proved. Indeed, as indicated in (7), let us look now at:

$$\frac{a}{b}=\frac{m^2-n^2}{2mn} \quad , \quad \frac{c}{b}=\frac{m^2+n^2}{2mn}$$

Since all fractions appearing here are in reduced form, we obtain from this:

$$a=m^2-n^2 \quad , \quad b=2mn \quad , \quad c=m^2+n^2$$

And finally, as indicated in (1), by multiplying a,b,c by an arbitrary number d, we obtain the general solutions from the statement, namely:

$$a=d(m^2-n^2) \quad , \quad b=2dmn \quad , \quad c=d(m^2+n^2)$$

(12) At the level of the interesting examples now, there are of course many of them, and we have for instance a solution as follows:

$$40^2+9^2=1681=41^2$$

Observe that $40+9+41=90$, so as good news here, we have solved as well a quite difficult exercise left, namely the one at the end of the proof of Theorem 8.4. □

Many other things can be said, as a continuation of the above, notably with the general Fermat equation, which is as follows, involving an arbitary exponent $n\in\mathbb{N}$:

$$a^n+b^n=c^n$$

And we will leave some reading about this as an exercise, but with the comment however that this is quite difficult, and in relation for instance with the more advanced algebraic number theory, barely discussed at the end of the previous chapter.

Getting back now to congruences, we have seen that the proof of Theorem 8.5 uses Theorem 8.1, and with this providing us with some further motivation for congruences. So, time to study more in detail the equations of type $a^2=b(c)$, with $c=p^k$ prime power. At $p=2$ we have the following result, coming as a continuation of Theorem 8.1:

THEOREM 8.6. *The quadratic residues modulo 2^k are as follows:*

(1) *At $k=1$ there is no restriction, these are $0,1(2)$.*
(2) *At $k=2$ these are $0,1(4)$.*
(3) *At $k=3$ these are $0,1,4(8)$.*
(4) *At $k=4$ these are $0,1,4,9(16)$.*
(5) *At $k=5$ these are $0,1,4,9,16,17,25(32)$.*

PROOF. This is very elementary mathematics, as follows:

(1) Nothing to be done at $k = 1$, with both cases $0, 1(2)$ being obviously possible.

(2) At $k = 2$, this is something that we already know, from Theorem 8.1.

(3) At $k = 3$ now, the possible numbers modulo 8 are as follows:

$$a = 0, \pm 1, \pm 2, \pm 3, \pm 4$$

The squares of these numbers, taken modulo 8, are then as follows:

$$a^2 = 0, 1, 4, 1, 0$$

Thus, we are led to the conclusion in the statement, $a^2 = 0, 1, 4(8)$.

(4) At $k = 4$ now, the possible numbers modulo 16 are as follows:

$$a = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, 8$$

The squares of these numbers, taken modulo 16, are then as follows:

$$a^2 = 0, 1, 4, 9, 0, 9, 4, 1, 0$$

Thus, we are led to the conclusion in the statement, $a^2 = 0, 1, 4, 9(16)$.

(5) At $k = 5$ now, the possible numbers modulo 32 are as follows:

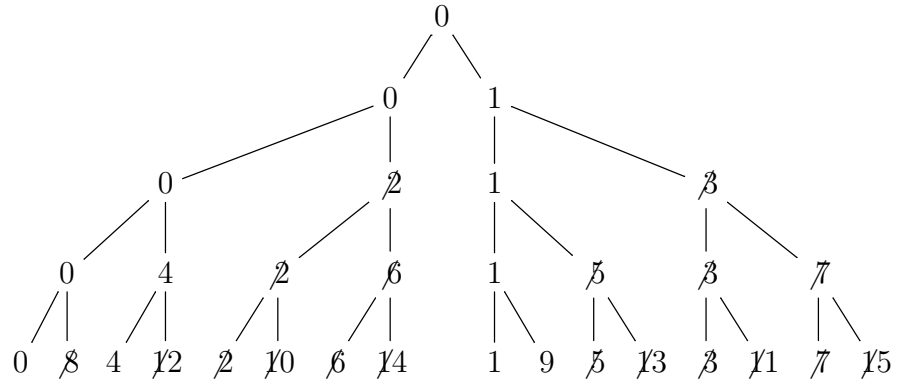
$$a = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12, \pm 13, \pm 14, \pm 15, 16$$

The squares of these numbers, taken modulo 32, are then as follows:

$$a^2 = 0, 1, 4, 9, 16, 25, 4, 17, 0, 17, 4, 25, 16, 9, 4, 1, 0$$

Thus, we are led to the conclusion in the statement, $a^2 = 0, 1, 4, 9, 16, 17, 25(32)$. □

The above result is certainly good to know as such, but at the level of the mathematics, it is quite unclear what happens there. One way of interpreting the results is by drawing a tree, as follows, with the quadratic residues which are forbidden being barred:



Let us see as well what happens at $p = 3$. Here the result is as follows:

THEOREM 8.7. *The quadratic residues modulo 3^k are as follows:*

(1) *At $k = 1$ these are $0, 1(3)$.*
(2) *At $k = 2$ these are $0, 1, 4, 7(9)$.*
(3) *At $k = 3$ these are $0, 1, 4, 7, 9, 10, 13, 16, 19, 22, 25(27)$.*

PROOF. This is again very elementary mathematics, as follows:

(1) At $k = 1$ this comes from the following computation:

$$(3n \pm 1)^2 = 1(3)$$

(2) At $k = 2$ now, the possible numbers modulo 9 are as follows:

$$a = 0, \pm 1, \pm 2, \pm 3, \pm 4$$

The squares of these numbers, taken modulo 9, are then as follows:

$$a^2 = 0, 1, 4, 0, 7$$

Thus, we are led to the conclusion in the statement, $a^2 = 0, 1, 4, 7(9)$.

(3) At $k = 3$ now, the possible numbers modulo 27 are as follows:

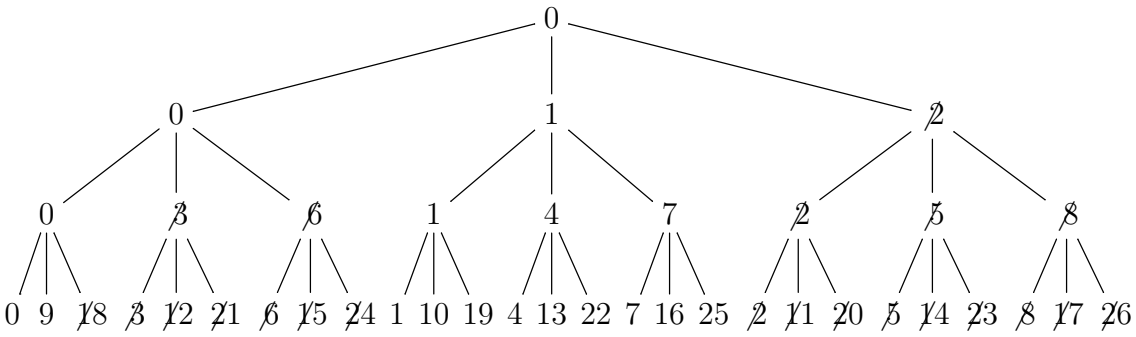
$$a = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12, \pm 13$$

The squares of these numbers, taken modulo 27, are then as follows:

$$a^2 = 0, 1, 4, 9, 16, 25, 9, 22, 10, 0, 19, 13, 9, 7$$

Thus, we are led to $a^2 = 0, 1, 4, 7, 9, 10, 13, 16, 19, 22, 25(27)$, as stated. □

As before with $p = 2$, rhe above result is certainly good to know, but at the level of the mathematics, it is quite unclear what happens there. We can interpret the results by drawing a tree, as follows, with the forbidden quadratic residues being barred:



Let us see as well what happens at $p = 5$. Here the result is as follows:

THEOREM 8.8. *The quadratic residues modulo 5^k are as follows:*

(1) *At $k = 1$ these are $0, 1, 4(5)$.*
(2) *At $k = 2$ these are $0, 1, 4, 9, 11, 14, 16, 19, 21, 24(25)$.*

PROOF. This is again very elementary mathematics, as follows:

(1) At $k = 1$, the possible numbers modulo 5 are as follows:

$$a = 0, \pm 1, \pm 2$$

The squares of these numbers, taken modulo 5, are then as follows:

$$a^2 = 0, 1, 4$$

Thus, we are led to the conclusion in the statement, $a^2 = 0, 1, 4(5)$.

(2) At $k = 2$ now, the possible numbers modulo 25 are as follows:

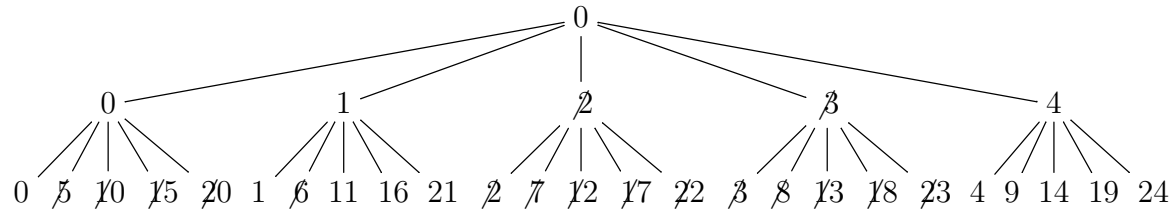
$$a = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11, \pm 12$$

The squares of these numbers, taken modulo 25, are then as follows:

$$a^2 = 0, 1, 4, 9, 16, 0, 11, 24, 14, 6, 0, 21, 19$$

Thus, we are led to $a^2 = 0, 1, 4, 9, 11, 14, 16, 19, 21, 24(25)$, as stated. □

As before with $p = 2, 3$, at the level of the mathematics, it is quite unclear what is going on. We can interpret the above results by drawing a tree, as follows:



And we will stop our elementary study here, all this being obviously a bit too amateurish. We will be back in a moment to such things, which more advanced tools.

8b. Legendre symbol

Getting back now to what we wanted to do in this chapter, understand $a = b^2(c)$, we have the following definition, putting everything on a solid basis:

DEFINITION 8.9. *The Legendre symbol is defined as follows,*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } \exists\, b \neq 0, a = b^2(p) \\ 0 & \text{if } a = 0(p) \\ -1 & \text{if } \nexists\, b, a = b^2(p) \end{cases}$$

with $p \geq 3$ prime.

Now leaving aside all sorts of nice and amateurish things that can be said about $a = b^2(c)$, and going straight to the point, what we want to do is to compute this symbol. I mean, if we manage to have this symbol computed, that would be a big win.

As a first result on the subject, due to Euler, we have:

THEOREM 8.10. *The Legendre symbol is given by the formula*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}}(p)$$

called Euler formula for the Legendre symbol.

PROOF. This is something not that complicated, the idea being as follows:

(1) We know from the Fermat theorem that we have $a^p = a(p)$, and leaving aside the case $a = 0(p)$, which is trivial, and therefore solved, this tells us that we have:

$$a^{p-1} = 1(p)$$

Now since our prime number p was assumed to be odd, $p \geq 3$, we can write this formula by factorizing, as follows:

$$\left(a^{\frac{p-1}{2}} - 1\right)\left(a^{\frac{p-1}{2}} + 1\right) = 0(p)$$

(2) Now let us think a bit at the elements of $\mathbb{F}_p - \{0\}$, which can be a quadratic residue, and which cannot. Since the squares b^2 with $b \neq 0$ are invariant under $b \to -b$, and give different b^2 values modulo p, up to this symmetry, we conclude that there are exactly $(p-1)/2$ quadratic residues, and with the remaining $(p-1)/2$ elements of $\mathbb{F}_p - \{0\}$ being non-quadratic residues. So, as a conclusion, $\mathbb{F}_p - \{0\}$ splits as follows:

$$\mathbb{F}_p - \{0\} = \left\{\frac{p-1}{2}\ \ squares\right\} \bigsqcup \left\{\frac{p-1}{2}\ \ non-squares\right\}$$

(3) Now by comparing what we have in (1) and in (2), the splits there must correspond to each other, so we are led to the following formula, valid for any $a \in \mathbb{F}_p - \{0\}$:

$$a^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } \exists\, b, a = b^2 \\ -1 & \text{if } \nexists\, b, a = b^2 \end{cases}$$

By comparing now with Definition 8.9, we obtain the formula in the statement. □

As a first consequence of the Euler formula, we have the following result:

PROPOSITION 8.11. *We have the following formula, valid for any $a, b \in \mathbb{Z}$:*

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$$

That is, the Legendre symbol is multiplicative in its upper variable.

PROOF. This follows from what we have, the idea being as follows:

(1) To start with, this is clear indeed indeed from the Euler formula from Theorem 8.10, because the quantity $a^{\frac{p-1}{2}}(p)$ is obviously multiplicative in $a \in \mathbb{Z}$.

(2) Alternatively, this can be proved as well directly, with no need for the Fermat formula used in the proof of Euler, just by thinking at what is quadratic residue and what is not in $\mathbb{F}_p$, along the lines of (2) in the proof of Theorem 8.10. □

The above result looks quite conceptual, and as consequences, we have:

PROPOSITION 8.12. *We have the following formula, telling us that modulo any prime number p, a product of non-squares is a square:*

$$\left(\frac{a}{p}\right) = -1 \ , \ \left(\frac{b}{p}\right) = -1 \implies \left(\frac{ab}{p}\right) = 1$$

Also, the Legendre symbol, regarded as a function

$$\chi : \mathbb{F}_p - \{0\} \to \{-1, 1\} \quad , \quad \chi(a) = \left(\frac{a}{p}\right)$$

is a character, in the sense that it is multiplicative.

PROOF. The first asssertion is a consequence of Proposition 8.11, more or less equivalent to it, and with the remark that this formally holds at $p = 2$ too, as $\emptyset \implies \emptyset$. As for the second assertion, this is just a fancy reformulation of Proposition 8.11. □

Question now, how to compute the Legendre symbol? There are many things to be known here, and all must be known, for efficient application, to the real life. We have opted to present them all in this chapter, of course with full proofs, but with some of these proofs being a bit formal and mysterious, in need of more study, that we will leave for later. As a first and main result, which is something heavy, we have:

THEOREM 8.13. *We have the quadratic reciprocity formula*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

valid for any primes $p, q \geq 3$.

PROOF. This is something quite tricky, one proof being as follows:

(1) First we have a combinatorial formula for the Legendre symbol, called Gauss lemma. Given a prime number $q \geq 3$, and $a \neq 0(q)$, consider the following sequence:

$$a \ , \ 2a \ , \ 3a \ , \ \ldots \ , \ \frac{q-1}{2}\, a$$

The Gauss lemma tells us that if we look at these numbers modulo q, and denote by n the number of residues modulo q which are greater than $q/2$, then:

$$\left(\frac{a}{q}\right) = (-1)^n$$

(2) In order to prove this lemma, the idea is to look at the following product:

$$Z = a \times 2a \times 3a \times \ldots \times \frac{q-1}{2}\, a$$

Indeed, on one hand we have the following formula, with Euler used at the end:

$$Z = a^{\frac{q-1}{2}} \left(\frac{q-1}{2}\right)! = \left(\frac{a}{q}\right) \left(\frac{q-1}{2}\right)!$$

(3) On the other hand, we can compute Z in more complicated way, but leading to a simpler answer. Indeed, let us define the following function:

$$|x| = \begin{cases} x & \text{if } 0 < x < q/2 \\ q - x & \text{if } q/2 < x < q \end{cases}$$

With this convention, our product Z is given by the following formula, with n being as in (1), namely the number of residues modulo q which are greater than $q/2$:

$$Z = (-1)^n \times |a| \times |2a| \times |3a| \times \ldots \times \left|\frac{q-1}{2}\, a\right|$$

(4) But, the numbers $|ra|$ appearing in the above formula are all distinct, so up to a permutation, these must be exactly the numbers $1, 2, \ldots, \frac{q-1}{2}$. That is, we have:

$$\left\{|a|\,,\ |2a|\,,\ |3a|\,,\ \ldots\,,\ \left|\frac{q-1}{2}\, a\right|\right\} = \left\{1, 2, 3, \ldots, \frac{q-1}{2}\right\}$$

Now by multiplying all these numbers, we obtain, via the formula in (3):

$$Z = (-1)^n \left(\frac{q-1}{2}\right)!$$

(5) But this is what we need, because when comparing with what we have in (2), we obtain the following formula, which is exactly the one claimed by the Gauss lemma:

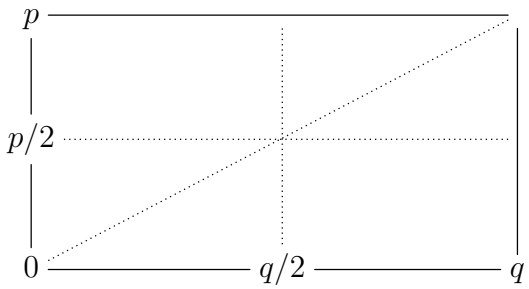
$$\left(\frac{a}{q}\right) = (-1)^n$$

(6) Next, we have a variation of this formula, due to Eisenstein. His formula for the Legendre symbol, this time involving a prime number numerator $p \geq 3$ in the symbol, is

as follows, with the quantities on the right being integer parts, and with the proof being very similar to the proof of the Gauss lemma, that we will leave here as an exercise:

$$\left(\frac{p}{q}\right) = (-1)^n \quad , \quad n = \sum_{k=0}^{(q-1)/2} \left[\frac{2kp}{q}\right]$$

(7) The key point now is that, in this latter formula of Eisenstein, the number n itself counts the points of the lattice $\mathbb{Z}^2$ lying in the triangle $(0,0),(q,0),(q,p)$. So, based on this observation, let us draw a picture, as follows:



(8) We must count the points of $\mathbb{Z}^2$ lying in the triangle $(0,0),(q,0),(q,p)$, modulo 2. This triangle has 3 components, when split by the dotted lines above. Since the points at right, in the small rectangle, and in the small triangle above it, will cancel modulo 2, we are left with the points at left, in the small triangle there, and the conclusion is that, if we denote by m the number of integer points there, we have the following formula:

$$\left(\frac{p}{q}\right) = (-1)^m$$

(9) Now by flipping the diagram, we have as well the following formula, with r being the number of integer points in the small triangle above the small triangle in (8):

$$\left(\frac{q}{p}\right) = (-1)^r$$

(10) But, since our two small triangles add up to a small rectangle, we have:

$$m + r = \frac{p-1}{2} \cdot \frac{q-1}{2}$$

Thus, by multiplying the formulae in (8) and (9), we are led to the result. □

As a comment now, the above result is extremely powerful, here being an illustration, computing the seemingly uncomputable number on the left in a matter of seconds:

$$\begin{aligned}\left(\frac{3}{173}\right) &= (-1)^{\frac{3-1}{2}\cdot\frac{173-1}{2}}\left(\frac{173}{3}\right)\\ &= \left(\frac{173}{3}\right)\\ &= \left(\frac{2}{3}\right)\\ &= -1\end{aligned}$$

In fact, when combining Theorem 8.13 with Proposition 8.11, it is quite clear that, no matter how big p is, if a has only small prime factors, we are saved.

8c. Further results

Besides Proposition 8.11, the quadratic reciprocity formula comes accompanied by two other statements, which are very useful in practice. First, at $a=-1$, we have:

PROPOSITION 8.14. *We have the following formula,*

$$\left(\frac{-1}{p}\right)=\begin{cases}1 & \text{if } p=1(4)\\ -1 & \text{if } p=3(4)\end{cases}$$

solving in practice the equation $b^2=-1(p)$.

PROOF. This follows from the Euler formula, which at $a=-1$ reads:

$$\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}(p)$$

Thus, we are led to the formula in the statement. □

As a second useful result, this time at $a=2$, we have:

THEOREM 8.15. *We have the following formula,*

$$\left(\frac{2}{p}\right)=\begin{cases}1 & \text{if } p=1,7(8)\\ -1 & \text{if } p=3,5(8)\end{cases}$$

solving in practice the equation $b^2=2(p)$.

PROOF. This is something quite tricky, the idea being as follows:

(1) As a first observation, the Euler formula at $a=2$ is as follows, obviously well below the quality of the very precise formula in the statement:

$$\left(\frac{2}{p}\right)=2^{\frac{p-1}{2}}(p)$$

As a second observation, the quadratic reciprocity formula, assuming that known, cannot help either, because in that formula $p, q \geq 3$ are odd primes.

(2) Thus, we must improvise, and prove the result. The proof will come via the following formula, which is equivalent to the formula in the statement:

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

Let us also mention that, despite 2 being an even prime, the problematics here is a bit similar to the one of quadratic reciprocity, and the proof below will contain many good ideas, that we will use later, in an alternative proof of quadratic reciprocity.

(3) Getting started now, let us introduce a formal number i satisfying $i^2 = -1$, then a formal number w satisfying $w^2 = i$, and then let us set $t = w + w^{-1}$:

$$i^2 = -1 \quad , \quad w^2 = i \quad , \quad t = w + w^{-1}$$

Quite crazy all this, you would say, and in answer, no worries, we will get to know more, about where all this comes from, as this book further develops.

(4) As a second remark that you might have, in case you know a bit about complex numbers, you might say that the above is not crazy, but rather stupid, because $t = \sqrt{2}$. In answer, yes I know, but it is better to forget this, and do formal arithmetic instead, with integers as scalars, based on our rules above, and the following computation:

$$\begin{aligned} t^2 &= 2 + w^2 + w^{-2} \\ &= 2 + i - i \\ &= 2 \end{aligned}$$

(5) Now by using the Euler formula for the Legendre symbol, we have:

$$\begin{aligned} \left(\frac{2}{p}\right) &= 2^{\frac{p-1}{2}} \ (p) \\ &= (t^2)^{\frac{p-1}{2}} \ (p) \\ &= t^{p-1} \ (p) \end{aligned}$$

(6) By multiplying now by t we obtain from this, in a formal sense, and I will leave it you to clarify all the details here, namely what this formal sense exactly means:

$$\left(\frac{2}{p}\right) t = t^p \ (p)$$

(7) On the other hand, by using the binomial formula, and the standard fact that all non-trivial binomial coefficients are multiples of p, we obtain, again formally:

$$\begin{aligned} t^p &= (w+w^{-1})^p \\ &= \sum_{k=0}^{p}\binom{k}{p}w^kw^{k-p} \\ &= w^p+w^{-p}\ (p) \end{aligned}$$

(8) Now let us look at the quantity w^p+w^{-p}. Since we have $w^4=-1$, this quantity will depend only on p modulo 8, and more precisely, we have:

$$w^p+w^{-p}=\begin{cases} w+w^{-1} & \text{if } p=\pm1(8)\\ -w-w^{-1} & \text{if } p=\pm3(8)\end{cases}$$

Thus $w^p+w^{-p}=\pm t$, with the sign depending on p modulo 8, and more specifically:

$$w^p+w^{-p}=(-1)^{\frac{p^2-1}{8}}t$$

(9) Time now to put everything together. By combining (6,7,8) we obtain:

$$\left(\frac{2}{p}\right)t=(-1)^{\frac{p^2-1}{8}}t\ (p)$$

By dividing by t, this gives the following formula:

$$\left(\frac{2}{p}\right)=(-1)^{\frac{p^2-1}{8}}\ (p)$$

But the mod p symbol can now be dropped, because our equality is between two ±1 quantities, and we obtain the formula in the statement. □

As already mentioned, the above result was something quite tricky, and we will be back to this later in this book, with some further explanations on all this.

As a continuation of this, speaking Legendre symbol for small values of the upper variable, we can try to compute these for $a=\pm\ 3,4,5,6,7,8,\ldots$ But by multiplicativity plus Proposition 8.14 plus Theorem 8.15 we are left with the case where $a=q$ is an odd prime, and we can solve the problem with quadratic reciprocity, so done.

Let us record however a few statements here, which can be useful in practice, and with this being mostly for illustration purposes, for Theorem 8.13. We first have:

PROPOSITION 8.16. *We have the following formula,*

$$\left(\frac{3}{p}\right)=\begin{cases}1 & \text{if } p=1,11(12)\\ -1 & \text{if } p=5,7(12)\end{cases}$$

valid for any prime $p\geq5$.

PROOF. By quadratic reciprocity, we have the following formula:

$$\left(\frac{3}{p}\right) = (-1)^{\frac{3-1}{2}\cdot\frac{p-1}{2}}\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)$$

Now since the sign depends on p modulo 4, and the symbol on the right depends on p modulo 3, we conclude that our symbol depends on p modulo 12, and the computation gives the formula in the statement. Finally, we have the following formula too:

$$\left(\frac{3}{p}\right) = (-1)^{\left[\frac{p+1}{6}\right]}$$

Indeed, the quantity on the right is something which depends on p modulo 12, and is in fact the simplest functional implementation of the formula in the statement. □

Along the same lines, we have as well the following result:

PROPOSITION 8.17. *We have the following formula,*

$$\left(\frac{5}{p}\right) = \begin{cases} 1 & \text{if } p = 1, 4(5) \\ -1 & \text{if } p = 2, 3(5) \end{cases}$$

valid for any odd prime $p \neq 5$.

PROOF. By quadratic reciprocity, we have the following formula:

$$\left(\frac{5}{p}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{p-1}{2}}\left(\frac{p}{5}\right) = \left(\frac{p}{5}\right)$$

Thus, we have the result. Alternatively, we have the following formula:

$$\left(\frac{5}{p}\right) = (-1)^{\left[\frac{2p+2}{5}\right]}$$

Indeed, this is the simplest implementation of the formula in the statement. □

Moving ahead now, we have the following interesting generalization of the Legendre symbol, to the case of denominators not necessarily prime, due to Jacobi:

THEOREM 8.18. *The theory of Legendre symbols can be extended by multiplicativity into a theory of Jacobi symbols, according to the formula*

$$\left(\frac{a}{p_1^{s_1}\ldots p_k^{s_k}}\right) = \left(\frac{a}{p_1}\right)^{s_1}\ldots\left(\frac{a}{p_k}\right)^{s_k}$$

with the denominator being not necessarily prime, but just an arbitrary odd number, and this theory has as results those imported from the Legendre theory.

PROOF. This is something self-explanatory, and we will leave listing the basic properties of the Jacobi symbols, based on the theory of Legendre symbols, as an exercise. □

The story is not over with Jacobi, because the denominator there is still odd, and positive. So, we have a problem to be solved, the solution to it being as follows:

THEOREM 8.19. *The theory of Jacobi symbols can be further extended into a theory of Kronecker symbols, according to the formula*

$$\left(\frac{a}{\pm p_1^{s_1}\ldots p_k^{s_k}}\right)=\left(\frac{a}{\pm 1}\right)\left(\frac{a}{p_1}\right)^{s_1}\ldots\left(\frac{a}{p_k}\right)^{s_k}$$

with the denominator being an arbitrary integer, via suitable values for

$$\left(\frac{a}{2}\right)\quad,\quad\left(\frac{a}{-1}\right)\quad,\quad\left(\frac{a}{0}\right)$$

and this theory has as results those imported from the Jacobi theory.

PROOF. Unlike the extension from Legendre to Jacobi, which was something straightforward, here we have some work to be done, in order to figure out the correct values of the 3 symbols in the statement. The answer for the first symbol is as follows:

$$\left(\frac{a}{2}\right)=\begin{cases}1 & \text{if } a=\pm 1(8)\\ 0 & \text{if } a=0(2)\\ -1 & \text{if } a=\pm 3(8)\end{cases}$$

The answer for the second symbol is as follows:

$$\left(\frac{a}{-1}\right)=\begin{cases}1 & \text{if } a\geq 0\\ -1 & \text{if } a<0\end{cases}$$

As for the answer for the third symbol, this is as follows:

$$\left(\frac{a}{0}\right)=\begin{cases}1 & \text{if } a=\pm 1\\ 0 & \text{if } a\neq\pm 1\end{cases}$$

And we will leave this as an instructive exercise, to figure out what the puzzle exactly is, and why these are the correct answers. And for an even better exercise, cover with a cloth the present proof, and try to figure out everything by yourself. □

As a further plot to the story, the theory of Kronecker symbols can be further generalized into a theory of Hilbert symbols. But then, you guessed it right, Hilbert was not the last one in the series, which contains some other illustrious mathematicians.

So bad these times are over. More recently, Prince made an attempt to join the series, with a very interesting symbol, which was however not accepted by the community.

8d. Some applications

Time now for some applications. Some very interesting objects in linear algebra are the Hadamard matrices, which are defined in a very simple way, as follows:

DEFINITION 8.20. *An Hadamard matrix is a square binary matrix*

$$H \in M_N(\pm 1)$$

whose rows are pairwise orthogonal.

Observe that the orthogonality condition states that, when comparing any two rows, the number of matches must equal the numbers of mismatches. As a basic example now, at $N = 2$ we have the following matrix, called first Walsh matrix:

$$W_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This matrix is quite trivial, of size 2×2, but by taking tensor powers of it, we have as examples the higher Walsh matrices as well, having size $2^k \times 2^k$, given by:

$$W_{2^k} = W_2^{\otimes k}$$

As an example here, the second Walsh matrix, obtained by tensoring W_2 with itself, is as follows, with respect to the lexicographic order on the double indices:

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Which raises the question, what happens then in arbitrary size $N \times N$? Do we still have there Hadamard matrices, that we can use for various applied linear algebra purposes?

In order to discuss this question, as a first remark, it is clear that we must have $2|N$. Along the same lines, it is easy to see, by playing around with the first rows, that once your matrix has $N \geq 3$ rows, we must have $4|N$, the precise result being as follows:

PROPOSITION 8.21. *The size of an Hadamard matrix $H \in M_N(\pm 1)$ must satisfy*

$$N \in \{2\} \cup 4\mathbb{N}$$

with this coming from the orthogonality condition between the first 3 rows.

PROOF. By permuting the rows and columns or by multiplying them by -1, as to rearrange the first 3 rows, we can always assume that our matrix looks as follows:

$$H=\begin{pmatrix}\underbrace{1\ldots\ldots1}_{x} & \underbrace{1\ldots\ldots1}_{y} & \underbrace{1\ldots\ldots1}_{z} & \underbrace{1\ldots\ldots1}_{t}\\ 1\ldots\ldots1 & 1\ldots\ldots1 & -1\ldots-1 & -1\ldots-1\\ 1\ldots\ldots1 & -1\ldots-1 & 1\ldots\ldots1 & -1\ldots-1\\ \vdots\ldots\vdots & \vdots\ldots\vdots & \vdots\ldots\vdots & \vdots\ldots\vdots\end{pmatrix}$$

Now if we denote by x,y,z,t the sizes of the block columns, as indicated, the orthogonality conditions between the first 3 rows give the following system of equations:

$$(1\perp 2)\quad:\quad x+y=z+t$$

$$(1\perp 3)\quad:\quad x+z=y+t$$

$$(2\perp 3)\quad:\quad x+t=y+z$$

The numbers x,y,z,t being such that the average of any two equals the average of the other two, and so equals the global average, the solution of our system is:

$$x=y=z=t$$

Thus the matrix size $N=x+y+z+t$ must be a multiple of 4, as claimed. □

The above result is something quite interesting, and a similar analysis with 4 rows or more does not give any further restriction on the possible values of the size $N\in\mathbb{N}$.

In fact, we are led in this way to the following famous conjecture:

CONJECTURE 8.22 (Hadamard). *There is an Hadamard matrix of order N,*

$$H\in M_N(\pm1)$$

for any $N\in4\mathbb{N}$.

Normally this is an analytic question, because in practice the number of Hadamard matrices grows exponentially with N, and so in order to prove the conjecture, you just need a modest lower estimate on this number. But, no one knows how to do this, and this despite the Hadamard conjecture being open for more than 100 years.

This being said, what we can do with our number theory methods is to verify at least the Hadamard conjecture at small values of $N\in4\mathbb{N}$. And here, with $N=4,8$ being solved by the Walsh matrices, we are faced with constructing a matrix at $N=12$.

In order to solve this question, let $q = p^k$ be an odd prime power, and consider the quadratic character of $\mathbb{F}_q$, given by the following formula:

$$\chi(a) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a = b^2, b \neq 0 \\ -1 & \text{otherwise} \end{cases}$$

Next, consider as well the following matrix, with indices in $\mathbb{F}_q$:

$$Q_{ab} = \chi(b - a)$$

With these conventions, the Paley construction of Hadamard matrices, which works well at $N = 12$, and at many other values of N, is as follows:

THEOREM 8.23. *Given an odd prime power $q = p^k$, construct $Q_{ab} = \chi(b-a)$ as above. We have then constructions of Hadamard matrices, as follows:*

(1) *Paley 1: if $q = 3(4)$ we have a matrix of size $N = q + 1$, as follows:*

$$P_N^1 = 1 + \begin{pmatrix} 0 & 1 & \ldots & 1 \\ -1 & & & \\ \vdots & & Q & \\ -1 & & & \end{pmatrix}$$

(2) *Paley 2: if $q = 1(4)$ we have a matrix of size $N = 2q + 2$, as follows:*

$$P_N^2 = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & & & \\ \vdots & & Q & \\ 1 & & & \end{pmatrix} \quad : \quad 0 \to \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad , \quad \pm 1 \to \pm \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

These matrices are skew-symmetric ($H + H^t = 2$), respectively symmetric ($H = H^t$).

PROOF. In order to simplify the presentation, we denote by 1 all the identity matrices, of any size, and by $\mathbb{I}$ all the rectangular all-one matrices, of any size as well. It is elementary to check that the matrix $Q_{ab} = \chi(a - b)$ has the following properties:

$$QQ^t = q1 - \mathbb{I} \quad , \quad Q\mathbb{I} = \mathbb{I}Q = 0$$

In addition, we have the following formulae, which are elementary as well, coming from the fact that -1 is a square in $\mathbb{F}_q$ precisely when $q = 1(4)$:

$$q = 1(4) \implies Q = Q^t \quad , \quad q = 3(4) \implies Q = -Q^t$$

With these observations in hand, the proof goes as follows:

(1) With our above conventions for 1 and $\mathbb{I}$, the matrix in the statement is:

$$P_N^1 = \begin{pmatrix} 1 & \mathbb{I} \\ -\mathbb{I} & 1 + Q \end{pmatrix}$$

With this formula in hand, the Hadamard matrix condition follows from:

$$\begin{aligned} P_N^1(P_N^1)^t &= \begin{pmatrix} 1 & \mathbb{I} \\ -\mathbb{I} & 1+Q \end{pmatrix} \begin{pmatrix} 1 & -\mathbb{I} \\ \mathbb{I} & 1-Q \end{pmatrix} \\ &= \begin{pmatrix} N & 0 \\ 0 & \mathbb{I}+1-Q^2 \end{pmatrix} \\ &= \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \end{aligned}$$

(2) If we denote by G, F the 2×2 matrices in the statement, which replace respectively the $0, 1$ entries, then we have the following formula for our matrix:

$$P_N^2 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix} \otimes F + 1 \otimes G$$

With this formula in hand, the Hadamard matrix condition follows from:

$$\begin{aligned} (P_N^2)^2 &= \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix}^2 \otimes F^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes G^2 + \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix} \otimes (FG+GF) \\ &= \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \otimes 2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes 2 + \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & Q \end{pmatrix} \otimes 0 \\ &= \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \end{aligned}$$

Finally, the last assertion is clear, from the above formulae relating Q, Q^t. □

In practice, with Walsh and Paley, the next problem is at $N = 92$. But here, we have:

THEOREM 8.24. *Assuming that $A, B, C, D \in M_K(\pm 1)$ are circulant, symmetric, pairwise commute and satisfy the condition*

$$A^2 + B^2 + C^2 + D^2 = 4K$$

the following $4K \times 4K$ matrix is Hadamard, called of Williamson type:

$$H = \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}$$

Moreover, matrices A, B, C, D as above exist at $K = 23$, where $4K = 92$.

PROOF. Consider the standard quaternion units $1, i, j, k \in M_4(0, 1)$, which describe the positions of the A, B, C, D entries in the matrix H from the statement:

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the above matrix H can be written as follows, in terms of $1, i, j, k$:

$$H = A \otimes 1 + B \otimes i + C \otimes j + D \otimes k$$

Assuming now that A, B, C, D are symmetric, we have:

$$\begin{aligned} HH^t &= (A \otimes 1 + B \otimes i + C \otimes j + D \otimes k) \\ &\quad (A \otimes 1 - B \otimes i - C \otimes j - D \otimes k) \\ &= (A^2 + B^2 + C^2 + D^2) \otimes 1 - ([A, B] - [C, D]) \otimes i \\ &\quad -([A, C] - [B, D]) \otimes j - ([A, D] - [B, C]) \otimes k \end{aligned}$$

Now assume that our matrices A, B, C, D pairwise commute, and satisfy the condition in the statement. In this case, it follows from the above formula that we have:

$$HH^t = 4K$$

Thus, we obtain indeed an Hadamard matrix, as claimed. However, finding such matrices is in general a difficult task, and this is where Williamson's extra assumption in the statement, that A, B, C, D should be taken circulant, comes from. Finally, regarding the $K = 23$ and $N = 92$ example, this comes via a computer search. □

At higher N things become more technical, and more complicated constructions, along the lines of those of Paley and Williamson, are needed. Quite curiously, as of now, early 21th century, the human knowledge stops at the number of the beast, namely:

$$\mathfrak{N} = 666$$

That is, explicit examples of Hadamard matrices have been constructed for all multiples of four $N \leq 664$, but no such matrix is known so far at $N = 668$.

8e. Exercises

This was another quite advanced arithmetic chapter, and as exercises here, we have:

EXERCISE 8.25. *Work out the numerics for the solutions of the Pythagoras equation.*

EXERCISE 8.26. *Compute more quadratic residues with respect to $q = p^k$ numbers.*

EXERCISE 8.27. *Learn some other proofs for the quadratic reciprocity formula.*

EXERCISE 8.28. *Fill in all the details, for the theory of the Jacobi symbols.*

EXERCISE 8.29. *Fill in all the details for the theory of Kronecker symbols too.*

EXERCISE 8.30. *And do not forget to learn a bit about more general symbols too.*

EXERCISE 8.31. *Learn more about the Hadamard conjecture, and its story.*

EXERCISE 8.32. *Learn also about the circulant Hadamard matrices.*

As bonus exercise, find a nice book on algebraic number theory, and start reading.

Part III

Advanced tools

No no limits, we'll reach for the sky
No valley too deep, no mountain too high
No no limits, won't give up the fight
We do what we want and we do it with pride

CHAPTER 9

Complex numbers

9a. Complex numbers

We have already learned many interesting things, and in order now to further upgrade our knowledge in arithmetic, let us discuss the complex numbers. There is a lot of magic here, and we will carefully explain this material. Their definition is as follows:

DEFINITION 9.1. *The complex numbers are variables of the form*

$$x = a + ib$$

with $a, b \in \mathbb{R}$, which add in the obvious way, and multiply according to the following rule:

$$i^2 = -1$$

Each real number can be regarded as a complex number, $a = a + i \cdot 0$.

In other words, we consider variables as above, without bothering for the moment with their precise meaning. Now consider two such complex numbers:

$$x = a + ib \quad , \quad y = c + id$$

The formula for the sum is then the obvious one, as follows:

$$x + y = (a + c) + i(b + d)$$

As for the formula of the product, by using the rule $i^2 = -1$, we obtain:

$$\begin{aligned} xy &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Thus, the complex numbers as introduced above are well-defined. The multiplication formula is of course quite tricky, and hard to memorize, but we will see later some alternative ways, which are more conceptual, for performing the multiplication.

The advantage of using the complex numbers comes from the fact that the equation $x^2 = 1$ has now a solution, $x = i$. In fact, this equation has two solutions, namely:

$$x = \pm i$$

This is of course very good news. More generally, we have the following result, regarding the arbitrary degree 2 equations, with real coefficients:

THEOREM 9.2. *The complex solutions of $ax^2+bx+c=0$ with $a,b,c\in\mathbb{R}$ are*

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

with the square root of negative real numbers being defined as

$$\sqrt{-m}=\pm i\sqrt{m}$$

and with the square root of positive real numbers being the usual one.

PROOF. We can write our equation in the following way:

$$\begin{aligned}
ax^2+bx+c=0 &\iff x^2+\frac{b}{a}x+\frac{c}{a}=0\\
&\iff \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0\\
&\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\\
&\iff x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We will see later that any degree 2 complex equation has solutions as well, and that more generally, any polynomial equation, real or complex, has solutions. Moving ahead now, we can represent the complex numbers in the plane, in the following way:

PROPOSITION 9.3. *The complex numbers, written as usual*

$$x=a+ib$$

can be represented in the plane, according to the following identification:

$$x=\begin{pmatrix}a\\b\end{pmatrix}$$

With this convention, the sum of complex numbers is the usual sum of vectors.

PROOF. Consider indeed two arbitrary complex numbers:

$$x=a+ib\quad,\quad y=c+id$$

Their sum is then by definition the following complex number:

$$x+y=(a+c)+i(b+d)$$

Now let us represent x, y in the plane, as in the statement:

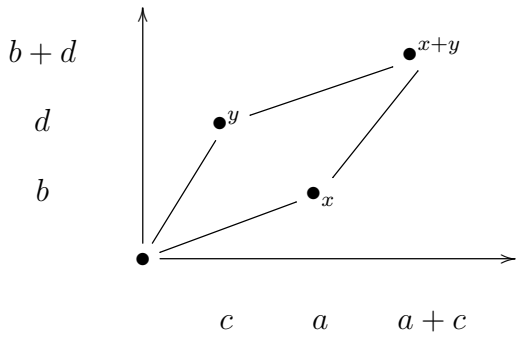
$$x = \begin{pmatrix} a \\ b \end{pmatrix} \quad , \quad y = \begin{pmatrix} c \\ d \end{pmatrix}$$

In this picture, their sum is given by the following formula:

$$x + y = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

But this is indeed the vector corresponding to $x + y$, so we are done. □

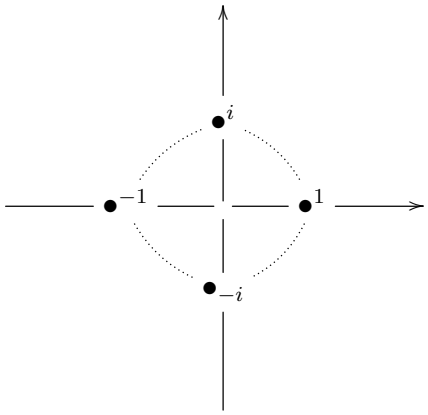
Here we have assumed that you are a bit familiar with vector calculus. If not, no problem, the idea is simply that vectors add by forming a parallelogram, as follows:



Observe that in our geometric picture from Proposition 9.3, the real numbers correspond to the numbers on the Ox axis. As for the purely imaginary numbers, these lie on the Oy axis, with the number i itself being given by the following formula:

$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

As an illustration for this, let us record now a basic picture, with some key complex numbers, namely $1, i, -1, -i$, represented according to our conventions:



You might perhaps wonder why I chose to draw that circle, connecting the numbers $1, i, -1, -i$, which does not look very useful. More on this in a moment, the idea being that that circle can be extremely useful, and coming in advance, some advice:

ADVICE 9.4. *When drawing complex numbers, always begin with the coordinate axes Ox, Oy, and with a copy of the unit circle.*

We have so far a quite good understanding of their complex numbers, and their addition. In order to understand now the multiplication operation, we must do something more complicated, namely using polar coordinates. Let us start with:

DEFINITION 9.5. *The complex numbers $x = a + ib$ can be written in polar coordinates,*

$$x = r(\cos t + i \sin t)$$

with the connecting formulae being as follows,

$$a = r \cos t \quad , \quad b = r \sin t$$

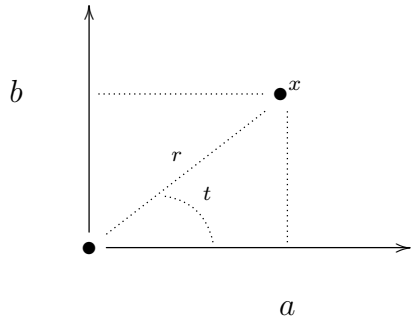
and in the other sense being as follows,

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

and with r, t being called modulus, and argument.

There is a clear relation here with the vector notation from Proposition 9.3, because r is the length of the vector, and t is the angle made by the vector with the Ox axis. To

be more precise, the picture for what is going on in Definition 9.5 is as follows:



As a basic example here, the number i takes the following form:

$$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

The point now is that in polar coordinates, the multiplication formula for the complex numbers, which was so far something quite opaque, takes a very simple form:

THEOREM 9.6. *Two complex numbers written in polar coordinates,*

$$x = r(\cos s + i\sin s) \quad , \quad y = p(\cos t + i\sin t)$$

multiply according to the following formula:

$$xy = rp(\cos(s+t) + i\sin(s+t))$$

In other words, the moduli multiply, and the arguments sum up.

PROOF. This follows from the following formulae, that we know well:

$$\cos(s+t) = \cos s\cos t - \sin s\sin t$$

$$\sin(s+t) = \cos s\sin t + \sin s\cos t$$

Indeed, we can assume that we have $r = p = 1$, by dividing everything by these numbers. Now with this assumption made, we have the following computation:

$$\begin{aligned} xy &= (\cos s + i\sin s)(\cos t + i\sin t) \\ &= (\cos s\cos t - \sin s\sin t) + i(\cos s\sin t + \sin s\cos t) \\ &= \cos(s+t) + i\sin(s+t) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result, which is based on some non-trivial trigonometry, is quite powerful. As a basic application of it, we can now compute powers, as follows:

THEOREM 9.7. *The powers of a complex number, written in polar form,*

$$x = r(\cos t + i \sin t)$$

are given by the following formula, valid for any exponent $k \in \mathbb{N}$:

$$x^k = r^k(\cos kt + i \sin kt)$$

Moreover, this formula holds in fact for any $k \in \mathbb{Z}$, and even for any $k \in \mathbb{Q}$.

PROOF. Given a complex number x, written in polar form as above, and an exponent $k \in \mathbb{N}$, we have indeed the following computation, with k terms everywhere:

$$\begin{aligned} x^k &= x \ldots x \\ &= r(\cos t + i \sin t) \ldots r(\cos t + i \sin t) \\ &= r^k([\cos(t + \ldots + t) + i \sin(t + \ldots + t)) \\ &= r^k(\cos kt + i \sin kt) \end{aligned}$$

Thus, we are done with the case $k \in \mathbb{N}$. Regarding now the generalization to the case $k \in \mathbb{Z}$, it is enough here to do the verification for $k = -1$, where the formula is:

$$x^{-1} = r^{-1}(\cos(-t) + i \sin(-t))$$

But this number x^{-1} is indeed the inverse of x, as shown by:

$$\begin{aligned} xx^{-1} &= r(\cos t + i \sin t) \cdot r^{-1}(\cos(-t) + i \sin(-t)) \\ &= \cos(t - t) + i \sin(t - t) \\ &= \cos 0 + i \sin 0 \\ &= 1 \end{aligned}$$

Finally, regarding the generalization to the case $k \in \mathbb{Q}$, it is enough to do the verification for exponents of type $k = 1/n$, with $n \in \mathbb{N}$. The claim here is that:

$$x^{1/n} = r^{1/n}\left[\cos\left(\frac{t}{n}\right) + i \sin\left(\frac{t}{n}\right)\right]$$

In order to prove this, let us compute the n-th power of this number. We can use the power formula for the exponent $n \in \mathbb{N}$, that we already established, and we obtain:

$$\begin{aligned} (x^{1/n})^n &= (r^{1/n})^n\left[\cos\left(n \cdot \frac{t}{n}\right) + i \sin\left(n \cdot \frac{t}{n}\right)\right] \\ &= r(\cos t + i \sin t) \\ &= x \end{aligned}$$

Thus, we have indeed a n-th root of x, and our proof is now complete. □

We should mention that there is a bit of ambiguity in the above, in the case of the exponents $k \in \mathbb{Q}$, due to the fact that the square roots, and the higher roots as well, can take multiple values, in the complex number setting. We will be back to this.

As a basic application of Theorem 9.7, we have the following result:

PROPOSITION 9.8. *Each complex number, written in polar form,*

$$x = r(\cos t + i \sin t)$$

has two square roots, given by the following formula:

$$\sqrt{x} = \pm\sqrt{r}\left[\cos\left(\frac{t}{2}\right) + i\sin\left(\frac{t}{2}\right)\right]$$

When $x > 0$, these roots are $\pm\sqrt{x}$. When $x < 0$, these roots are $\pm i\sqrt{-x}$.

PROOF. The first assertion is clear indeed from the general formula in Theorem 9.7, at $k = 1/2$. As for its particular cases with $x \in \mathbb{R}$, these are clear from it. □

As a comment here, for $x > 0$ we are very used to call the usual $\sqrt{x}$ square root of x. However, for $x < 0$, or more generally for $x \in \mathbb{C} - \mathbb{R}_+$, there is less interest in choosing one of the possible $\sqrt{x}$ and calling it "the" square root of x, because all this is based on our convention that i comes up, instead of down, which is something rather arbitrary. Actually, clocks turning clockwise, i should be rather coming down. All this is a matter of taste, but in any case, for our math, the best is to keep some ambiguity, as above.

With the above results in hand, and notably with the square root formula from Proposition 9.8, we can now go back to the degree 2 equations, and we have:

THEOREM 9.9. *The complex solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{C}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of complex numbers being defined as above.

PROOF. This is clear, the computations being the same as in the real case. To be more precise, our degree 2 equation can be written as follows:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now since we know from Proposition 9.8 that any complex number has a square root, we are led to the conclusion in the statement. □

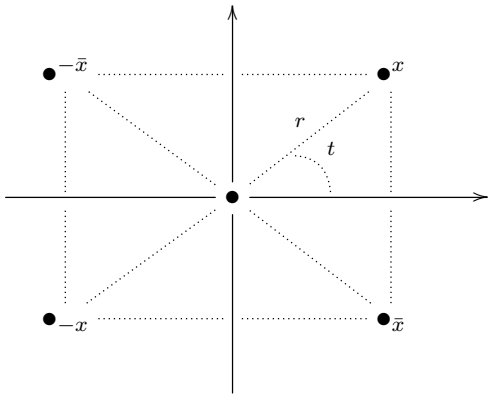
As a last general topic regarding the complex numbers, let us discuss conjugation. This is something quite tricky, complex number specific, as follows:

DEFINITION 9.10. *The complex conjugate of $x = a + ib$ is the following number,*

$$\bar{x} = a - ib$$

obtained by making a reflection with respect to the Ox axis.

As before with other such operations on complex numbers, a quick picture says it all. Here is the picture, with the numbers $x, \bar{x}, -x, -\bar{x}$ being all represented:



Observe that the conjugate of a real number $x \in \mathbb{R}$ is the number itself, $x = \bar{x}$. In fact, the equation $x = \bar{x}$ characterizes the real numbers, among the complex numbers. At the level of non-trivial examples now, we have the following formula:

$$\bar{i} = -i$$

There are many things that can be said about the conjugation of the complex numbers, and here is a summary of basic such things that can be said:

THEOREM 9.11. *The conjugation operation $x \to \bar{x}$ has the following properties:*

(1) *$x = \bar{x}$ precisely when x is real.*
(2) *$x = -\bar{x}$ precisely when x is purely imaginary.*
(3) *$x\bar{x} = |x|^2$, with $|x| = r$ being as usual the modulus.*
(4) *With $x = r(\cos t + i \sin t)$, we have $\bar{x} = r(\cos t - i \sin t)$.*
(5) *We have the formula $\overline{xy} = \bar{x}\bar{y}$, for any $x, y \in \mathbb{C}$.*
(6) *The solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$ are conjugate.*

PROOF. These results are all elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something clear too, because with $x = a + ib$ our equation $x = -\bar{x}$ reads $a + ib = -a + ib$, and so $a = 0$, which amounts in saying that x is purely imaginary.

(3) This is a key formula, which can be proved as follows, with $x = a + ib$:

$$\begin{aligned} x\bar{x} &= (a+ib)(a-ib) \\ &= a^2 + b^2 \\ &= |x|^2 \end{aligned}$$

(4) This is clear indeed from the picture following Definition 9.10.

(5) This is something quite magic, which can be proved as follows:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd)-i(ad+bc) \\ &= (a-ib)(c-id) \end{aligned}$$

However, what we have been doing here is not very clear, geometrically speaking, and our formula is worth an alternative proof. Here is that proof, which after inspection contains no computations at all, making it clear that the polar writing is the best:

$$\begin{aligned} & \overline{r(\cos s + i\sin s)\cdot p(\cos t + i\sin t)} \\ = \ & \overline{rp(\cos(s+t) + i\sin(s+t))} \\ = \ & rp(\cos(-s-t) + i\sin(-s-t)) \\ = \ & r(\cos(-s) + i\sin(-s))\cdot p(\cos(-t) + i\sin(-t)) \\ = \ & \overline{r(\cos s + i\sin s)}\cdot\overline{p(\cos t + i\sin t)} \end{aligned}$$

(6) This comes from the formula of the solutions, that we know from Theorem 9.2, but we can deduce this as well directly, without computations. Indeed, by using our assumption that the coefficients are real, $a, b, c \in \mathbb{R}$, we have:

$$\begin{aligned} ax^2 + bx + c = 0 &\implies \overline{ax^2 + bx + c} = 0 \\ &\implies \bar{a}\bar{x}^2 + \bar{b}\bar{x} + \bar{c} = 0 \\ &\implies a\bar{x}^2 + b\bar{x} + c = 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

9b. Exponential writing

Let us discuss now the theory of complex functions $f : \mathbb{C} \to \mathbb{C}$, in analogy with the theory of the real functions $f : \mathbb{R} \to \mathbb{R}$. We will see that many results that we know from the real setting extend to the complex setting. Before starting, two remarks on this:

(1) Most of the real functions $f : \mathbb{R} \to \mathbb{R}$ that we know, such as $\sin, \cos, \exp, \log$, extend into complex functions $f : \mathbb{C} \to \mathbb{C}$, and the study of these latter extensions brings some new light on the original real functions. Thus, what we will be doing here will be, in a certain sense, a refinement of the theory that we developed in chapter 6.

(2) On the other hand, since we have $\mathbb{C} \simeq \mathbb{R}^2$, the complex functions $f : \mathbb{C} \to \mathbb{C}$ that we will study here can be regarded as functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. This is something quite subtle, but in any case, what we will be doing here will stand as well as an introduction to the functions of type $f : \mathbb{R}^N \to \mathbb{R}^M$, which are something more complicated.

In short, one complex variable is something in between one real variable, and two or more real variables, and we can only expect to end up with a mysterious mixture of surprising and unsurprising results. Welcome to complex analysis. Let us start with:

DEFINITION 9.12. *A complex function $f : \mathbb{C} \to \mathbb{C}$, or more generally $f : X \to \mathbb{C}$, with $X \subset \mathbb{C}$ being a subset, is called continuous when, for any $x_n, x \in X$:*

$$x_n \to x \implies f(x_n) \to f(x)$$

where the convergence of the sequences of complex numbers, $x_n \to x$, means by definition that for n big enough, the quantity $|x_n - x|$ becomes arbitrarily small.

Observe that in real coordinates, $x = (a, b)$, the distances appearing in the definition of the convergence $x_n \to x$ are given by the following formula:

$$|x_n - x| = \sqrt{(a_n - a)^2 + (b_n - b)^2}$$

Thus $x_n \to x$ in the complex sense means that $(a_n, b_n) \to (a, b)$ in the usual, intuitive sense, with respect to the usual distance in the plane $\mathbb{R}^2$, and as a consequence, a function $f : \mathbb{C} \to \mathbb{C}$ is continuous precisely when it is continuous, in an intuitive sense, when regarded as function $f : \mathbb{R}^2 \to \mathbb{R}^2$. But more on this, later in this book.

At the level of examples, we first have the following result:

THEOREM 9.13. *We can exponentiate the complex numbers, according to the formula*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and the function $x \to e^x$ is continuous, and satisfies $e^{x+y} = e^x e^y$.

PROOF. We must first prove that the series converges. But this follows from:

$$\begin{aligned} |e^x| &= \left| \sum_{k=0}^{\infty} \frac{x^k}{k!} \right| \\ &\leq \sum_{k=0}^{\infty} \left| \frac{x^k}{k!} \right| \\ &= \sum_{k=0}^{\infty} \frac{|x|^k}{k!} \\ &= e^{|x|} < \infty \end{aligned}$$

Regarding the formula $e^{x+y}=e^xe^y$, this follows too as in the real case, as follows:

$$\begin{aligned}
e^{x+y} &= \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!}\\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\binom{k}{s}\cdot\frac{x^sy^{k-s}}{k!}\\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{x^sy^{k-s}}{s!(k-s)!}\\
&= e^xe^y
\end{aligned}$$

Finally, the continuity of $x\to e^x$ comes at $x=0$ from the following computation:

$$\begin{aligned}
|e^t-1| &= \left|\sum_{k=1}^{\infty}\frac{t^k}{k!}\right|\\
&\leq \sum_{k=1}^{\infty}\left|\frac{t^k}{k!}\right|\\
&= \sum_{k=1}^{\infty}\frac{|t|^k}{k!}\\
&= e^{|t|}-1
\end{aligned}$$

As for the continuity of $x\to e^x$ in general, this can be deduced now as follows:

$$\lim_{t\to0}e^{x+t}=\lim_{t\to0}e^xe^t=e^x\lim_{t\to0}e^t=e^x\cdot1=e^x$$

Thus, we are led to the conclusions in the statement. □

We will be back to more functions later. As an important fact, however, let us point out that, contrary to what the above might suggest, everything does not always extend trivially from the real to the complex case. For instance, we have:

PROPOSITION 9.14. *We have the following formula, valid for any $|x|<1$,*

$$\frac{1}{1-x}=1+x+x^2+\ldots$$

but, unlike in the real case, the geometric meaning of this formula is quite unclear.

PROOF. Here the formula in the statement holds indeed, by multiplying and cancelling terms, and with the convergence being justified by the following estimate:

$$\left|\sum_{n=0}^{\infty}x^n\right|\leq\sum_{n=0}^{\infty}|x|^n=\frac{1}{1-|x|}$$

As for the last assertion, this is something quite informal. To be more precise, for $x = 1/2$ our formula is clear, by cutting the interval $[0, 2]$ into half, and so on:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

More generally, for $x \in (-1, 1)$ the meaning of the formula in the statement is something quite clear and intuitive, geometrically speaking, by using a similar argument. However, when x is complex, and not real, we are led into a kind of mysterious spiral there, and the only case where the formula is "obvious", geometrically speaking, is that when $x = rw$, with $r \in [0, 1)$, and with w being a root of unity. To be more precise here, by anticipating a bit, assume that we have a number $w \in \mathbb{C}$ satisfying $w^N = 1$, for some $N \in \mathbb{N}$. We have then the following formula, for our infinite sum:

$$\begin{aligned} 1 + rw + r^2w^2 + \ldots &= (1 + rw + \ldots + r^{N-1}w^{N-1}) \\ &+ (r^N + r^{N+1}w \ldots + r^{2N-1}w^{N-1}) \\ &+ (r^{2N} + r^{2N+1}w \ldots + r^{3N-1}w^{N-1}) \\ &+ \ldots \end{aligned}$$

Thus, by grouping the terms with the same argument, our infinite sum is:

$$\begin{aligned} 1 + rw + r^2w^2 + \ldots &= (1 + r^N + r^{2N} + \ldots) \\ &+ (r + r^{N+1} + r^{2N+1} + \ldots)w \\ &+ \ldots \\ &+ (r^{N-1} + r^{2N-1} + r^{3N-1} + \ldots)w^{N-1} \end{aligned}$$

But the sums of each ray can be computed with the real formula for geometric series, that we know and understand well, and with an extra bit of algebra, we get:

$$\begin{aligned} 1 + rw + r^2w^2 + \ldots &= \frac{1}{1 - r^N} + \frac{rw}{1 - r^N} + \ldots + \frac{r^{N-1}w^{N-1}}{1 - r^N} \\ &= \frac{1}{1 - r^N}\left(1 + rw + \ldots + r^{N-1}w^{N-1}\right) \\ &= \frac{1}{1 - r^N} \cdot \frac{1 - r^N}{1 - rw} \\ &= \frac{1}{1 - rw} \end{aligned}$$

Summarizing, as claimed above, the geometric series formula can be understood, in a purely geometric way, for variables of type $x = rw$, with $r \in [0, 1)$, and with w being a root of unity. In general, however, this formula tells us that the numbers on a certain infinite spiral sum up to a certain number, which remains something quite mysterious. □

Getting back now to less mysterious mathematics, which in fact will turn to be quite mysterious as well, as is often the case with things involving complex numbers, as an

application of all this, let us discuss the final and most convenient writing of the complex numbers, which is a variation on the polar writing, as follows:

$$x = re^{it}$$

The point with this formula comes from the following deep result:

THEOREM 9.15. *We have the following formula,*

$$e^{it} = \cos t + i \sin t$$

valid for any $t \in \mathbb{R}$.

PROOF. There are two possible proofs here, as follows:

(1) Physics proof. We have indeed the following elegant computation, and I will leave it to you to figure out what is wrong with it, and what the fix is:

$$\begin{aligned} e^{it} &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \\ &= \sum_{k=2l} \frac{(it)^k}{k!} + \sum_{k=2l+1} \frac{(it)^k}{k!} \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l}}{(2l)!} + i \sum_{l=0}^{\infty} (-1)^l \frac{t^{2l+1}}{(2l+1)!} \\ &= \cos t + i \sin t \end{aligned}$$

(2) Mathematics proof. Consider the following function $f : \mathbb{R} \to \mathbb{C}$:

$$f(t) = \frac{\cos t + i \sin t}{e^{it}}$$

By using $\sin' = \cos$ and $\cos' = -\sin$, that we know since chapter 6, we have:

$$\begin{aligned} f'(t) &= (e^{-it}(\cos t + i \sin t))' \\ &= -ie^{-it}(\cos t + i \sin t) + e^{-it}(-\sin t + i \cos t) \\ &= e^{-it}(-i \cos t + \sin t) + e^{-it}(-\sin t + i \cos t) \\ &= 0 \end{aligned}$$

We conclude that $f : \mathbb{R} \to \mathbb{C}$ is constant, equal to $f(0) = 1$, as desired. □

As a main application of the above formula, we have:

THEOREM 9.16. *We have the following formula,*

$$e^{\pi i} = -1$$

and we have $E = mc^2$ *as well.*

PROOF. We have two assertions here, the idea being as follows:

(1) The first formula, $e^{\pi i}=-1$, which is actually the main formula in mathematics, comes from Theorem 9.15, by setting $t=\pi$. Indeed, we obtain:

$$\begin{aligned}e^{\pi i}&=\cos\pi+i\sin\pi\\&=-1+i\cdot 0\\&=-1\end{aligned}$$

(2) As for $E=mc^2$, which is the main formula in physics, this is something deep too. Although we will not really need it here, we recommend learning it as well, for symmetry reasons between math and physics, say from Feynman [34]. □

Now back to our $x=re^{it}$ objectives, with the above theory in hand we can indeed use from now on this notation, the complete statement being as follows:

THEOREM 9.17. *The complex numbers $x=a+ib$ can be written in polar coordinates,*

$$x=re^{it}$$

with the connecting formulae being

$$a=r\cos t\quad,\quad b=r\sin t$$

and in the other sense being

$$r=\sqrt{a^2+b^2}\quad,\quad\tan t=\frac{b}{a}$$

and with r,t being called modulus, and argument.

PROOF. This is a reformulation of our previous Definition 9.5, by using the formula $e^{it}=\cos t+i\sin t$ from Theorem 9.15, and multiplying everything by r. □

With this in hand, we can now go back to the basics, namely the addition and multiplication of the complex numbers. We have the following result:

THEOREM 9.18. *In polar coordinates, the complex numbers multiply as*

$$re^{is}\cdot pe^{it}=rp\,e^{i(s+t)}$$

with the arguments s,t being taken modulo 2π.

PROOF. This is something that we already know, from Theorem 9.6, reformulated by using the notations from Theorem 9.17. Observe that this follows as well directly, from the fact that we have $e^{a+b}=e^ae^b$, that we know from analysis. □

The above formula is obviously very powerful. However, in polar coordinates we do not have a simple formula for the sum. Thus, this formalism has its limitations.

We can investigate as well more complicated operations, as follows:

THEOREM 9.19. *We have the following operations on the complex numbers, written in polar form, as above:*

(1) *Inversion:* $(re^{it})^{-1} = r^{-1}e^{-it}$.
(2) *Square roots:* $\sqrt{re^{it}} = \pm\sqrt{r}e^{it/2}$.
(3) *Powers:* $(re^{it})^a = r^a e^{ita}$.
(4) *Conjugation:* $\overline{re^{it}} = re^{-it}$.

PROOF. This is something that we already know, from Theorem 9.7, but we can now discuss all this, from a more conceptual viewpoint, the idea being as follows:

(1) We have indeed the following computation, using Theorem 9.18:

$$\begin{aligned}(re^{it})(r^{-1}e^{-it}) &= rr^{-1}\cdot e^{i(t-t)}\\ &= 1\cdot 1\\ &= 1\end{aligned}$$

(2) Once again by using Theorem 9.18, we have:

$$(\pm\sqrt{r}e^{it/2})^2 = (\sqrt{r})^2 e^{i(t/2+t/2)} = re^{it}$$

(3) Given an arbitrary number $a \in \mathbb{R}$, we can define, as stated:

$$(re^{it})^a = r^a e^{ita}$$

Due to Theorem 9.18, this operation $x \to x^a$ is indeed the correct one.

(4) This comes from the fact, that we know from Theorem 9.11, that the conjugation operation $x \to \bar{x}$ keeps the modulus, and switches the sign of the argument. □

9c. Equations, roots

Getting back to algebra, recall from Theorem 9.9 that any degree 2 equation has 2 complex roots. We can in fact prove that any polynomial equation, of arbitrary degree $N \in \mathbb{N}$, has exactly N complex solutions, counted with multiplicities:

THEOREM 9.20. *Any polynomial $P \in \mathbb{C}[X]$ decomposes as*

$$P = c(X - a_1)\ldots(X - a_N)$$

with $c \in \mathbb{C}$ and with $a_1, \ldots, a_N \in \mathbb{C}$.

PROOF. The problem is that of proving that our polynomial has at least one root, because afterwards we can proceed by recurrence. We prove this by contradiction. So, assume that P has no roots, and pick a number $z \in \mathbb{C}$ where $|P|$ attains its minimum:

$$|P(z)| = \min_{x\in\mathbb{C}} |P(x)| > 0$$

Since $Q(t) = P(z+t) - P(z)$ is a polynomial which vanishes at $t = 0$, this polynomial must be of the form ct^k + higher terms, with $c \neq 0$, and with $k \geq 1$ being an integer. We obtain from this that, with $t \in \mathbb{C}$ small, we have the following estimate:

$$P(z+t) \simeq P(z) + ct^k$$

Now let us write $t = rw$, with $r > 0$ small, and with $|w| = 1$. Our estimate becomes:

$$P(z+rw) \simeq P(z) + cr^k w^k$$

Now recall that we assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$\begin{aligned} |P(z+rw)| &\simeq |P(z) + cr^k w^k| \\ &= |P(z)|(1 - |c|r^k) \end{aligned}$$

Now by choosing $r > 0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(z+rw)| < |P(z)|$$

But this contradicts our definition of $z \in \mathbb{C}$, as a point where $|P|$ attains its minimum. Thus P has a root, and by recurrence it has N roots, as stated. □

All this is very nice, and we will see applications in a moment. As a word of warning, however, we should mention that the above result remains something quite theoretical. Indeed, the proof is by contradiction, and there is no way of recycling the material there into something explicit, that can be used for effectively computing the roots.

We will be back to this in the next chapter, with a number of sharp results on the subject, extending to degree 3 and 4 what we know well in degree 2.

We have kept the best for the end. As a last topic regarding the equations and roots, which is something really beautiful, we have the roots of unity. Let us start with:

THEOREM 9.21. *The equation $x^N = 1$ has N complex solutions, namely*

$$\left\{ w^k \middle| k = 0, 1, \ldots, N-1 \right\} \quad , \quad w = e^{2\pi i/N}$$

which are called roots of unity of order N.

PROOF. This follows from the general multiplication formula for complex numbers from Theorem 9.18. Indeed, with $x = re^{it}$ our equation reads:

$$r^N e^{itN} = 1$$

Thus $r = 1$, and $t \in [0, 2\pi)$ must be a multiple of $2\pi/N$, as stated. □

As an illustration here, the roots of unity of small order, along with some of their basic properties, which are very useful for computations, are as follows:

$\underline{N=1}$. Here the unique root of unity is 1.

$\underline{N=2}$. Here we have two roots of unity, namely 1 and -1.

$\underline{N=3}$. Here we have 1, then $w=e^{2\pi i/3}$, and then $w^2=\bar{w}=e^{4\pi i/3}$.

$\underline{N=4}$. Here the roots of unity, read as usual counterclockwise, are $1,i,-1,-i$.

$\underline{N=5}$. Here, with $w=e^{2\pi i/5}$, the roots of unity are $1,w,w^2,w^3,w^4$.

$\underline{N=6}$. Here a useful alternative writing is $\{\pm 1,\pm w,\pm w^2\}$, with $w=e^{2\pi i/3}$.

$\underline{N=7}$. Here, with $w=e^{2\pi i/7}$, the roots of unity are $1,w,w^2,w^3,w^4,w^5,w^6$.

$\underline{N=8}$. Here the roots of unity, read as usual counterclockwise, are the numbers $1,w,i,iw,-1,-w,-i,-iw$, with $w=e^{\pi i/4}$, which is also given by $w=(1+i)/\sqrt{2}$.

The roots of unity are very useful variables, and have many interesting properties. As a first application, we can now solve the ambiguity questions related to the extraction of N-th roots, from Theorem 9.7 and Theorem 9.19, the statement being as follows:

THEOREM 9.22. *Any nonzero complex number, written as*

$$x=re^{it}$$

has exactly N roots of order N, which appear as

$$y=r^{1/N}e^{it/N}$$

multiplied by the N roots of unity of order N.

PROOF. We must solve the equation $z^N=x$, over the complex numbers. Since the number y in the statement clearly satisfies $y^N=x$, our equation is equivalent to:

$$z^N=y^N$$

Now observe that we can write this equation as follows:

$$\left(\frac{z}{y}\right)^N=1$$

We conclude that the solutions z appear by multiplying y by the solutions of $t^N=1$, which are the N-th roots of unity, as claimed. □

The roots of unity appear in connection with many other interesting questions, and there are many useful formulae relating them, which are good to know. Here is a basic such formula, very beautiful, to be used many times in what follows:

THEOREM 9.23. *The roots of unity, $\{w^k\}$ with $w = e^{2\pi i/N}$, have the property*

$$\sum_{k=0}^{N-1}(w^k)^s = N\delta_{N|s}$$

for any exponent $s \in \mathbb{N}$, where on the right we have a Kronecker symbol.

PROOF. The numbers in the statement, when written more conveniently as $(w^s)^k$ with $k = 0, \ldots, N-1$, form a certain regular polygon in the plane P_s. Thus, if we denote by C_s the barycenter of this polygon, we have the following formula:

$$\frac{1}{N}\sum_{k=0}^{N-1} w^{ks} = C_s$$

Now observe that in the case $N\not|\, s$ our polygon P_s is non-degenerate, circling around the unit circle, and having center $C_s = 0$. As for the case $N|s$, here the polygon is degenerate, lying at 1, and having center $C_s = 1$. Thus, we have the following formula:

$$C_s = \delta_{N|s}$$

Thus, we obtain the formula in the statement. □

As an interesting philosophical fact, regarding the roots of unity, and the complex numbers in general, we can now solve the following equation, in a "uniform" way:

$$x_1 + \ldots + x_N = 0$$

With this being not a joke. Frankly, can you find some nice-looking family of real numbers $x_1, \ldots, x_N$ satisfying $x_1 + \ldots + x_N = 0$? Certainly not. But with complex numbers we have now our answer, the sum of the N-th roots of unity being zero.

9d. Fourier, Poisson

As a basic application of our complex number technology, we can talk about Fourier transforms. Many things can be said here, and we will just provide an introduction to all this, with focus on probability. Let us start with the following basic fact:

THEOREM 9.24. *Assuming that $f, g \in L^\infty(X)$ are independent, we have*

$$F_{f+g} = F_f F_g$$

where $F_f(x) = E(e^{ixf})$ is the Fourier transform.

PROOF. We have the following computation, using the Fubini theorem:

$$\begin{aligned} F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixz} d\mu_{f+g}(z) \\ &= \int_{\mathbb{R}} e^{ixz} d(\mu_f * \mu_g)(z) \\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{ix(z+t)} d\mu_f(z) d\mu_g(t) \\ &= \int_{\mathbb{R}} e^{ixz} d\mu_f(z) \int_{\mathbb{R}} e^{ixt} d\mu_g(t) \\ &= F_f(x) F_g(x) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Now with this machinery in hand, let us go back to the Poisson laws, that we met in chapter 5 in relation with counting questions. It is convenient to enlarge the attention to the whole family of Poisson laws p_t, which are constructed as follows:

DEFINITION 9.25. *The Poisson law of parameter* 1 *is the following measure,*

$$p_1 = \frac{1}{e} \sum_{k\in\mathbb{N}} \frac{\delta_k}{k!}$$

and the Poisson law of parameter $t > 0$ *is the following measure,*

$$p_t = e^{-t} \sum_{k\in\mathbb{N}} \frac{t^k}{k!} \delta_k$$

with the letter "p" standing for Poisson.

As a first observation, our Poisson laws as constructed above have indeed mass 1, as they should, and this due to the following formula:

$$e^t = \sum_{k\in\mathbb{N}} \frac{t^k}{k!}$$

We will see in the moment why these measures appear a bit everywhere, in discrete contexts, the reasons for this coming from the Poisson Limit Theorem (PLT). Let us first develop some general theory. We first have the following result:

PROPOSITION 9.26. *We have the following formula, for any* $s, t > 0$*,*

$$p_s * p_t = p_{s+t}$$

so the Poisson laws form a convolution semigroup.

PROOF. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula, we obtain:

$$\begin{aligned} p_s * p_t &= e^{-s} \sum_k \frac{s^k}{k!} \delta_k * e^{-t} \sum_l \frac{t^l}{l!} \delta_l \\ &= e^{-s-t} \sum_n \delta_n \sum_{k+l=n} \frac{s^k t^l}{k!l!} \\ &= e^{-s-t} \sum_n \frac{\delta_n}{n!} \sum_{k+l=n} \frac{n!}{k!l!} s^k t^l \\ &= e^{-s-t} \sum_n \frac{(s+t)^n}{n!} \delta_n \\ &= p_{s+t} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Next in line, we have the following result, which is fundamental as well:

PROPOSITION 9.27. *The Poisson laws appear as formal exponentials*

$$p_t = \sum_k \frac{t^k (\delta_1 - \delta_0)^{*k}}{k!}$$

with respect to the convolution of measures $*$.

PROOF. By using the binomial formula, the measure on the right is:

$$\begin{aligned} \mu &= \sum_k \frac{t^k}{k!} \sum_{r+s=k} (-1)^s \frac{k!}{r!s!} \delta_r \\ &= \sum_k t^k \sum_{r+s=k} (-1)^s \frac{\delta_r}{r!s!} \\ &= \sum_r \frac{t^r \delta_r}{r!} \sum_s \frac{(-1)^s}{s!} \\ &= \frac{1}{e} \sum_r \frac{t^r \delta_r}{r!} \\ &= p_t \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Fourier transform computation, this is as follows:

PROPOSITION 9.28. *The Fourier transform of p_t is given by*

$$F_{p_t}(y) = \exp\left((e^{iy} - 1)t\right)$$

for any $t > 0$.

PROOF. We have indeed the following computation:

$$\begin{aligned}
F_{p_t}(y) &= e^{-t}\sum_k \frac{t^k}{k!}F_{\delta_k}(y)\\
&= e^{-t}\sum_k \frac{t^k}{k!}\,e^{iky}\\
&= e^{-t}\sum_k \frac{(e^{iy}t)^k}{k!}\\
&= \exp(-t)\exp(e^{iy}t)\\
&= \exp\big((e^{iy}-1)t\big)
\end{aligned}$$

Thus, we obtain the formula in the statement. □

Observe that the above formula gives an alternative proof for Proposition 9.26, by using the fact that the logarithm of the Fourier transform linearizes the convolution. As another application, we can now establish the Poisson Limit Theorem, as follows:

THEOREM 9.29 (PLT). *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\,\delta_1\right)^{*n}\to p_t$$

for any $t>0$.

PROOF. Let us denote by ν_n the measure under the convolution sign, namely:

$$\nu_n=\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\,\delta_1$$

We have the following computation, for the Fourier transform of the limit:

$$\begin{aligned}
F_{\delta_r}(y)=e^{iry} &\implies F_{\nu_n}(y)=\left(1-\frac{t}{n}\right)+\frac{t}{n}\,e^{iy}\\
&\implies F_{\nu_n^{*n}}(y)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}\,e^{iy}\right)^n\\
&\implies F_{\nu_n^{*n}}(y)=\left(1+\frac{(e^{iy}-1)t}{n}\right)^n\\
&\implies F(y)=\exp\big((e^{iy}-1)t\big)
\end{aligned}$$

Thus, we obtain indeed the Fourier transform of p_t, as desired. □

All this is very nice, making a link with our Bernoulli and binomial law considerations from chapter 4. Also, getting back now to our counting questions from chapter 2, and chapter 5, we have the following generalization of the main results there:

THEOREM 9.30. *The number of partial fixed points of permutations*

$$\chi_t : S_N \to \mathbb{N}$$

given by the following formula, with $t > 0$ being a parameter,

$$\chi_t(\sigma) = \#\left\{i \in \{1, \ldots, [tN]\} \middle| \sigma(i) = i\right\}$$

follows with $N \to \infty$ the Poisson law p_t.

PROOF. This is something that we know from chapters 2 and 5 at $t = 1$, and the proof in general is similar. Indeed, by inclusion-exclusion we obtain:

$$\chi_t(0) \simeq \frac{1}{e^t}$$

More generally, again by inclusion-exclusion, we have, for any $k \in \mathbb{N}$:

$$\chi_t(k) \simeq \frac{t^k}{e^t k!}$$

Thus, we are led to the conclusion in the statement. □

And with this, good news, many things from chapters 2,4,5,7 unified, and we can now say that we know what e is, from the perspective of group theory and probability.

9e. Exercises

Welcome to the complex numbers, and as exercises about them, we have:

EXERCISE 9.31. *What happens to $\mathbb{C}$ when representing it upside-down?*

EXERCISE 9.32. *What about drawing $\mathbb{R}$ from right to left? Or doing both?*

EXERCISE 9.33. *Further explore the basic geometric series, for complex numbers.*

EXERCISE 9.34. *Study more $e^{it} = \cos t + i \sin t$, including Taylor series discussion.*

EXERCISE 9.35. *And, as mentioned above, do not forget to learn about $E = mc^2$ too.*

EXERCISE 9.36. *Study the real and complex solutions of $x_1 + \ldots + x_N = 0$.*

EXERCISE 9.37. *Learn more about the Fourier transform, and its basic properties.*

EXERCISE 9.38. *Learn more about the various versions of the Poisson laws.*

As bonus exercise, you can start reading some complex analysis. All good learning.

CHAPTER 10

Polynomials

10a. Resultant, discriminant

We have seen that many questions lead us into computing roots of polynomials. Let us start with something that we know well, but is always good to remember:

PROPOSITION 10.1. *The solutions of $ax^2+bx+c=0$ with $a,b,c\in\mathbb{C}$ are*

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

with the square root of complex numbers being defined as $\sqrt{re^{it}}=\sqrt{r}e^{it/2}$.

PROOF. This is indeed something that we know well, coming from:

$$\begin{aligned}ax^2+bx+c=0 &\iff x^2+\frac{b}{a}x+\frac{c}{a}=0\\ &\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\\ &\iff x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In degree 3 and higher, we would first like to understand what the analogue of the discriminant $\Delta=b^2-4ac$ is. In order to discuss this question, let us start with:

THEOREM 10.2. *Given a monic polynomial $P\in\mathbb{C}[X]$, factorized as*

$$P=(X-a_1)\ldots(X-a_k)$$

the following happen:

(1) *The coefficients of P are symmetric functions in $a_1,\ldots,a_k$.*
(2) *The symmetric functions in $a_1,\ldots,a_k$ are polynomials in the coefficients of P.*

PROOF. This is something standard, the idea being as follows:

(1) By expanding our polynomial, we have the following formula:

$$P=\sum_{r=0}^{k}(-1)^r\sum_{i_1<\ldots<i_r}a_{i_1}\ldots a_{i_r}\cdot X^{k-r}$$

Thus the coefficients of P are, up to some signs, the following functions:

$$f_r = \sum_{i_1<\ldots<i_r} a_{i_1} \ldots a_{i_r}$$

But these are indeed symmetric functions in $a_1, \ldots, a_k$, as claimed.

(2) Conversely now, let us look at the symmetric functions in the roots $a_1, \ldots, a_k$. These appear as linear combinations of the basic symmetric functions, given by:

$$S_r = \sum_i a_i^r$$

Moreover, when allowing polynomials instead of linear combinations, we need in fact only the first k such sums, namely $S_1, \ldots, S_k$. That is, the symmetric functions $\mathcal{F}$ in our variables $a_1, \ldots, a_k$, with integer coefficients, appear as follows:

$$\mathcal{F} = \mathbb{Z}[S_1, \ldots, S_k]$$

(3) The point now is that, alternatively, the symmetric functions in our variables $a_1, \ldots, a_k$ appear as well as linear combinations of the functions f_r that we found in (1), and that when allowing polynomials instead of linear combinations, we need in fact only the first k functions, namely $f_1, \ldots, f_k$. That is, we have as well:

$$\mathcal{F} = \mathbb{Z}[f_1, \ldots, f_k]$$

But this gives the result, because we can pass from $\{S_r\}$ to $\{f_r\}$, and vice versa. □

Getting back now to our original question, namely deciding whether two polynomials $P, Q \in \mathbb{C}[X]$ have a common root or not, this has the following nice answer:

THEOREM 10.3. *Given two polynomials $P, Q \in \mathbb{C}[X]$, written as*

$$P = c(X - a_1) \ldots (X - a_k) \quad , \quad Q = d(X - b_1) \ldots (X - b_l)$$

the following quantity, which is called resultant of P, Q,

$$R(P, Q) = c^l d^k \prod_{ij} (a_i - b_j)$$

is a certain polynomial in the coefficients of P, Q, with integer coefficients, and we have $R(P, Q) = 0$ precisely when P, Q have a common root.

PROOF. This is something quite tricky, the idea being as follows:

(1) Given two polynomials $P, Q \in \mathbb{C}[X]$, we can certainly construct the quantity $R(P, Q)$ in the statement, with the role of the normalization factor $c^l d^k$ to become clear later on, and then we have $R(P, Q) = 0$ precisely when P, Q have a common root:

$$R(P, Q) = 0 \iff \exists i, j, a_i = b_j$$

(2) As bad news, however, this quantity $R(P, Q)$, defined in this way, is a priori not very useful in practice, because it depends on the roots a_i, b_j of our polynomials P, Q,

that we cannot compute in general. However, and here comes our point, as we will prove below, it turns out that $R(P,Q)$ is in fact a polynomial in the coefficients of P,Q, with integer coefficients, and this is where the power of $R(P,Q)$ comes from.

(3) Getting started now, let us expand the formula of $R(P,Q)$, by making all the multiplications there, abstractly, in our head. Everything being symmetric in $a_1,\ldots,a_k$, we obtain in this way certain symmetric functions in these variables, which will be therefore certain polynomials in the coefficients of P. Moreover, due to our normalization factor c^l, these polynomials in the coefficients of P will have integer coefficients.

(4) With this done, let us look now what happens with respect to the remaining variables $b_1,\ldots,b_l$, which are the roots of Q. Once again what we have here are certain symmetric functions in these variables $b_1,\ldots,b_l$, and these symmetric functions must be certain polynomials in the coefficients of Q. Moreover, due to our normalization factor d^k, these polynomials in the coefficients of Q will have integer coefficients.

(5) Thus, we are led to the conclusion in the statement, that $R(P,Q)$ is a polynomial in the coefficients of P,Q, with integer coefficients, and with the remark that the c^ld^k factor is there for these latter coefficients to be indeed integers, instead of rationals. □

All the above might seem a bit complicated, so as an illustration, let us work out an example. Consider the case of a polynomial of degree 2, and a polynomial of degree 1:

$$P=ax^2+bx+c\quad,\quad Q=dx+e$$

In order to compute the resultant, let us factorize our polynomials:

$$P=a(x-p)(x-q)\quad,\quad Q=d(x-r)$$

The resultant can be then computed as follows, by using the method above:

$$\begin{aligned}R(P,Q)&=ad^2(p-r)(q-r)\\&=ad^2(pq-(p+q)r+r^2)\\&=cd^2+bd^2r+ad^2r^2\\&=cd^2-bde+ae^2\end{aligned}$$

Finally, observe that $R(P,Q)=0$ corresponds indeed to the fact that P,Q have a common root. Indeed, the root of Q is $r=-e/d$, and we have:

$$\begin{aligned}P(r)&=\frac{ae^2}{d^2}-\frac{be}{d}+c\\&=\frac{R(P,Q)}{d^2}\end{aligned}$$

Regarding now the explicit formula of the resultant $R(P,Q)$, this is something quite complicated, and there are several methods for dealing with this problem. We have:

THEOREM 10.4. *The resultant of two polynomials, written as*

$$P = p_k X^k + \ldots + p_1 X + p_0 \quad , \quad Q = q_l X^l + \ldots + q_1 X + q_0$$

appears as the determinant of an associated matrix, as follows,

$$R(P,Q) = \begin{vmatrix} p_k & & & q_l & & \\ \vdots & \ddots & & \vdots & \ddots & \\ p_0 & & p_k & q_0 & & q_l \\ & \ddots & \vdots & & \ddots & \vdots \\ & & p_0 & & & q_0 \end{vmatrix}$$

with the matrix having size $k+l$, and having 0 coefficients at the blank spaces.

PROOF. This is something clever, due to Sylvester, as follows:

(1) Consider the vector space $\mathbb{C}_k[X]$ formed by the polynomials of degree $< k$:

$$\mathbb{C}_k[X] = \left\{ P \in \mathbb{C}[X] \middle| \deg P < k \right\}$$

This is a vector space of dimension k, having as basis the monomials $1, X, \ldots, X^{k-1}$. Now given polynomials P, Q as in the statement, consider the following linear map:

$$\Phi : \mathbb{C}_l[X] \times \mathbb{C}_k[X] \to \mathbb{C}_{k+l}[X] \quad , \quad (A,B) \to AP + BQ$$

(2) Our first claim is that with respect to the standard bases for all the vector spaces involved, namely those consisting of the monomials $1, X, X^2, \ldots$, the matrix of Φ is the matrix in the statement. But this is something which is clear from definitions.

(3) Our second claim is that $\det \Phi = 0$ happens precisely when P, Q have a common root. Indeed, our polynomials P, Q having a common root means that we can find A, B such that $AP + BQ = 0$, and so that $(A, B) \in \ker \Phi$, which reads $\det \Phi = 0$.

(4) Finally, our claim is that we have $\det \Phi = R(P,Q)$. But this follows from the uniqueness of the resultant, up to a scalar, and with this uniqueness property being elementary to establish, along the lines of the proofs of Theorems 10.2 and 10.3. □

As an illustration for this, consider our favorite polynomials, as before:

$$P = ax^2 + bx + c \quad , \quad Q = dx + e$$

According to the above result, the resultant should be then, as it should:

$$R(P,Q) = \begin{vmatrix} a & d & 0 \\ b & e & d \\ c & 0 & e \end{vmatrix} = ae^2 - bde + cd^2$$

We can go back now to our original question, regarding Δ, and we have:

THEOREM 10.5. *Given a polynomial $P \in \mathbb{C}[X]$, written as*

$$P(X) = aX^N + bX^{N-1} + cX^{N-2} + \ldots$$

its discriminant, defined as being the following quantity,

$$\Delta(P) = \frac{(-1)^{\binom{N}{2}}}{a} R(P, P')$$

is a polynomial in the coefficients of P, with integer coefficients, and $\Delta(P) = 0$ happens precisely when P has a double root.

PROOF. The fact that the discriminant $\Delta(P)$ is a polynomial in the coefficients of P, with integer coefficients, comes from Theorem 10.3, coupled with the fact that the division by the leading coefficient a is indeed possible, under $\mathbb{Z}$, as being shown by the following formula, which is written a bit informally, coming from Theorem 10.4:

$$R(P, P') = \begin{vmatrix} a & & & Na & & \\ \vdots & \ddots & & \vdots & \ddots & \\ z & & a & y & & Na \\ & \ddots & \vdots & & \ddots & \vdots \\ & & z & & & y \end{vmatrix}$$

Also, the fact that we have $\Delta(P) = 0$ precisely when P has a double root is clear from Theorem 10.3. Finally, let us mention that the sign $(-1)^{\binom{N}{2}}$ is there for various reasons, including the compatibility with some well-known formulae, at small values of $N \in \mathbb{N}$, such as $\Delta(P) = b^2 - 4ac$ in degree 2, that we will discuss in a moment. □

As a first illustration, let us see what happens in degree 2. We have:

$$P = aX^2 + bX + c \quad , \quad P' = 2aX + b$$

Thus, the discriminant is given by the following formula, as it should:

$$\begin{aligned} \Delta(P) &= -\frac{1}{a} \begin{vmatrix} a & 2a & \\ b & b & 2a \\ c & & b \end{vmatrix} \\ &= -\begin{vmatrix} 1 & 2 & \\ b & b & 2a \\ c & & b \end{vmatrix} \\ &= -b^2 + 2(b^2 - 2ac) \\ &= b^2 - 4ac \end{aligned}$$

We will be back later to such formulae, in degree 3, and in degree 4 as well, with the comment however, coming in advance, that these formulae are not very beautiful.

At the theoretical level now, we have the following result, which is not trivial:

THEOREM 10.6. *The discriminant of a polynomial P is given by the formula*

$$\Delta(P) = a^{2N-2}\prod_{i<j}(r_i - r_j)^2$$

where a is the leading coefficient, and $r_1, \ldots, r_N$ are the roots.

PROOF. This is something quite tricky, the idea being as follows:

(1) The first thought goes to the formula in Theorem 10.3, so let us see what that formula teaches us, in the case $Q = P'$. Let us write P, P' as follows:

$$P = a(x - r_1)\ldots(x - r_N)$$

$$P' = Na(x - p_1)\ldots(x - p_{N-1})$$

According to Theorem 10.3, the resultant of P, P' is then given by:

$$R(P, P') = a^{N-1}(Na)^N\prod_{ij}(r_i - p_j)$$

And bad news, this is not exactly what we wished for, namely the formula in the statement. That is, we are on the good way, but certainly have to work some more.

(2) Obviously, we must get rid of the roots $p_1, \ldots, p_{N-1}$ of the polynomial P'. In order to do this, let us rewrite the formula that we found in (1) in the following way:

$$\begin{aligned} R(P, P') &= N^N a^{2N-1}\prod_i\left(\prod_j(r_i - p_j)\right) \\ &= N^N a^{2N-1}\prod_i\frac{P'(r_i)}{Na} \\ &= a^{N-1}\prod_i P'(r_i) \end{aligned}$$

(3) In order to compute now P', and more specifically the values $P'(r_i)$ that we are interested in, we can use the Leibnitz rule. So, consider our polynomial:

$$P(x) = a(x - r_1)\ldots(x - r_N)$$

The Leibnitz rule for derivatives tells us that $(fg)' = f'g + fg'$, but then also that $(fgh)' = f'gh + fg'h + fgh'$, and so on. Thus, for our polynomial, we obtain:

$$P'(x) = a\sum_i(x - r_1)\ldots\underbrace{(x - r_i)}_{missing}\ldots(x - r_N)$$

Now when applying this formula to one of the roots r_i, we obtain:

$$P'(r_i) = a(r_i - r_1) \ldots \underbrace{(r_i - r_i)}_{missing} \ldots (r_i - r_N)$$

By making now the product over all indices i, this gives the following formula:

$$\prod_i P'(r_i) = a^N \prod_{i \neq j} (r_i - r_j)$$

(4) Time now to put everything together. By taking the formula in (2), making the normalizations in Theorem 10.5, and then using the formula found in (3), we obtain:

$$\begin{aligned} \Delta(P) &= (-1)^{\binom{N}{2}} a^{N-2} \prod_i P'(r_i) \\ &= (-1)^{\binom{N}{2}} a^{2N-2} \prod_{i \neq j} (r_i - r_j) \end{aligned}$$

(5) This is already a nice formula, which is very useful in practice, and that we can safely keep as a conclusion, to our computations. However, we can do slightly better, by grouping opposite terms. Indeed, this gives the following formula:

$$\begin{aligned} \Delta(P) &= (-1)^{\binom{N}{2}} a^{2N-2} \prod_{i \neq j} (r_i - r_j) \\ &= (-1)^{\binom{N}{2}} a^{2N-2} \prod_{i<j} (r_i - r_j) \cdot \prod_{i>j} (r_i - r_j) \\ &= (-1)^{\binom{N}{2}} a^{2N-2} \prod_{i<j} (r_i - r_j) \cdot (-1)^{\binom{N}{2}} \prod_{i<j} (r_i - r_j) \\ &= a^{2N-2} \prod_{i<j} (r_i - r_j)^2 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As applications now, the formula in Theorem 10.6 is quite useful for the real polynomials $P \in \mathbb{R}[X]$ in small degree, because it allows to say when the roots are real, or complex, or at least have some partial information about this. For instance, we have:

PROPOSITION 10.7. *Consider a polynomial with real coefficients, $P \in \mathbb{R}[X]$, assumed for simplicity to have nonzero discriminant, $\Delta \neq 0$.*

(1) *In degree 2, the roots are real when $\Delta > 0$, and complex when $\Delta < 0$.*
(2) *In degree 3, all roots are real precisely when $\Delta > 0$.*

PROOF. This is very standard, the idea being as follows:

(1) The first assertion is something that we certainly know, coming from Proposition 10.1, but let us see how this comes via the formula in Theorem 10.6, namely:

$$\Delta(P) = a^{2N-2}\prod_{i<j}(r_i - r_j)^2$$

In degree $N = 2$, this formula looks as follows, with r_1, r_2 being the roots:

$$\Delta(P) = a^2(r_1 - r_2)^2$$

Thus $\Delta > 0$ amounts in saying that we have $(r_1 - r_2)^2 > 0$. Now since r_1, r_2 are conjugate, and with this being something trivial, meaning no need here for the computations in Proposition 10.1, we conclude that $\Delta > 0$ means that r_1, r_2 are real, as stated.

(2) In degree $N = 3$ now, we know from analysis that P has at least one real root, and the problem is whether the remaining 2 roots are real, or complex conjugate. For this purpose, we can use the formula in Theorem 10.6, which in degree 3 reads:

$$\Delta(P) = a^4(r_1 - r_2)^2(r_1 - r_3)^2(r_2 - r_3)^2$$

We can see that in the case $r_1, r_2, r_3 \in \mathbb{R}$, we have $\Delta(P) > 0$. Conversely now, assume that $r_1 = r$ is the real root, coming from analysis, and that the other roots are $r_2 = z$ and $r_3 = \bar{z}$, with z being a complex number, which is not real. We have then:

$$\begin{aligned}\Delta(P) &= a^4(r - z)^2(r - \bar{z})^2(z - \bar{z})^2\\ &= a^4|r - z|^4(2iIm(z))^2\\ &= -4a^4|r - z|^4Im(z)^2\\ &< 0\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Finally, as yet another application of this, worth mentioning, we have:

THEOREM 10.8. *The diagonalizable matrices $A \in M_N(\mathbb{C})$ are dense.*

PROOF. There are actually two standard proofs here, as follows:

(1) Via the pedestrian way, by using the Jordan form. Here you have to learn well the Jordan form, and good luck with that, and once that done, you can argue that by perturbing the Jordan blocks, in the obvious way, you can arrange up to epsilon as for your matrix to have distinct eigenvalues, and so to be diagonalizable.

(2) As a geometry king, using the discriminant. Indeed, for a matrix $A \in M_N(\mathbb{C})$, with characteristic polynomial P_A, having distinct eigenvalues means:

$$\Delta(P_A) \neq 0$$

But this is the complement of a hypersurface, which is dense, and since all these matrices are diagonalizable, the diagonalizable matrices are dense too. Just like that. □

10b. Cardano formula

Let us work out now what happens in degree 3. We must first compute the discriminant of arbitrary degree 3 polynomials, and the result here is as follows:

THEOREM 10.9. *The discriminant of a degree* 3 *polynomial,*

$$P = aX^3 + bX^2 + cX + d$$

is given by $\Delta(P) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$.

PROOF. We have two methods here, both instructive, and we will try them both:

(1) Let us first go the pedestrian way, based on the definition of the resultant, from Theorem 10.3. Consider two polynomials, of degree 3 and degree 2, written as follows:

$$P = aX^3 + bX^2 + cX + d$$

$$Q = eX^2 + fX + g = e(X - s)(X - t)$$

The resultant of these two polynomials is then given by:

$$\begin{aligned} R(P,Q) &= a^2e^3(p-s)(p-t)(q-s)(q-t)(r-s)(r-t) \\ &= a^2 \cdot e(p-s)(p-t) \cdot e(q-s)(q-t) \cdot e(r-s)(r-t) \\ &= a^2Q(p)Q(q)Q(r) \\ &= a^2(ep^2+fp+g)(eq^2+fq+g)(er^2+fr+g) \end{aligned}$$

Next, we can use the following formulae, in order to get rid of p, q, r:

$$p + q + r = -\frac{b}{a} \quad , \quad pq + pr + qr = \frac{c}{a} \quad , \quad pqr = -\frac{d}{a}$$

Indeed, by using these formulae, we obtain, after some routine work:

$$\begin{aligned} R(P,Q) &= d^2e^3 - cde^2f + c^2e^2g - 2bde^2g + bdef^2 - bcefg + 3adefg \\ &- adf^3 + b^2eg^2 - 2aceg^2 + acf^2g - abfg^2 + a^2g^3 \end{aligned}$$

Getting back now to our discriminant problem, with $Q = P'$, which corresponds to $e = 3a$, $f = 2b$, $g = c$, we obtain the following formula:

$$\begin{aligned} R(P,P') &= 27a^3d^2 - 18a^2bcd + 9a^2c^3 - 18a^2bcd + 12ab^3d - 6ab^2c^2 + 18a^2bcd \\ &- 8ab^3d + 3ab^2c^2 - 6a^2c^3 + 4ab^2c^2 - 2ab^2c^2 + a^2c^3 \end{aligned}$$

By simplifying now terms, and dividing by $-a$, we obtain, as desired:

$$\Delta(P) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

(2) Let us see as well how the computation goes, by using Theorem 10.4, which is our most advanced tool, so far. Consider a polynomial of degree 3, and its derivative:

$$P = aX^3 + bX^2 + cX + d$$

$$P' = 3aX^2 + 2bX + c$$

By using Theorem 10.4 and computing the determinant, we obtain:

$$\begin{aligned}
R(P,P') &= \begin{vmatrix} a & & 3a & & \\ b & a & 2b & 3a & \\ c & b & c & 2b & 3a \\ d & c & & c & 2b \\ & d & & & c \end{vmatrix} \\
&= \begin{vmatrix} a & & & & \\ b & a & -b & 3a & \\ c & b & -2c & 2b & 3a \\ d & c & -3d & c & 2b \\ & d & & & c \end{vmatrix} \\
&= a\begin{vmatrix} a & -b & 3a & \\ b & -2c & 2b & 3a \\ c & -3d & c & 2b \\ d & & & c \end{vmatrix} \\
&= -ad\begin{vmatrix} -b & 3a & \\ -2c & 2b & 3a \\ -3d & c & 2b \end{vmatrix} + ac\begin{vmatrix} a & -b & 3a \\ b & -2c & 2b \\ c & -3d & c \end{vmatrix} \\
&= -ad(-4b^3 - 27a^2d + 12abc + 3abc) \\
&\quad +ac(-2ac^2 - 2b^2c - 9abd + 6ac^2 + b^2c + 6abd) \\
&= a(4b^3d + 27a^2d^2 - 15abcd + 4ac^3 - b^2c^2 - 3abcd) \\
&= a(4b^3d + 27a^2d^2 - 18abcd + 4ac^3 - b^2c^2)
\end{aligned}$$

Now according to Theorem 10.5, the discriminant of our polynomial is given by:

$$\begin{aligned}
\Delta(P) &= -\frac{R(P,P')}{a} \\
&= -4b^3d - 27a^2d^2 + 18abcd - 4ac^3 + b^2c^2 \\
&= b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd
\end{aligned}$$

Thus, we have again obtained the formula in the statement. □

Still talking degree 3 equations, let us try now to solve such an equation $P = 0$, with $P = aX^3 + bX^2 + cX + d$ as above. By linear transformations we can assume $a = 1, b = 0$, and then it is convenient to write $c = 3p, d = 2q$. Thus, our equation becomes:

$$x^3 + 3px + 2q = 0$$

Regarding such equations, many things can be said, and to start with, we have the following famous result, dealing with real roots, due to Cardano:

THEOREM 10.10. *For a normalized degree 3 equation, namely*

$$x^3+3px+2q=0$$

the discriminant is $\Delta=-108(p^3+q^2)$. Assuming $p,q\in\mathbb{R}$ and $\Delta<0$, the number

$$x=\sqrt[3]{-q+\sqrt{p^3+q^2}}+\sqrt[3]{-q-\sqrt{p^3+q^2}}$$

is a real solution of our equation.

PROOF. The formula of Δ is clear from the above, and with $108=4\times 27$. Now with x as in the statement, by using $(a+b)^3=a^3+b^3+3ab(a+b)$, we have:

$$\begin{aligned}x^3&=\left(\sqrt[3]{-q+\sqrt{p^3+q^2}}+\sqrt[3]{-q-\sqrt{p^3+q^2}}\right)^3\\&=-2q+3\sqrt[3]{-q+\sqrt{p^3+q^2}}\cdot\sqrt[3]{-q-\sqrt{p^3+q^2}}\cdot x\\&=-2q+3\sqrt[3]{q^2-p^3-q^2}\cdot x\\&=-2q-3px\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding the other roots, we know from Proposition 10.7 that these are both real when $\Delta<0$, and complex conjugate when $\Delta<0$. Thus, in the context of Theorem 10.10, the other two roots are complex conjugate, the formula for them being as follows:

PROPOSITION 10.11. *For a normalized degree 3 equation, namely*

$$x^3+3px+2q=0$$

with $p,q\in\mathbb{R}$ and discriminant $\Delta=-108(p^3+q^2)$ negative, $\Delta<0$, the numbers

$$z=w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}$$

$$\bar{z}=w^2\sqrt[3]{-q+\sqrt{p^3+q^2}}+w\sqrt[3]{-q-\sqrt{p^3+q^2}}$$

with $w=e^{2\pi i/3}$ are the complex conjugate solutions of our equation.

PROOF. As before, by using $(a+b)^3=a^3+b^3+3ab(a+b)$, we have:

$$\begin{aligned}z^3&=\left(w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}\right)^3\\&=-2q+3\sqrt[3]{-q+\sqrt{p^3+q^2}}\cdot\sqrt[3]{-q-\sqrt{p^3+q^2}}\cdot z\\&=-2q+3\sqrt[3]{q^2-p^3-q^2}\cdot z\\&=-2q-3pz\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a conclusion, we have the following statement, unifying the above:

THEOREM 10.12. *For a normalized degree* 3 *equation, namely*

$$x^3+3px+2q=0$$

the discriminant is $\Delta=-108(p^3+q^2)$. *Assuming* $p,q\in\mathbb{R}$ *and* $\Delta<0$, *the numbers*

$$x=w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}$$

with $w=1,e^{2\pi i/3},e^{4\pi i/3}$ *are the solutions of our equation.*

PROOF. This follows indeed from Theorem 10.10 and Proposition 10.11. Alternatively, we can redo the computation in their proof, which was nearly identical anyway, in the present setting, with x being given by the above formula, by using $w^3=1$. □

As a comment here, the formula in Theorem 10.12 holds of course in the case $\Delta>0$ too, and also when the coefficients are complex numbers, $p,q\in\mathbb{C}$, and this due to the fact that the proof rests on the nearly trivial computation from the proof of Theorem 10.10, or of Proposition 10.11. However, these extensions are quite often not very useful, because when it comes to extract all the above square and cubic roots, for complex numbers, you can well end up with the initial question, the one that you started with.

Thus, as a conclusion to this, Theorem 10.12 as formulated above is what can be best said about the degree 3 equations. There are of course many versions of it, and slight generalizations, but in practice, Theorem 10.12 is what mostly matters.

10c. Higher degree

In higher degree things become quite complicated. In degree 4, to start with, we first have the following result, dealing with the discriminant and its applications:

THEOREM 10.13. *The discriminant of* $P=ax^4+bx^3+cx^2+dx+e$ *is given by the following formula:*

$$\begin{aligned}\Delta &= 256a^3e^3-192a^2bde^2-128a^2c^2e^2+144a^2cd^2e-27a^2d^4\\ &\quad +144ab^2ce^2-6ab^2d^2e-80abc^2de+18abcd^3+16ac^4e\\ &\quad -4ac^3d^2-27b^4e^2+18b^3cde-4b^3d^3-4b^2c^3e+b^2c^2d^2\end{aligned}$$

In the case $\Delta<0$ *we have* 2 *real roots and* 2 *complex conjugate roots, and in the case* $\Delta>0$ *the roots are either all real or all complex.*

PROOF. The formula of Δ follows from the definition of the discriminant, from Theorem 3.10, with the resultant computed via Theorem 3.10, as follows:

$$\Delta=\frac{1}{a}\begin{vmatrix}
a & & & 4a & & & \\
b & a & & 3b & 4a & & \\
c & b & a & 2c & 3b & 4a & \\
d & c & b & d & 2c & 3b & 4a\\
e & d & c & & d & 2c & 3b\\
 & e & d & & & d & 2c\\
 & & e & & & & d
\end{vmatrix}$$

As for the last assertion, the study here is routine, a bit as in degree 3. □

In practice, as in degree 3, we can do first some manipulations on our polynomials, as to have them in simpler form, and we have the following version of Theorem 10.13:

PROPOSITION 10.14. *The discriminant of $P=x^4+cx^2+dx+e$, normalized degree 4 polynomial, is given by the following formula:*

$$\Delta=16c^4e-4c^3d^2-128c^2e^2+144cd^2e-27d^4+256e^3$$

As before, if $\Delta<0$ we have 2 real roots and 2 complex conjugate roots, and if $\Delta>0$ the roots are either all real or all complex.

PROOF. This is a consequence of Theorem 10.13, with $a=1,b=0$, but we can deduce this as well directly. Indeed, the formula of Δ follows, quite easily, from:

$$\Delta=\begin{vmatrix}
1 & & & 4 & & & \\
 & 1 & & & 4 & & \\
c & & 1 & 2c & & 4 & \\
d & c & & d & 2c & & 4\\
e & d & c & & d & 2c & \\
 & e & d & & & d & 2c\\
 & & e & & & & d
\end{vmatrix}$$

As for the last assertion, this is something that we know, from Theorem 10.13. □

We still have some work to do. Indeed, looking back at what we did in degree 3, the passage there from Theorem 10.9 to Theorem 10.10 was made of two operations, namely "depressing" the equation, that is, getting rid of the next-to-highest term, and then rescaling the coefficients, as for the formula of Δ to become as simple as possible.

In our present setting now, degree 4, with the depressing done as above, in Proposition 10.14, it remains to rescale the coefficients, as for the formula of Δ to become as simple as possible. And here, a bit of formula hunting, in relation with $2,3$ powers, leads to:

THEOREM 10.15. *The discriminant of a normalized degree* 4 *polynomial, written as*

$$P = x^4 + 6px^2 + 4qx + 3r$$

is given by the following formula:

$$\Delta = 256 \times 27 \times \left(9p^4r - 2p^3q^2 - 6p^2r^2 + 6pq^2r - q^4 + r^3\right)$$

In the case $\Delta < 0$ *we have* 2 *real roots and* 2 *complex conjugate roots, and in the case* $\Delta > 0$ *the roots are either all real or all complex.*

PROOF. This follows from Proposition 10.14, with $c = 6p, d = 4q, e = 3r$, but we can deduce this as well directly. Indeed, the formula of Δ follows, quite easily, from:

$$\Delta = \begin{vmatrix} 1 & & & 4 & & & \\ & 1 & & & 4 & & \\ 6p & & 1 & 12p & & 4 & \\ 4q & 6p & & 4q & 12p & & 4 \\ 3r & 4q & 6p & & 4q & 12p & \\ & 3r & 4q & & & 4q & 12p \\ & & 3r & & & & 4q \end{vmatrix}$$

As for the last assertion, this is something that we know from Theorem 10.13. □

Time now to get to the real thing, solving the equation. We have here:

THEOREM 10.16. *The roots of a normalized degree* 4 *equation, written as*

$$x^4 + 6px^2 + 4qx + 3r = 0$$

are as follows, with y *satisfying the equation* $(y^2 - 3r)(y - 3p) = 2q^2$*,*

$$x_1 = \frac{1}{\sqrt{2}}\left(-\sqrt{y - 3p} + \sqrt{-y - 3p + \frac{4q}{\sqrt{2y - 6p}}}\right)$$

$$x_2 = \frac{1}{\sqrt{2}}\left(-\sqrt{y - 3p} - \sqrt{-y - 3p + \frac{4q}{\sqrt{2y - 6p}}}\right)$$

$$x_3 = \frac{1}{\sqrt{2}}\left(\sqrt{y - 3p} + \sqrt{-y - 3p - \frac{4q}{\sqrt{2y - 6p}}}\right)$$

$$x_4 = \frac{1}{\sqrt{2}}\left(\sqrt{y - 3p} - \sqrt{-y - 3p - \frac{4q}{\sqrt{2y - 6p}}}\right)$$

and with y *being computable via the Cardano formula.*

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, let us write our equation in the following form:

$$x^4 = -6px^2 - 4qx - 3r$$

The idea will be that of adding a suitable common term, to both sides, as to make square on both sides, as to eventually end with a sort of double quadratic equation. For this purpose, our claim is that what we need is a number y satisfying:

$$(y^2 - 3r)(y - 3p) = 2q^2$$

Indeed, assuming that we have this number y, our equation becomes:

$$\begin{aligned}(x^2 + y)^2 &= x^4 + 2x^2y + y^2\\ &= -6px^2 - 4qx - 3r + 2x^2y + y^2\\ &= (2y - 6p)x^2 - 4qx + y^2 - 3r\\ &= (2y - 6p)x^2 - 4qx + \frac{2q^2}{y - 3p}\\ &= \left(\sqrt{2y - 6p}\cdot x - \frac{2q}{\sqrt{2y - 6p}}\right)^2\end{aligned}$$

(2) Which looks very good, leading us to the following degree 2 equations:

$$x^2 + y + \sqrt{2y - 6p}\cdot x - \frac{2q}{\sqrt{2y - 6p}} = 0$$

$$x^2 + y - \sqrt{2y - 6p}\cdot x + \frac{2q}{\sqrt{2y - 6p}} = 0$$

Now let us write these two degree 2 equations in standard form, as follows:

$$x^2 + \sqrt{2y - 6p}\cdot x + \left(y - \frac{2q}{\sqrt{2y - 6p}}\right) = 0$$

$$x^2 - \sqrt{2y - 6p}\cdot x + \left(y + \frac{2q}{\sqrt{2y - 6p}}\right) = 0$$

(3) Regarding the first equation, the solutions there are as follows:

$$x_1 = \frac{1}{2}\left(-\sqrt{2y - 6p} + \sqrt{-2y - 6p + \frac{8q}{\sqrt{2y - 6p}}}\right)$$

$$x_2 = \frac{1}{2}\left(-\sqrt{2y - 6p} - \sqrt{-2y - 6p + \frac{8q}{\sqrt{2y - 6p}}}\right)$$

As for the second equation, the solutions there are as follows:

$$x_3 = \frac{1}{2}\left(\sqrt{2y-6p} + \sqrt{-2y-6p-\frac{8q}{\sqrt{2y-6p}}}\right)$$

$$x_4 = \frac{1}{2}\left(\sqrt{2y-6p} - \sqrt{-2y-6p-\frac{8q}{\sqrt{2y-6p}}}\right)$$

(4) Now by cutting a $\sqrt{2}$ factor from everything, this gives the formulae in the statement. As for the last claim, regarding the nature of y, this comes from Cardano. □

We still have to compute the number y appearing in the above via Cardano, and the result here, adding to what we already have in Theorem 10.16, is as follows:

THEOREM 10.17 (continuation). *The value of y in the previous theorem is*

$$y = t + p + \frac{a}{t}$$

where the number t is given by the formula

$$t = \sqrt[3]{b + \sqrt{b^2 - a^3}}$$

with $a = p^2 + r$ and $b = 2p^2 - 3pr + q^2$.

PROOF. The legend has it that this is what comes from Cardano, but depressing and normalizing and solving $(y^2 - 3r)(y - 3p) = 2q^2$ makes it for too many operations, so the most pragmatic is to simply check this equation. With y as above, we have:

$$\begin{aligned} y^2 - 3r &= t^2 + 2pt + (p^2 + 2a) + \frac{2pa}{t} + \frac{a^2}{t^2} - 3r \\ &= t^2 + 2pt + (3p^2 - r) + \frac{2pa}{t} + \frac{a^2}{t^2} \end{aligned}$$

With this in hand, we have the following computation:

$$\begin{aligned} (y^2 - 3r)(y - 3p) &= \left(t^2 + 2pt + (3p^2 - r) + \frac{2pa}{t} + \frac{a^2}{t^2}\right)\left(t - 2p + \frac{a}{t}\right) \\ &= t^3 + (a - 4p^2 + 3p^2 - r)t + (2pa - 6p^3 + 2pr + 2pa) \\ &\quad + (3p^2 a - ra - 4p^2 a + a^2)\frac{1}{t} + \frac{a^3}{t^3} \\ &= t^3 + (a - p^2 - r)t + 2p(2a - 3p^2 + r) + a(a - p^2 - r)\frac{1}{t} + \frac{a^3}{t^3} \\ &= t^3 + 2p(-p^2 + 3r) + \frac{a^3}{t^3} \end{aligned}$$

Now by using the formula of t in the statement, this gives:

$$\begin{aligned}(y^2-3r)(y-3p) &= b+\sqrt{b^2-a^3}-4p^2+6pr+\frac{a^3}{b+\sqrt{b^2-a^3}}\\ &= b+\sqrt{b^2-a^3}-4p^2+6pr+b-\sqrt{b^2-a^3}\\ &= 2b-4p^2+6pr\\ &= 2(2p^2-3pr+q^2)-4p^2+6pr\\ &= 2q^2\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

10d. Plane curves

No discussion about polynomials would be complete without a word on plane curves, and basic algebraic geometry in general, so let us get now into this. Let us start with something very standard, which is as old as modern science, namely:

THEOREM 10.18. *The conics, which are the degree* 2 *curves in the plane,*

$$C=\left\{(x,y)\in\mathbb{R}^2\Big|P(x,y)=0\right\}$$

with $\deg P\leq 2$*, have the following properties, modulo degeneration:*

(1) *They appear by cutting a* 2*-sided cone with a plane.*
(2) *They can be classified into ellipses, parabolas and hyperbolas.*
(3) *Planets and other celestial bodies move around the Sun on conics.*

PROOF. Tough theorem that we have here, with countless generations of humans thinking about this, since the Stone Age, then eventually with the old Greeks clarifying (1) and (2), then again with countless people thinking about this, with some of them ending up being burned at stake, and finally with Kepler and Newton clarifying (3). So, these are our heroes, and as the least of things, please learn about their findings:

(1) By suitably choosing our coordinate axes (x,y,z), we can assume that our cone is given by an equation as follows, with $k>0$:

$$x^2+y^2=kz^2$$

In order to prove the result, we must in principle intersect this cone with an arbitrary plane, which has an equation as follows, with $(a,b,c)\neq(0,0,0)$:

$$ax+by+cz=d$$

However, before getting into computations, observe that what we want to find is a certain degree 2 equation in the above plane, for the intersection. Thus, it is convenient to change the coordinates, as for our plane to be given by the following equation:

$$z=0$$

But with this done, what we have to do is to see how the cone equation $x^2+y^2=kz^2$ changes, under this change of coordinates, and then set $z=0$, as to get the (x,y) equation of the intersection. But this leads to the conclusion that the cone equation $x^2+y^2=kz^2$ becomes in this way a degree 2 equation in (x,y), which can be arbitrary, as claimed.

(2) Let us first classify the conics up to non-degenerate linear transformations of the plane. So, consider an arbitrary conic, written as follows, with $a,b,c,d,e,f\in\mathbb{R}$:

$$ax^2+by^2+cxy+dx+ey+f=0$$

Assume first $a\neq 0$. By making a square out of ax^2, up to a linear transformation in (x,y), we can get rid of the term cxy, and we are left with:

$$ax^2+by^2+dx+ey+f=0$$

In the case $b\neq 0$ we can make two obvious squares, and again up to a linear transformation in (x,y), we are left with an equation as follows:

$$x^2\pm y^2=k$$

In the case of positive sign, $x^2+y^2=k$, the solutions are the circle, when $k\geq 0$, the point, when $k=0$, and $\emptyset$, when $k<0$. As for the case of negative sign, $x^2-y^2=k$, which reads $(x-y)(x+y)=k$, here once again by linearity our equation becomes $xy=l$, which is a hyperbola when $l\neq 0$, and two lines when $l=0$. The case $b\neq 0$ being similar, we are left with the case $a=b=0$. Here our conic is as follows, with $c,d,e,f\in\mathbb{R}$:

$$cxy+dx+ey+f=0$$

If $c\neq 0$, by linearity our equation becomes $xy=l$, which produces a hyperbola or two lines, as explained before. As for the remaining case, $c=0$, here our equation is:

$$dx+ey+f=0$$

But this is generically the equation of a line, unless we are in the case $d=e=0$, where our equation is $f=0$, having as solutions $\emptyset$ when $f\neq 0$, and $\mathbb{R}^2$ when $f=0$. Thus, done, and our classification up to linear transformations leads to the classification in general too, by applying linear transformations to the solutions that we found.

(3) According to Kepler and Newton, the force of attraction between two bodies of masses M,m at distance d is given by the following formula, G being a constant:

$$||F||=G\cdot\frac{Mm}{d^2}$$

Thus the equation of motion of m, assuming that M is fixed at 0, is:

$$\ddot{x}=-\frac{GMx}{||x||^3}$$

And the point is that this equation can be solved by using polar coordinates, good exercise here for you, and we are led in this way to the conclusion in the statement. □

Summarizing, the conics are at the core of everything, mathematics, number theory, physics, life. But, what is next? A natural answer to this question comes from:

DEFINITION 10.19. *An algebraic curve in $\mathbb{R}^2$ is the vanishing set*

$$C = \left\{(x,y) \in \mathbb{R}^2 \middle| P(x,y) = 0\right\}$$

of a polynomial $P \in \mathbb{R}[X,Y]$ of arbitrary degree.

As a first question now, what results from what we know about conics have a chance to be relevant to the arbitrary algebraic curves? And here, normally none, because the conics are obviously very particular curves, having very particular properties.

Let us record however a useful statement here, as follows:

PROPOSITION 10.20. *The conics can be written in cartesian, polar, parametric or complex coordinates, with the equations for the unit circle being*

$$x^2 + y^2 = 1 \quad , \quad r = 1 \quad , \quad x = \cos t \, , \, y = \sin t \quad , \quad |z| = 1$$

and with the equations for ellipses, parabolas and hyperbolas being similar.

PROOF. The equations for the circle are clear, and we will leave as an instructive exercise working out those for ellipses, parabolas and hyperbolas. □

As a true answer to our question now, coming this time from a very modest conic, namely $xy = 0$, that we dismissed in the above as being "degenerate", we have:

THEOREM 10.21. *The following happen, for curves C defined by polynomials P:*

(1) *In degree $d = 2$, curves can have singularities, such as $xy = 0$ at $(0,0)$.*
(2) *In general, assuming $P = P_1 \ldots P_k$, we have $C = C_1 \cup \ldots\ldots \cup C_k$.*
(3) *A union of curves $C_i \cup C_j$ is generically non-smooth, unless disjoint.*
(4) *Due to this, we say that C is non-degenerate when P is irreducible.*

PROOF. All this is self-explanatory, the details being as follows:

(1) This is something obvious, just the story of two lines crossing.

(2) This comes from the following trivial fact, with the notation $z = (x,y)$:

$$P_1 \ldots P_k(z) = 0 \iff P_1(z) = 0, \text{ or } P_2(z) = 0, \ \ldots \ , \text{ or } P_k(z) = 0$$

(3) This is something very intuitive, and it actually takes a bit of time to imagine a situation where $C_1 \cap C_2 \neq \emptyset$, $C_1 \not\subset C_2$, $C_2 \not\subset C_1$, but $C_1 \cup C_2$ is smooth. In practice now, "generically" has of course a mathematical meaning, in relation with probability, and our assertion does say something mathematical, that we are supposed to prove. But, we will not insist on this, and leave this as an instructive exercise, precise formulation of the claim, and its proof, in the case you are familiar with probability theory.

(4) This is just a definition, based on the above, that we will use in what follows. □

With degree 1 and 2 investigated, and our conclusions recorded, let us get now to degree 3, see what new phenomena appear here. And here, to start with, we have the following remarkable curve, well-known from calculus, because 0 is not a maximum or minimum of the function $x \to y$, despite the derivative vanishing there:

$$x^3 = y$$

Also, in relation with set theory and logic, and with the foundations of mathematics in general, we have the following curve, which looks like the empyset $\emptyset$:

$$(x - y)(x^2 + y^2 - 1) = 0$$

But, it is not about counterexamples to calculus, or about logic, that we want to talk about here. As a first truly remarkable degree 3 curve, or cubic, we have the cusp:

PROPOSITION 10.22. *The standard cusp, which is the cubic given by*

$$x^3 = y^2$$

has a singularity at $(0, 0)$*, with only* 1 *tangent line at that singularity.*

PROOF. The two branches of the cusp are indeed both tangent to Ox, because:

$$y' = \pm\frac{3}{2}\sqrt{x} \implies y'(0) = 0$$

Observe also that what happens for the cusp is different from what happens for $xy = 0$, precisely because we have 1 line tangent at the singularity, instead of 2. □

As a second remarkable cubic, which gets the crown, and the right to have a Theorem about it, we have the Tschirnhausen curve, which is as follows:

THEOREM 10.23. *The Tschirnhausen cubic, given by the following equation,*

$$x^3 = x^2 - 3y^2$$

makes the dream of $xy = 0$ *come true, by self-intersecting, and being non-degenerate.*

PROOF. This is something self-explanatory, by drawing a picture, but there are several other interesting things that can be said about this curve, as follows:

(1) Let us start with the curve written in polar coordinates as follows:

$$r\cos^3\left(\frac{\theta}{3}\right) = a$$

With $t = \tan(\theta/3)$, the equations of the coordinates are as follows:

$$x = a(1 - 3t^2) \quad , \quad y = at(3 - t^2)$$

Now by eliminating t, we reach to the following equation:

$$(a - x)(8a + x)^2 = 27ay^2$$

(2) By translating horizontally by $8a$, and changing signs of variables, we have:

$$x = 3a(3 - t^2) \quad , \quad y = at(3 - t^2)$$

Now by eliminating t, we reach to the following equation:

$$x^3 = 9a(x^2 - 3y^2)$$

But with $a = 1/9$ this is precisely the equation in the statement. □

In degree 4 now, quartics, we have enough dimensions for "improving" the cusp and the Tschirnhausen curve. First we have the cardioid, which is as follows:

PROPOSITION 10.24. *The cardioid, which is a quartic, given in polar coordinates by*

$$2r = a(1 - \cos\theta)$$

makes the dream of $x^3 = y^2$ come true, by being a closed curve, with a cusp.

PROOF. As before with the Tschirnhausen curve, this is something self-explanatory, by drawing a picture, but there are several things that must be said, as follows:

(1) The cardioid appears by definition by rolling a circle of radius $c > 0$ around another circle of same radius $c > 0$. With θ being the rolling angle, we have:

$$x = 2c(1 - \cos\theta)\cos\theta \quad , \quad y = 2c(1 - \cos\theta)\sin\theta$$

(2) Thus, in polar coordinates we get the equation in the statement, with $a = 4c$:

$$r = 2c(1 - \cos\theta)$$

(3) Finally, in cartesian coordinates, the equation is as follows:

$$(x^2 + y^2)^2 + 4cx(x^2 + y^2) = 4c^2y^2$$

Thus, what we have is indeed a degree 4 curve, as claimed. □

Still in degree 4, the crown gets to the Bernoulli lemniscate, which is as follows:

THEOREM 10.25. *The Bernoulli lemniscate, a quartic, which is given by*

$$r^2 = a^2\cos 2\theta$$

makes the dream of $x^3 = x^2 - 3y^2$ come true, by being closed, and self-intersecting.

PROOF. As usual, this is something self-explanatory, by drawing a picture, which looks like ∞, but there are several other things that must be said, as follows:

(1) In cartesian coordinates, the equation is as follows, with $a^2 = 2c^2$:

$$(x^2 + y^2)^2 = c^2(x^2 - y^2)$$

(2) Also, we have the following nice complex reformulation of this equation:

$$|z + c| \cdot |z - c| = c^2$$

Thus, we are led to the conclusions in in the statement. □

In degree 5, in the lack of any spectacular quintic, let us record:

THEOREM 10.26. *Unlike in degree* $3, 4$*, where equations can be solved, by the Cardano formula, in degree* 5 *this generically does not happen, an example being*

$$x^5 - x - 1 = 0$$

having Galois group S_5*, not solvable. Geometrically, this tells us that the intersection of the quintic* $y = x^5 - x - 1$ *with the line* $y = 0$ *cannot be computed.*

PROOF. Obviously off-topic, but with no good quintic available, and still a few more minutes before the bell ringing, I had to improvise a bit, and tell you about this. □

Back now to our usual business, in degree 6, sextics, we first have here:

PROPOSITION 10.27. *The trefoil sextic, or Kiepert curve, which is given by*

$$r^3 = a^3 \cos 3\theta$$

looks like a trefoil, closed curve, with a triple self-intersection.

PROOF. As before, drawing a picture is mandatory. With $z = re^{i\theta}$ we have:

$$\begin{aligned}
r^3 = a^3 \cos 3\theta &\iff r^3 \cos 3\theta = \left(\frac{r^2}{a}\right)^3 \\
&\iff z^3 + \bar{z}^3 = 2\left(\frac{z\bar{z}}{a}\right)^3 \\
&\iff (x+iy)^3 + (x-iy)^3 = 2\left(\frac{x^2+y^2}{a}\right)^3 \\
&\iff x^3 - 3xy^2 = \left(\frac{x^2+y^2}{a}\right)^3 \\
&\iff (x^2+y^2)^3 = a^3(x^3 - 3xy^2)
\end{aligned}$$

Thus, we have indeed a sextic, as claimed. □

We also have in degree 6 the most beautiful of curves them all, the Cayley sextic:

THEOREM 10.28. *The Cayley sextic, given in polar coordinates by*

$$r = a\cos^3\left(\frac{\theta}{3}\right)$$

makes the dream of everyone come true, by looking like a self-intersecting heart.

PROOF. As before, picture mandatory. With $z = re^{i\theta}$ and $u = z^{1/3}$ we have:

$$\begin{aligned}
r = a\cos^3\left(\frac{\theta}{3}\right) &\iff ar\cos^3\left(\frac{\theta}{3}\right) = r^2\\
&\iff a\left(\frac{u+\bar{u}}{2}\right)^3 = r^2\\
&\iff a(u^3+\bar{u}^3+3u\bar{u}(u+\bar{u})) = 8r^2\\
&\iff 3au\bar{u}\cdot\frac{u+\bar{u}}{2} = 4r^2 - ax\\
&\iff 27a^3r^6\cdot\frac{r^2}{a} = (4r^2-ax)^3\\
&\iff 27a^2(x^2+y^2)^2 = (4x^2+4y^2-ax)^3
\end{aligned}$$

Thus, we have indeed a sextic, as claimed. □

Quite remarkably, most of the above curves are sinusoidal spirals, in the following sense, and with actually the term "sinusoidal spiral" being a bit unfortunate:

THEOREM 10.29. *The sinusoidal spirals, which are as follows,*

$$r^n = a^n\cos n\theta$$

with $a \neq 0$ and $n \in \mathbb{Q} - \{0\}$, include the following curves:

(1) $n = -1$ *line.*
(2) $n = 1$ *circle,* $n = -1/2$ *parabola,* $n = -2$ *hyperbola.*
(3) $n = -3$ *Humbert cubic,* $n = -1/3$ *Tschirnhausen curve.*
(4) $n = 1/2$ *cardioid,* $n = 2$ *Bernoulli lemniscate.*
(5) $n = 3$ *Kiepert trefoil,* $n = 1/3$ *Cayley sextic.*

PROOF. We first have to prove that the sinusoidal spirals are indeed algebraic curves. But this is best done by using the complex coordinate $z = re^{i\theta}$, as follows:

$$\begin{aligned}
r^n = a^n\cos n\theta &\iff r^n\cos n\theta = \left(\frac{r^2}{a}\right)^n\\
&\iff z^n + \bar{z}^n = 2\left(\frac{z\bar{z}}{a}\right)^n\\
&\iff (x+iy)^n + (x-iy)^n = 2\left(\frac{x^2+y^2}{a}\right)^n
\end{aligned}$$

As a first observation now, in the case $n \in \mathbb{N}$ we can simply use the binomial formula, and we get an algebraic equation of degree $2n$, as follows:

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k} = \left(\frac{x^2+y^2}{a}\right)^n$$

In general, things are a bit more complicated, as shown for instance by our computation for the Cayley sextic. However, the same idea as there applies, and we are led in this way to the equation of an algebraic curve, as claimed. Regarding now the examples:

(1) At $n=-1$ the equation is as follows, producing a line:

$$r\cos\theta=a\iff x=a$$

(2) At $n=1$ the equation is as follows, producing a circle:

$$r=a\cos\theta\iff r^2=ax\iff x^2+y^2=ax$$

(3) At $n=-1/2$ the equation is as follows, producing a parabola:

$$a=r\cos^2(\theta/2)\iff r+x=2a\iff y^2=4a(a-x)$$

(4) At $n=-2$ the equation is as follows, producing a hyperbola:

$$a^2=r\cos^2 2\theta\iff a^2=2x^2-r^2\iff (x+y)(x-y)=a^2$$

(5) At $n=-3$ the equation is as follows, producing a curve with 3 components, which looks like some sort of "trivalent hyperbola", called Humbert cubic:

$$r^3\cos 3\theta=a^3\iff z^3+\bar{z}^3=2a^3\iff x^3-3xy^2=a^3$$

(6) As for the other curves, this follows from our various formulae above. □

10e. Exercises

This was a quite tricky, and very useful chapter. As exercises, we have:

Exercise 10.30. *Learn more tricks for degree 2 equations, which yes, do exist.*

Exercise 10.31. *Clarify what we said, in relation with symmetric polynomials.*

Exercise 10.32. *Compute more resultants, using both the available methods.*

Exercise 10.33. *Clarify the various normalizations, in the definition of $\Delta(P)$.*

Exercise 10.34. *Learn more about degree 3, including trigonometric aspects.*

Exercise 10.35. *Learn more about degree 4 too, including trigonometric aspects.*

Exercise 10.36. *Learn more about sinusoidal spirals, and their properties.*

Exercise 10.37. *Learn as well about polynomial lemniscates, and stelloids.*

As bonus exercise, find and start reading a nice, old-style algebraic geometry book.

CHAPTER 11

Gauss sums

11a. Gauss sums

Time for the complex numbers to strike again, with some non-trivial applications to the Legendre symbols. We recall from chapter 8 that these symbols, motivated by the equation $a = b^2(p)$ over the integers, with p prime, were constructed as follows:

DEFINITION 11.1. *The Legendre symbol is defined as follows,*

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } \exists\, b \neq 0, a = b^2(p) \\ 0 & \text{if } a = 0(p) \\ -1 & \text{if } \nexists\, b, a = b^2(p) \end{cases}$$

with $p \geq 3$ prime.

We have learned many things about these symbols, in chapter 8, with the summary of what we know from there about them being as follows:

THEOREM 11.2. *The Legendre symbols are given by the Euler formula:*

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}}(p)$$

They are subject to upper multiplicativity and quadratic reciprocity,

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \quad , \quad \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

and together with the Euler formula at $a = -1$, and with the extra formula

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p = 1, 7(8) \\ -1 & \text{if } p = 3, 5(8) \end{cases}$$

this fully computes them, in practice.

PROOF. This is something that we know from chapter 8, the idea being as follows:

(1) The Euler formula follows from Fermat, $a^{p-1} = 1(p)$ for $a \neq 0(p)$, by factorizing, $(a^{\frac{p-1}{2}} - 1)(a^{\frac{p-1}{2}} + 1) = 0(p)$, and doing some counting work inside $\mathbb{F}_p$.

(2) The multiplicativity of the Legendre symbol in its upper variable is clear from the Euler formula, because $a^{\frac{p-1}{2}}(p)$ is obviously multiplicative in $a \in \mathbb{Z}$.

(3) The quadratic reciprocity formula is something non-trivial, but we eventually managed to prove that in chapter 8, following Gauss, Eisenstein and others.

(4) Regarding now the explicit computation of the Legendre symbol, the Euler formula remains something theoretical, except at $a=-1$, where this formula gives:

$$\left(\frac{-1}{p}\right)=\begin{cases}1 & \text{if } p=1(4)\\ -1 & \text{if } p=3(4)\end{cases}$$

(5) In general, the Legendre symbol can be computed by recurrence, using multiplicativity and quadratic reciprocity, with a sample computation here being as follows:

$$\begin{aligned}\left(\frac{15}{37}\right) &= \left(\frac{3}{37}\right)\left(\frac{5}{37}\right)\\ &= (-1)^{\frac{3-1}{2}\cdot\frac{37-1}{2}}\left(\frac{37}{3}\right)(-1)^{\frac{5-1}{2}\cdot\frac{37-1}{2}}\left(\frac{37}{5}\right)\\ &= \left(\frac{37}{3}\right)\left(\frac{37}{5}\right)\\ &= \left(\frac{1}{3}\right)\left(\frac{2}{5}\right)\\ &= -1\end{aligned}$$

(6) However, and as illustrated by the above example, when doing the recurrence we need as input an explicit formula at $a=2$. And the formula here, which is the one in the statement, can be obtained by using some wild tricks, explained in chapter 8. □

Summarizing, we have a quite complete theory for the Legendre symbols, ready to be used in practice, but when looking at the above proofs, some of them remain a bit mysterious, and we are led in this way to the following theoretical question:

QUESTION 11.3. *Can we have a better understanding of the $a=2$ formula? And, why not, of the quadratic reciprocity formula too?*

These two questions are in fact related, as we will soon discover, and we will discuss them here. Let us start with the $a=2$ formula. The result here is as follows:

THEOREM 11.4. *We have the following formula, as announced above,*

$$\left(\frac{2}{p}\right)=\begin{cases}1 & \text{if } p=1,7(8)\\ -1 & \text{if } p=3,5(8)\end{cases}$$

providing the missing piece for the full computation of the Legendre symbols.

PROOF. This is something from chapter 8, coming via a quite mysterious proof, but now that we know about complex numbers, we can slightly improve that, as follows:

(1) Let us set $w = e^{\pi i/4}$, so that $w^2 = i$, and $t = w + w^{-1}$. We have of course $t = \sqrt{2}$, but it is better to forget this, and do formal arithmetic instead. We first have:

$$t^2 = 2 + w^2 + w^{-2} = 2$$

By using the Euler formula for the Legendre symbol, we obtain:

$$\left(\frac{2}{p}\right) = 2^{\frac{p-1}{2}}(p) = t^{p-1}(p)$$

Now by multiplying by t we obtain from this, again in a formal sense:

$$\left(\frac{2}{p}\right) t = t^p(p)$$

(2) On the other hand, by using the binomial formula we have, again formally:

$$t^p = (w + w^{-1})^p = w^p + w^{-p}(p)$$

Since $w^8 = 1$, this will depend only on p modulo 8. More precisely, we have:

$$w^p + w^{-p} = (-1)^{\frac{p^2-1}{8}} t$$

Now by putting everything together, and dividing by t, we obtain:

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}(p)$$

But this gives the formula in the statement, as desired. □

The above proof still remains a bit mysterious, but do not worry, we will learn in a moment more about all this. Getting back now to Question 11.3, the point is that with the same idea, we can prove as well the quadratic reciprocity theorem, as follows:

THEOREM 11.5. *We have the quadratic reciprocity formula*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$$

valid for any primes $p, q \geq 3$, as announced above.

PROOF. This is something from chapter 8, coming there via a quite ad-hoc proof. However, thinking a bit, our $t = w + w^{-1}$ trick above can be adapted, as follows:

(1) To start with, we need an analogue of that $t = w + w^{-1}$ variable. For this purpose, let us set $w = e^{2\pi i/q}$, now that we have a prime $q \geq 3$ involved, and then:

$$t = \sum_{k=0}^{q-1} w^{k^2}$$

Observe that at $q = 2$, excluded by the statement, we have $w = -1$, and so $t = 1 + (-1) = 0$, instead of the $t = w + w^{-1}$ with $w = e^{\pi i/4}$ used before. However, believe me, this is due to some bizarre reasons, and the above t is the good variable, at $q \geq 3$.

(2) The above variable t is called Gauss sum, can be defined for any $q \in \mathbb{N}$, not necessarily prime, and can be explicitly computed, the formula being as follows:

$$t = \begin{cases} \sqrt{q} & \text{if } q = 1(4) \\ 0 & \text{if } q = 2(4) \\ \sqrt{q}\, i & \text{if } q = 3(4) \\ \sqrt{q}(1+i) & \text{if } q = 0(4) \end{cases}$$

In particular, assuming that q is odd, as is our $q \geq 3$ prime, we have:

$$t^2 = \begin{cases} q & \text{if } q = 1(4) \\ -q & \text{if } q = 3(4) \end{cases}$$

(3) In what follows we will only need this latter formula, for $q \geq 3$ prime, so let us prove this now, and with the comment that the proof of the first formula in (2) is something quite complicated, and better avoid that. We have, by definition of our variable t:

$$\begin{aligned} |t|^2 &= \sum_{kl} w^{k^2-l^2} \\ &= \sum_{kl} w^{(k+l)(k-l)} \\ &= \sum_{lr} w^{r(2l+r)} \\ &= \sum_{r} w^{r^2} \sum_{l} (w^{2r})^l \\ &= q \end{aligned}$$

(4) On the other hand, it is easy to see that t^2 is real, so $t^2 = \pm q$. With a bit more work it is possible to compute the sign too, $t^2 = (-1)^{\frac{q-1}{2}} q$, but we will not need this here, because the sign will come for free at the end of the proof, via a symmetry argument. So, as a conclusion, we have a formula as follows, for a certain $e_q \in \{0, 1\}$:

$$t^2 = (-1)^{e_q} q$$

(5) With this done, let us turn to the proof of our theorem, by using the variable t a bit as before, in the proof of Theorem 11.4. By using the Euler formula, we have:

$$\left(\frac{t^2}{p}\right) = (t^2)^{\frac{p-1}{2}}\ (p) = t^{p-1}\ (p)$$

By multiplying now by t we obtain from this, in a formal sense:

$$\left(\frac{t^2}{p}\right)t = t^p\ (p)$$

(6) In order to compute now t^p by other means, observe first that, if we denote by $\mathbb{Z}_q - \{0\} = S \sqcup N$ the partition into squares and non-squares, we have:

$$\begin{aligned} t &= \sum_{k=0}^{q-1} w^{k^2} \\ &= 1 + 2\sum_{s\in S} w^s \\ &= \sum_{s\in S} w^s - \sum_{s\in N} w^s \\ &= \sum_{r=0}^{k-1}\left(\frac{r}{q}\right)w^r \end{aligned}$$

(7) By using now the multinomial formula, with the observation that all the non-trivial multinomial coefficients are multiples of p, we obtain, in a formal sense:

$$\begin{aligned} t^p &= \left(\sum_r \left(\frac{r}{q}\right)w^r\right)^p \\ &= \sum_r \left(\frac{r}{q}\right)w^{rp}\ (p) \\ &= \sum_s \left(\frac{p^{-1}s}{q}\right)w^s\ (p) \\ &= \left(\frac{p^{-1}}{q}\right)\sum_s \left(\frac{s}{q}\right)w^s\ (p) \\ &= \left(\frac{p}{q}\right)t\ (p) \end{aligned}$$

(8) Time now to put everything together. By combining (5,7) we obtain:

$$\left(\frac{t^2}{p}\right)t = \left(\frac{p}{q}\right)t\ (p)$$

We can divide by t, and then drop the modulo p symbol, because our new equality, without t, is between two ± 1 quantities, and we obtain:

$$\left(\frac{t^2}{p}\right) = \left(\frac{p}{q}\right)$$

Now by taking into account the formula found in (4), this reads:

$$\left(\frac{(-1)^{e_q}}{p}\right)\left(\frac{q}{p}\right)=\left(\frac{p}{q}\right)$$

By using the Euler formula for the symbol on the left, we obtain from this:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{p-1}{2}\cdot e_q}$$

Now by symmetry we must have $e_q=\frac{q-1}{2}$, and this finishes the proof. □

11b. Further summing

We have seen in the above that the quadratic reciprocity theorem can be established via Gauss sums t, and this is certainly excellent news. However, we have mentioned in step (2) of our proof above a very nice, powerful and final formula for the Gauss sum t itself, and this even in the general case, where $q\in\mathbb{N}$ is not necessarily prime.

Time now to discuss all this. So, we want to solve the following question:

QUESTION 11.6. *What is the value of the Gauss quadratic sum*

$$t=\sum_{k=0}^{q-1}w^{k^2}$$

where $w=e^{2\pi i/q}$*, with* $q\in\mathbb{N}$*?*

Let us begin with some experiments, at small values of q. We have here:

PROPOSITION 11.7. *The first few Gauss sums are as follows:*

(1) *At* $q=1$ *we have* $t=1$.
(2) *At* $q=2$ *we have* $t=0$.
(3) *At* $q=3$ *we have* $t=\sqrt{3}\,i$.
(4) *At* $q=4$ *we have* $t=2(1+i)$.
(5) *At* $q=5$ *we have* $t=\sqrt{5}$.
(6) *At* $q=6$ *we have* $t=0$.
(7) *At* $q=7$ *we have* $t=\sqrt{7}\,i$.
(8) *At* $q=8$ *we have* $t=2\sqrt{2}(1+i)$.

PROOF. The computations are as follows, with $w=e^{2\pi i/q}$:

(1) At $q=1$ we have $w=1$, and $t=1$.

(2) At $q=2$ we have $w=-1$, and $t=1+(-1)=0$

(3) At $q=3$ we have $w=e^{2\pi i/3}$, and the computation goes as follows:

$$\begin{aligned} t &= 1+w+w^4\\ &= 1+2w\\ &= 1+2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)\\ &= \sqrt{3}\,i \end{aligned}$$

(4) At $q=4$ we have $w=i$, and the computation goes as follows:

$$\begin{aligned} t &= 1+i+i^4+i^9\\ &= 1+i+1+i\\ &= 2+2i\\ &= 2(1+i) \end{aligned}$$

(5) At $q=5$ we have $w=e^{2\pi i/5}$, and the computation goes as follows:

$$\begin{aligned} t &= 1+w+w^4+w^9+w^{16}\\ &= 1+w+w^4+w^4+w\\ &= 1+2(w+w^4)\\ &= 1+4\cos\left(\frac{2\pi}{5}\right)\\ &= \sqrt{5} \end{aligned}$$

Here we have used some crazy trigonometry at the end, which can be avoided, or rather proved, when thinking well, at where this trigonometry comes from, as follows:

$$\begin{aligned} t^2 &= (1+2w+2w^4)^2\\ &= 1+4w^2+4w^3+4w+4w^4+8\\ &= 5+4(1+w+w^2+w^3+w^4)\\ &= 5 \end{aligned}$$

Observe that there is actually still some work to be done here, when extracting the square root of $t^2=5$. But the picture shows that the root is positive, $t=\sqrt{5}$.

(6) At $q=6$ it is most convenient to use $w=e^{2\pi i/3}$ as variable, as it is customary, and with this convention our root of unity is $e^{2\pi i/6}=-w^2$, and we have:

$$\begin{aligned} t &= 1-w^2+w^8-w^{18}+w^{32}-w^{50}\\ &= 1-w^2+w^2-1+w^2-w^2\\ &= 0 \end{aligned}$$

(7) At $q=7$ we have $w=e^{2\pi i/7}$, and the computation goes as follows:

$$\begin{aligned} t &= 1+w+w^4+w^9+w^{16}+w^{25}+w^{36}\\ &= 1+w+w^4+w^2+w^2+w^4+w\\ &= 1+2(w+w^2+w^4)\\ &= \sqrt{7}\,i \end{aligned}$$

Here again we have used some crazy trigonometry, the justification being as follows, and with the correct root of $t^2=-7$, among $t=\pm\sqrt{7}\,i$, being $t=\sqrt{7}\,i$, as shown by the picture, with the components w,w^2,w^4 of our sum t tending to lie North-West:

$$\begin{aligned} t^2 &= (1+2w+2w^2+2w^4)^2\\ &= 1+4w^2+4w^4+4w\\ &\quad +4w+4w^2+4w^4\\ &\quad +8w^3+8w^5+8w^6\\ &= 1+8(w+w^2+w^3+w^4+w^5+w^6)\\ &= -7+8(1+w+w^2+w^3+w^4+w^5+w^6)\\ &= -7 \end{aligned}$$

(8) At $q=8$ we have $w=e^{\pi i/4}$, and the computation goes as follows:

$$\begin{aligned} t &= 1+w+w^4+w^9+w^{16}+w^{25}+w^{36}+w^{49}\\ &= 1+w-1+w+1+w-1+w\\ &= 4w\\ &= 2\sqrt{2}(1+i) \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

All the above is quite interesting, and we can formulate our conclusion as follows:

Conclusion 11.8. *The first few quadratic Gauss sums are given by*

q	1	2	3	4	5	6	7	8
t	1	0	$\sqrt{3}\,i$	$2(1+i)$	$\sqrt{5}$	0	$\sqrt{7}\,i$	$2\sqrt{2}(1+i)$

with everything coming from easy algebra, except for the signs.

Moving ahead now with the general case, there is some obvious periodicity in the above table, of order 4, and with everything working fine, I mean with the dependence on q being clear in all cases modulo 4, we are led to the following statement:

THEOREM 11.9. *We have the following formula for the Gauss sums,*

$$t=\begin{cases}\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \sqrt{q}\,i & \text{if } q=3(4)\\ \sqrt{q}(1+i) & \text{if } q=0(4)\end{cases}$$

valid for any $q\in\mathbb{N}$, not necessarily prime.

PROOF. This is straightforward, except for the signs, the idea being as follows:

(1) To start with, let us compute $|t|^2$. This is something that we did in the proof of Theorem 11.5, for $q\geq 3$ prime, and the computation there can be recycled, as follows:

$$\begin{aligned}|t|^2&=\sum_{kl}w^{k^2-l^2}=\sum_{kl}w^{(k+l)(k-l)}\\ &=\sum_{lr}w^{r(2l+r)}=\sum_r w^{r^2}\sum_l(w^{2r})^l\\ &=\sum_r w^{r^2}\times\delta_{2|2r}\,q=q\sum_{q|2r}w^{r^2}\end{aligned}$$

(2) We have some cases here. For q odd we get q, and for q even, we have:

$$\begin{aligned}|t|^2&=q(1+(w^{(q/2)^2})\\ &=q(1+(w^{q/2})^{q/2}\\ &=q(1+(-1)^{q/2})\end{aligned}$$

(3) We are therefore led to the following formula, for our variable $|t|^2$:

$$|t|^2=\begin{cases}q & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ q & \text{if } q=3(4)\\ 2q & \text{if } q=0(4)\end{cases}$$

(4) Now by extracting the square root, we have the following formula, for $|t|$:

$$|t|=\begin{cases}\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \sqrt{q} & \text{if } q=3(4)\\ \sqrt{2q} & \text{if } q=0(4)\end{cases}$$

(5) The question is now, shall we go ahead and compute t, or be less greedy, and compute t^2 first. And let us be modest, of course, and go with t^2 first. But here, it is pretty much clear, from the computations in the proof of Proposition 11.7, that we can

get away with some simple algebra, I mean with algebra a hair more complicated than that in (1,2) above. For this purpose, the best is to go with the following alternative definition of the Gauss sums, that we already met in the proof of Theorem 11.5:

$$t=\sum_{r=0}^{q-1}\left(\frac{r}{q}\right)w^r$$

(6) Now by taking the square of this quantity, and then working out what exactly happens at $q=1,2,3,0(4)$, exactly as in the proof of Proposition 11.7, and we will leave this as an instructive exercise, we are led to the following formula:

$$t^2=\begin{cases}q & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ -q & \text{if } q=3(4)\\ 2qi & \text{if } q=0(4)\end{cases}$$

(7) In what regards now t itself, by taking the square root, we must have:

$$t=\begin{cases}\pm\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \pm\sqrt{q}\,i & \text{if } q=3(4)\\ \pm\sqrt{q}(1+i) & \text{if } q=0(4)\end{cases}$$

(8) So, almost done, but thinking a bit, in fact we just got started. Indeed, remember from Proposition 11.7 that the computation of the signs is tricky business, done on pictures, more specifically at $q=5$ by arguing that the components of t tend to pull it East, and at $q=7$, by arguing that these components tend to pull it North-West.

(9) So, what kind of question is this, geography or something? Well, in answer, such things are called mathematical analysis. Normally, what we need are some estimates, with ε and everything, as to decide what is the approximate direction of the pull of the components of t, as to compute that missing sign. And, more on this in a moment. □

11c. The Gauss sign

Time to get into the real thing, which is the computation of the missing sign, for the Gauss sums t. As a first observation, since $t=0$ for $q=2(4)$, the problem is solved in this case, at least we know one thing. Thus, we are left with the cases $q=0,1,3(4)$.

In order to deal with these questions, let us introduce the following notion:

DEFINITION 11.10. *The Fourier matrix of order $q \in \mathbb{N}$ is*

$$F_q = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{q-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(q-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{q-1} & w^{2(q-1)} & \ldots & w^{(q-1)^2} \end{pmatrix}$$

with $w = e^{2\pi i/q}$. That is, $F_q = (w^{kl})$, with indices $k, l = 0, 1, \ldots, q-1$.

These matrices are well-known in discrete Fourier analysis, and more on this later, at the end of the present chapter. In the meantime, the interest in relation with our Gauss sum questions is quite obvious, coming from the following formula:

$$Tr(F_q) = \sum_k w^{k^2} = t$$

We are therefore led to the following question, whose answer would solve everything in relation with the Gauss sums, and in particular, our sign problem:

QUESTION 11.11. *What are the eigenvalues of F_q, and their multiplicities?*

So, this is the problem that we would like to solve. For certain reasons, that will become clear later, it is convenient to study, more generally, the following matrices:

DEFINITION 11.12. *Associated to any $p, q \in \mathbb{N}$ with $(p, q) = 1$ is the matrix*

$$F_{pq} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & z & z^2 & \ldots & z^{q-1} \\ 1 & z^2 & z^4 & \ldots & z^{2(q-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z^{q-1} & z^{2(q-1)} & \ldots & z^{(q-1)^2} \end{pmatrix}$$

with $z = e^{2\pi ip/q}$. That is, $F_{pq} = (w^{pkl})$, with indices $k, l = 0, 1, \ldots, q-1$.

Observe that we have $F_{1q} = F_q$. In what follows we will be mainly interested in this case, $p = 1$, but as mentioned above, it is technically convenient to deal with F_{pq}. As a first result now regarding F_{pq}, providing the key to its algebra, we have:

PROPOSITION 11.13. *The square of the matrix F_{pq} is given by*

$$F_{pq}^2 = \begin{pmatrix} q & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & q \\ 0 & 0 & \ldots & q & 0 \\ & & \vdots & & \\ 0 & q & \ldots & 0 & 0 \end{pmatrix}$$

and its fourth power is $F_{pq}^4 = q^2 1_q$.

PROOF. The first formula can be proved as follows, using $(p,q)=1$, which shows that $z=e^{2\pi ip/q}$ is a primitive q-root of unity, so that the usual barycenter trick applies:

$$\begin{aligned}(F_{pq}^2)_{kl} &= \sum_m (F_{pq})_{km}(F_{pq})_{ml}\\ &= \sum_m z^{km}z^{ml}\\ &= \sum_m z^{m(k+l)}\\ &= q\delta_{k+l,0}\end{aligned}$$

As for the second formula, $F_{pq}^4=q^2 1_q$, this comes from this, with the matrix representing F_{pq}^2 in the statement being q times the symmetry $e_k\leftrightarrow e_{-k}$. □

Getting now to diagonalization, we first have the following result:

PROPOSITION 11.14. *The following happen, in relation with the matrix F_{pq}:*

(1) *F_{pq}^4 has unique eigenvalue q^2.*
(2) *F_{pq}^2 has eigenvalues $q,-q$.*
(3) *F_{pq} has eigenvalues $\pm\sqrt{q},\pm i\sqrt{q}$.*

PROOF. This is indeed something self-explanatory, with (1) coming from the equality $F_{pq}^4=q^2 1_q$, then with (2) coming from (1), and then with (3) coming from (2). □

We can in fact say more, with a word on multiplicities. First we have:

PROPOSITION 11.15. *The multiplicity of q as eigenvalue of F_{pq}^2 is*

$$m_q=\begin{cases}\frac{q+1}{2} & (q \text{ odd})\\ \frac{q}{2}+1 & (q \text{ even})\end{cases}$$

and the multiplicity of $-q$ as eigenvalue of F_{pq}^2 is

$$m_{-q}=\begin{cases}\frac{q-1}{2} & (q \text{ odd})\\ \frac{q}{2}-1 & (q \text{ even})\end{cases}$$

with these numbers summing up to q.

PROOF. This follows from the formula for F_{pq}^2 from Proposition 11.13, namely:

$$F_{pq}^2=\begin{pmatrix}q & 0 & \ldots & 0 & 0\\ 0 & 0 & \ldots & 0 & q\\ 0 & 0 & \ldots & q & 0\\ & & \vdots & & \\ 0 & q & \ldots & 0 & 0\end{pmatrix}$$

Indeed, we are led in this way to the multiplicities from the statement. □

In what regards now the multiplicities for F_{pq}, we have here:

THEOREM 11.16. *The multiplicities a, b, c, d of the eigenvalues of F_{pq},*

$$\underbrace{\sqrt{q}}_{a}\ ,\ \underbrace{-\sqrt{q}}_{b}\ ,\ \underbrace{i\sqrt{q}}_{c}\ ,\ \underbrace{-i\sqrt{q}}_{d}$$

are subject to the following equations, when q is odd,

$$a+b=\frac{q+1}{2}\quad ,\quad c+d=\frac{q-1}{2}$$

and are subject to the following equations, when q is even:

$$a+b=\frac{q}{2}+1\quad ,\quad c+d=\frac{q}{2}-1$$

Also, $Tr(F_{pq})=\sqrt{q}[(a-b)+(c-d)i]$, with at $p=1$ this being the Gauss sum t.

PROOF. According to Proposition 11.14, the multiplicities of the eigenvalues for the matrices F_{pq} and F_{pq}^2 are related by the following formulae:

$$a+b=m_q\quad ,\quad c+d=m_{-q}$$

Now by using the formulae of m_q, m_{-q} from Proposition 11.15, this gives the formulae in the statement. Regarding now the last assertion, we have indeed:

$$\begin{aligned}Tr(F_{pq}) &= \sqrt{q}\times a-\sqrt{q}\times b+i\sqrt{q}\times c-i\sqrt{q}\times d\\ &= \sqrt{q}[(a-b)+(c-d)i]\end{aligned}$$

As for the very last statement, which is there for everything to be complete, this is something that we already know, coming from $Tr(F_q)=\sum_k w^{k^2}=t$. □

Let us further investigate the trace of F_{pq}. We have here the following result:

PROPOSITION 11.17. *The trace of the matrix F_{pq} satisfies*

$$|Tr(F_{pq})|=\begin{cases}\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \sqrt{q} & \text{if } q=3(4)\\ \sqrt{2q} & \text{if } q=0(4)\end{cases}$$

for any $(p,q)=1$.

PROOF. This is something that we basically know, from our previous investigations of the Gauss sums, but here is that computation again, in terms of our present formalism

and notations, which are a bit more general, than what we were doing before:

$$\begin{aligned}|Tr(F_{pq})|^2 &= \sum_{kl} z^{k^2-l^2} = \sum_{kl} z^{(k+l)(k-l)}\\ &= \sum_{lr} z^{r(2l+r)} = \sum_r z^{r^2}\sum_l (z^{2r})^l\\ &= \sum_r z^{r^2}\times\delta_{2|2r}\, q = q\sum_{q|2r} z^{r^2}\end{aligned}$$

We have some cases here. For q odd we get q, and for q even, we have:

$$\begin{aligned}|Tr(F_{pq})|^2 &= q(1+(z^{(q/2)^2})\\ &= q(1+(z^{q/2})^{q/2}\\ &= q(1+(-1)^{q/2})\end{aligned}$$

We are therefore led to the following formula, for our variable $|Tr(F_{pq})|^2$:

$$|Tr(F_{pq})|^2=\begin{cases}q & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ q & \text{if } q=3(4)\\ 2q & \text{if } q=0(4)\end{cases}$$

Now by extracting the square root, we are led to the formula in the statement. □

We can in fact say more about the trace, by using Theorem 11.16, as follows:

THEOREM 11.18. *The trace of the matrix F_{pq} satisfies*

$$Tr(F_{pq})=\begin{cases}\pm\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \pm\sqrt{q}\,i & \text{if } q=3(4)\\ \pm\sqrt{q}(1+i) & \text{if } q=0(4)\end{cases}$$

for any $(p,q)=1$.

PROOF. Again, this is something that we basically know, from our previous investigations of the Gauss sums, and in the present setting, which is a bit more general, this can be obtained either by recycling the computation there, or by merging what we found in Theorem 11.16 and Proposition 11.17. We will leave this as an exercise. □

As a last elementary result, regarding the traces, we have:

PROPOSITION 11.19. *We have the following multiplicativity formula,*

$$Tr(F_{p,qr})=Tr(F_{pq,r})Tr(F_{pr,q})$$

valid for any numbers p,q,r satisfying $(p,qr)=1$ and $(q,r)=1$.

PROOF. We have indeed the following computation, with $w = e^{2\pi i/qr}$:

$$\begin{aligned} Tr(F_{pq,r})Tr(F_{pr,q}) &= \sum_k (w^q)^{pqk^2} \sum_l (w^r)^{prl^2} \\ &= \sum_{kl} w^{p(q^2k^2+r^2l^2)} \end{aligned}$$

By some basic quadratic residue arithmetic, this equals the following quantity:

$$Tr(F_{p,qr}) = \sum_k w^{pk^2}$$

Thus, we are led to the conclusion in the statement. □

This was for the elementary part of the proof, connecting our problems regarding the Gauss sums t to the algebra of the Fourier matrices F_{pq}. The continuation of the story, bringing some truly new results, involves the computation of $\det(F_{1q})$, as follows:

THEOREM 11.20. *We have the following formula,*

$$\det(F_{1q}) = \begin{cases} i^{\binom{q}{2}} q^{q/2} & \text{for } q \text{ odd} \\ i^{\binom{q}{2}+1} q^{q/2} & \text{for } q \text{ even} \end{cases}$$

for the determinant of the Fourier matrix, $F_{1q} = (w^{kl})$ with $w = e^{2\pi i/q}$.

PROOF. To start with, the Fourier matrix in the statement looks as follows:

$$F_{1q} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{q-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(q-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{q-1} & w^{2(q-1)} & \ldots & w^{(q-1)^2} \end{pmatrix}$$

But this is a Vandermonde matrix, so we have the following formula:

$$\det(F_{1q}) = \prod_{k>l} (w^k - w^l)$$

On the other hand, recall from Proposition 11.13 that we have:

$$F_{1q}^2 = \begin{pmatrix} q & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & q \\ 0 & 0 & \ldots & q & 0 \\ & & \vdots & & \\ 0 & q & \ldots & 0 & 0 \end{pmatrix}$$

Thus we have $\det(F_{1q}^2) = \pm q^q$, and a more careful study gives:

$$\det(F_{1q}^2) = \begin{cases} (-1)^{\binom{q}{2}} q^q & \text{for } q \text{ odd} \\ (-1)^{\binom{q}{2}+1} q^q & \text{for } q \text{ even} \end{cases}$$

Now by extracting the square root, we obtain the following formula:

$$\det(F_{1q}) = \begin{cases} \pm i^{\binom{q}{2}} q^{q/2} & \text{for } q \text{ odd} \\ \pm i^{\binom{q}{2}+1} q^{q/2} & \text{for } q \text{ even} \end{cases}$$

In order to compute the sign, we can use the previous Vandermonde approach, which gives, in terms of the root of unity $x = e^{\pi i/q}$, satisfying $x^2 = w$:

$$\begin{aligned} \det(F_{1q}) &= \prod_{k>l} (x^{2k} - x^{2l}) \\ &= \prod_{k>l} x^{k+l}(x^{k-l} - x^{l-k}) \\ &= \prod_{k>l} x^{k+l} \prod_{k>l} 2i \sin\left(\frac{(k-l)\pi}{q}\right) \\ &= (2i)^{\binom{q}{2}} \prod_{k>l} x^{k+l} \prod_{k>l} \sin\left(\frac{(k-l)\pi}{q}\right) \\ &= (2i)^{\binom{q}{2}} \cdot i^{(q-1)^2} \cdot \prod_{k>l} \sin\left(\frac{(k-l)\pi}{q}\right) \end{aligned}$$

Here we have used the following computation, with $x = e^{\pi i/q}$, which is something elementary, and that we will leave here as an instructive exercise:

$$\prod_{k>l} x^{k+l} = i^{(q-1)^2}$$

Now observe that $2^{\binom{q}{2}}$ is positive, then $i^{(q-1)^2}$ is 1 when q is odd, and 0 when q is even, and the product of sines on the right, which are all positive, is positive too. We are therefore led to the conclusion that the missing sign that we were computing is $+$:

$$\det(F_{1q}) = \begin{cases} i^{\binom{q}{2}} q^{q/2} & \text{for } q \text{ odd} \\ i^{\binom{q}{2}+1} q^{q/2} & \text{for } q \text{ even} \end{cases}$$

Thus, we are led to the formula in the statement. □

We will need as well a key result regarding the traces, as follows:

THEOREM 11.21. *For q odd and $(p,q)=1$ we have*

$$Tr(F_{pq})=\left(\frac{p}{q}\right)Tr(F_{1q})$$

with on the right a Jacobi symbol.

PROOF. This is something quite tricky, coming in two steps, as follows:

(1) Let us first discuss the case where $q\geq 3$ is prime. Here we have, with $w=e^{2\pi i/q}$, by definition of the Legendre symbol, and since the sum of roots of unity vanishes:

$$\begin{aligned}Tr(F_{pq}) &= \sum_k w^{k^2p}\\ &= \sum_l\left[\left(\frac{l}{q}\right)+1\right]w^{lp}\\ &= \sum_l\left(\frac{l}{q}\right)w^{lp}+\sum_l w^{lp}\\ &= \sum_l\left(\frac{l}{q}\right)w^{lp}\end{aligned}$$

Now by using the multiplicativity of the Legendre symbol in its upper variable, we can deduce from this the formula in the statement, by using the following trick:

$$\begin{aligned}Tr(F_{pq}) &= \left(\frac{p}{q}\right)^2\sum_l\left(\frac{l}{q}\right)w^{lp}\\ &= \left(\frac{p}{q}\right)\sum_l\left(\frac{lp}{q}\right)w^{lp}\\ &= \left(\frac{p}{q}\right)\sum_k\left(\frac{k}{q}\right)w^{k}\\ &= \left(\frac{p}{q}\right)Tr(F_{1q})\end{aligned}$$

(2) In the general case now, observe first that we have a formula as follows, with the product being over all prime factors of q, each repeated with its multiplicity:

$$Tr(F_{1q})=\pm\prod_{r|q}Tr(F_{1r})$$

By applying to this the automorphism $z\to z^p$ we have, more generally:

$$Tr(F_{pq})=\pm\prod_{r|q}Tr(F_{pr})$$

But with this formula in hand we obtain, by using (1) for all factors $r|q$:

$$\begin{aligned} Tr(F_{pq}) &= \pm\prod_{r|q}\left(\frac{p}{r}\right)Tr(F_{1r}) \\ &= \prod_{r|q}\left(\frac{p}{r}\right)\cdot\left(\pm\prod_{r|q}Tr(F_{1r})\right) \\ &= \left(\frac{p}{q}\right)Tr(F_{1q}) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

And with this, good news, done with the preliminaries, and we can now prove:

THEOREM 11.22. *We have the following formula for the Gauss sums,*

$$t=\begin{cases}\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \sqrt{q}\,i & \text{if } q=3(4)\\ \sqrt{q}(1+i) & \text{if } q=0(4)\end{cases}$$

valid for any $q\in\mathbb{N}$, not necessarily prime.

PROOF. This follows indeed by using our various results above, as follows:

(1) To start with, let us recall that we have the following formula:

$$t=\begin{cases}\pm\sqrt{q} & \text{if } q=1(4)\\ 0 & \text{if } q=2(4)\\ \pm\sqrt{q}\,i & \text{if } q=3(4)\\ \pm\sqrt{q}(1+i) & \text{if } q=0(4)\end{cases}$$

(2) When q is odd, it follows from Theorem 11.20 that the sign is $+$. Thus, we know one thing, and with this being the main one, q odd being the case that matters.

(3) When q is even, and more specifically in the case left, $q=0(4)$, the study is a bit more complicated, involving the general Fourier matrices F_{pq}, and Proposition 11.19 and a recurrence argument, based on Theorem 11.21. Our claim here is that we have:

$$Tr(F_{pq})=\left(\frac{q}{p}\right)(1+i^p)\sqrt{q}$$

In order to prove this claim, let us write $q=2^ab$ with $a\geq 2$ and b odd. By using Proposition 11.19, then an elementary computation with the roots of unity of order 2^a, coupled with Theorem 11.21, then the multiplicativity property of the Jacobi symbols,

then the trace formula in the odd case, coming from (2), and finally the multiplicativity of Jacobi symbols again, coupled with quadratic reciprocity, we have:

$$\begin{aligned}
Tr(F_{pq}) &= Tr(F_{p,2^ab}) \\
&= Tr(F_{bp,2^a})Tr(F_{2^ap,b}) \\
&= \left(\frac{2^a}{bp}\right)(1+i^{bp})\sqrt{2^a}\times\left(\frac{2^ap}{b}\right)Tr(F_{1b}) \\
&= \sqrt{2^a}\left(\frac{2^a}{bp}\right)\left(\frac{2^ap}{b}\right)(1+i^{bp})Tr(F_{1b}) \\
&= \sqrt{2^a}\left(\frac{2^a}{b}\right)\left(\frac{2^a}{p}\right)\left(\frac{2^a}{b}\right)\left(\frac{p}{b}\right)(1+i^{bp})Tr(F_{1b}) \\
&= \sqrt{2^a}\left(\frac{2^a}{p}\right)\left(\frac{p}{b}\right)(1+i^{bp})Tr(F_{1b}) \\
&= \sqrt{2^a}\left(\frac{2^a}{p}\right)\left(\frac{p}{b}\right)(1+i^{bp})i^{(b-1)^2/4}\sqrt{b} \\
&= \left(\frac{2^a}{p}\right)\left(\frac{p}{b}\right)(1+i^{bp})i^{(b-1)^2/4}\sqrt{q} \\
&= \pm\left(\frac{2^ab}{p}\right)\cdot\Big[\pm(1+i^p)\Big]\sqrt{q} \\
&= \left(\frac{q}{p}\right)(1+i^p)\sqrt{q}
\end{aligned}$$

Thus, claim proved, and with $p=1$ we are led to the formula in the statement. □

11d. Some applications

Let us go back to the Hadamard matrix questions discussed chapter 8. As a further twist to the plot, bringing some sort of solution to that questions, we have:

THEOREM 11.23. *When enlarging the attention to the complex Hadamard matrices, $H\in M_N(\mathbb{T})$ having the rows pairwise orthogonal, the Fourier matrix,*

$$F_N=\begin{pmatrix}
1 & 1 & 1 & \ldots & 1\\
1 & w & w^2 & \ldots & w^{N-1}\\
1 & w^2 & w^4 & \ldots & w^{2(N-1)}\\
\vdots & \vdots & \vdots & & \vdots\\
1 & w^{N-1} & w^{2(N-1)} & \ldots & w^{(N-1)^2}
\end{pmatrix}$$

with $w=e^{2\pi i/N}$, provides an example of such a matrix, at any $N\in\mathbb{N}$. Thus, the Hadamard Conjecture problematics dissapears, in the complex setting.

PROOF. By using the standard fact that the averages of complex numbers correspond to barycenters, we conclude that the scalar products between the rows of F_N are:

$$< R_a, R_b >= \sum_j w^{(a-b)j} = N\delta_{ab}$$

Thus F_N is indeed a complex Hadamard matrix, as claimed. □

In view of this, let us study more in detail the complex Hadamard matrices. Many examples can be constructed, quite often by using the combinatorics of roots of unity, and as a basic example here, we have the tensor products of Fourier matrices:

THEOREM 11.24. *Given a finite abelian group G, with dual group $\widehat{G} = \{\chi : G \to \mathbb{T}\}$, consider the Fourier coupling $\mathcal{F}_G : G \times \widehat{G} \to \mathbb{T}$, given by $(i, \chi) \to \chi(i)$.*

(1) *Via the standard isomorphism $G \simeq \widehat{G}$, this Fourier coupling can be regarded as a square matrix, $F_G \in M_G(\mathbb{T})$, which is a complex Hadamard matrix.*
(2) *In the case of the cyclic group $G = \mathbb{Z}_N$ we obtain in this way, via the standard identification $\mathbb{Z}_N = \{1, \ldots, N\}$, the Fourier matrix F_N.*
(3) *In general, when using a decomposition $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$, the corresponding Fourier matrix is given by $F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$.*

PROOF. This follows indeed from some basic facts from group theory:

(1) With the identification $G \simeq \widehat{G}$ made our matrix is given by $(F_G)_{i\chi} = \chi(i)$, and the scalar products between the rows are then, as desired:

$$< R_i, R_j >= \sum_\chi \chi(i - j) = |G| \cdot \delta_{ij}$$

(2) This follows from the well-known and elementary fact that, via the identifications $\mathbb{Z}_N = \widehat{\mathbb{Z}_N} = \{1, \ldots, N\}$, the Fourier coupling here is $(i, j) \to w^{ij}$, with $w = e^{2\pi i/N}$.

(3) By decomposing G as a product of cyclic groups, as in the statement, and using (2) for the cyclic components, we obtain the formula in the statement. □

As an application of this, in relation with the matrices from chapter 8, we have:

PROPOSITION 11.25. *The Walsh matrix, W_N with $N = 2^n$, which is given by*

$$W_N = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n}$$

is the Fourier matrix of the finite abelian group $K_N = \mathbb{Z}_2^n$.

PROOF. We have indeed the following computation, for any $N = 2^n$:

$$W_N = F_2^{\otimes n} = F_{K_2}^{\otimes n} = F_{K_2^n} = F_{K_N}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, the complex Hadamard matrices are quite interesting objects. Getting now to classification matters, we first have the following result:

THEOREM 11.26. *The complex Hadamard matrices are as follows, up to equivalence coming by permuting rows or columns, or multiplying them by numbers in $\mathbb{T}$:*

(1) *At $N = 2$ we only have the Fourier matrix F_2.*
(2) *At $N = 3$ we only have the Fourier matrix F_3.*
(3) *At $N = 4$ we only have F_4, and its deformations.*
(4) *At $N = 5$ we only have the Fourier matrix F_5.*

PROOF. This is a mixture of trivial and non-trivial results, as follows:

(1) This is something trivial, coming from definitions.

(2) This is again elementary, that we will leave as an instructive exercise.

(3) Here the matrices are up to equivalence as follows, with $q \in \mathbb{T}$:

$$F_4^q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & q & -1 & -q \\ 1 & -q & -1 & q \end{pmatrix}$$

As for the proof, this is again elementary, and we will leave it as an exercise.

(4) This is something which is remarkably difficult, due to Haagerup. Consider indeed an Hadamard matrix $H \in M_5(\mathbb{T})$, chosen to be dephased, as follows:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & a & x & * & * \\ 1 & y & b & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \end{pmatrix}$$

The point is that some study, which remains something quite formal and mysterious, shows that the numbers a, b, x, y must satisfy in fact the following equation:

$$(x - y)(x - ab)(y - ab) = 0$$

But with this in hand, which is something powerful, the continuation is routine, and we are led to roots of unity, and to the conclusion in the statement. See [10]. □

At $N = 6$ things explode, with a summary of what happens here being as follows:

THEOREM 11.27. *The complex Hadamard matrices at $N = 6$ are as follows:*

(1) *As basic examples, we have F_6 and its two-parameter deformations.*
(2) *We have as well another deformable matrix H_6, and an isolated one T_6.*
(3) *The matrices F_6, H_6, T_6 and their deformations are the only regular ones.*
(4) *However, we have many exotic examples too, which are not classified yet.*

PROOF. The idea here is that the matrices which are regular, in the sense that the vanishing scalar products between their rows come from roots of unity, can be classified, as said in (1,2,3), and that passed this, things are quite complicated, as said in (4). As an illustration for (4), here is a beast found by Björck and Fröberg:

$$BF_6=\begin{pmatrix}1&ia&-a&-i&-\bar{a}&i\bar{a}\\ i\bar{a}&1&ia&-a&-i&-\bar{a}\\ -\bar{a}&i\bar{a}&1&ia&-a&-i\\ -i&-\bar{a}&i\bar{a}&1&ia&-a\\ -a&-i&-\bar{a}&i\bar{a}&1&ia\\ ia&-a&-i&-\bar{a}&i\bar{a}&1\end{pmatrix}$$

Indeed, assuming that $a\in\mathbb{T}$ is one of the roots of $a^2+(\sqrt{3}-1)a+1=0$, this matrix is Hadamard. For more on all this, $N=6$ and higher, you can check my book [10]. □

In view of the above examples and counterexamples, and thinking a bit at number theory, where primes are the kings, we are led to the question of finding the complex Hadamard matrices which are "isolated", in a geometric sense. And here, skipping some details, we have an interesting construction due to McNulty and Weigert, that we would like to explain now. This construction is based on the following simple fact:

THEOREM 11.28. *Assuming that $K\in M_N(\mathbb{C})$ is Hadamard, so is the matrix*

$$H_{ia,jb}=\frac{1}{\sqrt{Q}}K_{ij}(L_i^*R_j)_{ab}$$

provided that $\{L_1,\ldots,L_N\}\subset\sqrt{Q}U_Q$ and $\{R_1,\ldots,R_N\}\subset\sqrt{Q}U_Q$ are such that

$$\frac{1}{\sqrt{Q}}L_i^*R_j\in\sqrt{Q}U_Q$$

with $i,j=1,\ldots,N$, are complex Hadamard.

PROOF. The check of the unitarity of the matrix in the statement can be done as follows, by using our various assumptions on the various matrices involved:

$$\begin{aligned}<H_{ia},H_{kc}>&=\frac{1}{Q}\sum_{jb}K_{ij}(L_i^*R_j)_{ab}\bar{K}_{kj}\overline{(L_k^*R_j)}_{cb}\\ &=\sum_j K_{ij}\bar{K}_{kj}(L_i^*L_k)_{ac}\\ &=N\delta_{ik}(L_i^*L_k)_{ac}\\ &=NQ\delta_{ik}\delta_{ac}\end{aligned}$$

The entries of our matrix being in addition on the unit circle, we are done. □

The above construction is of course something quite abstract, but as a very concrete input for it, we can use the following well-known Fourier analysis construction:

PROPOSITION 11.29. *For $q \geq 3$ prime, the family of matrices*

$$\{F_q, DF_q, \ldots, D^{q-1}F_q\}$$

where F_q is the Fourier matrix, and where D is the diagonal matrix given by

$$D = diag\left(1, 1, w, w^3, w^6, w^{10}, \ldots, w^{\frac{q^2-1}{8}}, \ldots, w^{10}, w^6, w^3, w\right)$$

with $w = e^{2\pi i/q}$, are such that $\frac{1}{\sqrt{q}}E_i^*E_j$ is complex Hadamard, for any $i \neq j$.

PROOF. With by definition $0, 1, \ldots, q-1$ as indices for our matrices, as usual in a Fourier analysis context, the formula of the above matrix D is:

$$D_c = w^{0+1+\ldots+(c-1)} = w^{\frac{c(c-1)}{2}}$$

Since we have $\frac{1}{\sqrt{q}}E_i^*E_j \in \sqrt{q}U_q$, we just need to check that these matrices have entries belonging to $\mathbb{T}$, for any $i \neq j$. With $k = j - i$, these entries are given by:

$$\frac{1}{\sqrt{q}}(E_i^*E_j)_{ab} = \frac{1}{\sqrt{q}}(F_q^*D^kF_q)_{ab} = \frac{1}{\sqrt{q}}\sum_c w^{c(b-a)}D_c^k$$

But an elementary computation with roots of unity, that we will leave here as an exercise, shows that these entries are on the unit circle, as desired. □

Next, we have the following result, making use of Gauss sums:

PROPOSITION 11.30. *The matrices $G_k = \frac{1}{\sqrt{q}}F_q^*D^kF_q$, with $D = diag\left(w^{\frac{c(c-1)}{2}}\right)$, and with $k \neq 0$ are circulant, their first row vectors V^k being given by*

$$V_i^k = \delta_q\left(\frac{k/2}{q}\right)w^{\frac{q^2-1}{8}\cdot k} \cdot w^{-\frac{\frac{i}{k}(\frac{i}{k}-1)}{2}}$$

where $\delta_q = 1$ if $q = 1(4)$ and $\delta_q = i$ if $q = 3(4)$, and with all inverses being taken in $\mathbb{Z}_q$.

PROOF. The above matrices G_k are indeed circulant, their first vectors being:

$$V_i^k = \frac{1}{\sqrt{q}}\sum_c w^{\frac{c(c-1)}{2}\cdot k+ic}$$

But this is a Gauss sum, and by computing the square, we obtain:

$$(V_i^k)^2 = \delta_q^2 \cdot w^{\frac{q^2-1}{4}\cdot k} \cdot w^{-\frac{i}{k}(\frac{i}{k}-1)}$$

By extracting now the square root, we obtain a formula as follows:

$$V_i^k = \pm\delta_q \cdot w^{\frac{q^2-1}{8}\cdot k} \cdot w^{-\frac{\frac{i}{k}(\frac{i}{k}-1)}{2}}$$

And with Theorem 11.22 computing the sign, this leads to the above formula. □

Let us combine now the above results. We obtain the following statement:

THEOREM 11.31. *Let $q \geq 3$ be prime, consider subsets $S, T \subset \{0, 1, \ldots, q-1\}$ satisfying the conditions $|S| = |T|$ and $S \cap T = \emptyset$, and write:*

$$S = \{s_1, \ldots, s_N\} \quad , \quad T = \{t_1, \ldots, t_N\}$$

Then, with the matrix V being as above, the following matrix,

$$H_{ia,jb} = K_{ij} V_{b-a}^{t_j - s_i}$$

is complex Hadamard, provided that the matrix $K \in M_N(\mathbb{C})$ is complex Hadamard.

PROOF. This follows indeed by using the general construction in Theorem 11.28, with input coming from Proposition 11.29 and Proposition 11.30. □

The above construction covers many interesting examples of Hadamard matrices, known to be isolated, such as the Tao matrix, which is as follows, with $w = e^{2\pi i/3}$:

$$T_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & w & w & w^2 & w^2 \\ 1 & w & 1 & w^2 & w^2 & w \\ 1 & w & w^2 & 1 & w & w^2 \\ 1 & w^2 & w^2 & w & 1 & w \\ 1 & w^2 & w & w^2 & w & 1 \end{pmatrix}$$

For more on such matrices, there are many texts available, including my book [10].

11e. Exercises

This was a tricky number theory chapter, and as exercises on this, we have:

EXERCISE 11.32. *Compute more Gauss sums, as many as you can.*

EXERCISE 11.33. *When stuck with higher sums, develop more trigonometry.*

EXERCISE 11.34. *Learn about various generalizations of the Gauss sums.*

EXERCISE 11.35. *Fill in the details, in the computation of the Gauss sum sign.*

EXERCISE 11.36. *Learn as well some further proofs of quadratic reciprocity.*

EXERCISE 11.37. *Learn more about discrete Fourier analysis, in general.*

EXERCISE 11.38. *Learn about the classification of complex Hadamard matrices.*

EXERCISE 11.39. *Read about the other applications of the Gauss sums.*

As bonus exercise, review your trigonometry knowledge, and read more, if needed.

CHAPTER 12

Transcendence

12a. Abstract algebra

We have already met field theory in this book, on several occasions, with this being, obviously, deeply connected to arithmetic. In this chapter we get into a deeper study of this. Our goals will be multiple, on one hand understanding a bit of Galois theory, and on the other hand in order to have some tools for investigating some basic number theory questions, such as the irrationality and transcendence of e and π.

Let us start with something that we know well since chapter 3, namely:

DEFINITION 12.1. *A field is a set F with a sum operation $+$ and a product operation $\times$, subject to the following conditions:*

(1) *$a+b=b+a$, $a+(b+c)=(a+b)+c$, there exists $0\in F$ such that $a+0=0$, and any $a\in F$ has an inverse $-a\in F$, satisfying $a+(-a)=0$.*
(2) *$ab=ba$, $a(bc)=(ab)c$, there exists $1\in F$ such that $a1=a$, and any $a\neq 0$ has a multiplicative inverse $a^{-1}\in F$, satisfying $aa^{-1}=1$.*
(3) *The sum and product are compatible via $a(b+c)=ab+ac$.*

In other words, a field satisfies what we can normally expect from "numbers", and as basic examples, we have of course $\mathbb{Q},\mathbb{R},\mathbb{C}$. There are many other examples of fields, along the same lines. We can talk for instance about fields like $\mathbb{Q}[\sqrt{2}]$, as follows:

PROPOSITION 12.2. *The following is an intermediate field $\mathbb{Q}\subset F\subset\mathbb{R}$,*

$$\mathbb{Q}[\sqrt{2}]=\left\{a+b\sqrt{2}\middle|a,b\in\mathbb{Q}\right\}$$

and the same happens for any $\mathbb{Q}[\sqrt{n}]$, with $n\neq m^2$ being not a square.

PROOF. All the field axioms are clearly satisfied, except perhaps for the inversion axiom. But this axiom is satisfied too, due to the following formula:

$$\frac{1}{a+b\sqrt{2}}=\frac{a-b\sqrt{2}}{a^2-2b^2}$$

Observe that the denominator is indeed nonzero, due to $a^2\neq 2b^2$, which follows by reasoning modulo 2. As for the case of $\mathbb{Q}[\sqrt{n}]$ with $n\neq m^2$, this is similar. □

The above result is quite interesting, obviously in relation with arithmetic, and suggests looking into the intermediate fields of numbers, as follows:

$$\mathbb{Q} \subset F \subset \mathbb{C}$$

As another observation now, complementary to this, with our field theory we are not away from geometry, quite the opposite. Indeed, while the usual spaces of functions are obviously not fields, geometry and analysis remain around the corner, due to:

THEOREM 12.3. *The quotients of complex polynomials, called rational funtions, when written in reduced form, as follows, with P, Q prime to each other,*

$$f = \frac{P}{Q}$$

are well-defined and continuous outside the zeroes $P_f \subset \mathbb{C}$ of Q, called poles of f:

$$f : \mathbb{C} - P_f \to \mathbb{C}$$

Also, these functions are stable under summing, making products and taking inverses,

$$\frac{P}{Q} + \frac{R}{S} = \frac{PS + QR}{QS} \quad , \quad \frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS} \quad , \quad \left(\frac{P}{Q}\right)^{-1} = \frac{Q}{P}$$

so they form a field $\mathbb{C}(X)$, called field of rational functions.

PROOF. Almost everything here is clear from definitions, and with the comment that, in what regards the term "pole", this does not come from the Poles who invented this, but rather from the fact that, when trying to draw the graph of f, or rather imagine that graph, which takes place in $2+2 = 4$ real dimensions, we are faced with some sort of tent, which is suspended by infinite poles, which lie, guess where, at the poles of f. □

Back to arithmetic, there are many other interesting examples of fields, such as the integers modulo a prime p, which form a field $\mathbb{F}_p$, that we met in chapter 3, or the p-adic numbers, again with respect to a prime number p, that we met in chapter 7. We will be back to this, and to fields in general, in a moment, after developing some more tools.

As a second piece of abstract algebra, that we will need, we have:

DEFINITION 12.4. *We have notions of rings, modules and ideals, as follows:*

(1) *A ring R is a set with operations $+$ and $\times$, satisfying the usual conditions for such operations, except for $ab = ba$, and for $a \neq 0 \implies \exists a^{-1}$.*
(2) *A module V over a ring R is a vector space, but we will call it ring, and keep the name vector spaces for the modules over fields, $R = F$.*
(3) *An ideal $I \subset R$ is a subgroup with the left ideal property $i \in I, r \in R \implies ir \in I$, or the right ideal property $i \in I, r \in R \implies ri \in I$, or both.*

This was a quite crowded statement, but you get the point, with (1) and (2) we are sort of trying to do field and vector space mathematics, over things which are not necessarily fields and vector spaces over them, and (3) is something technical, non-field specific. At the level of examples, these abound, and we have three important ones, as follows:

(1) The integers form a ring, $R = \mathbb{Z}$, which in addition is commutative, $ab = ba$. As obvious module over $\mathbb{Z}$, we have the lattice $V = \mathbb{Z}^N$. Finally, since $R = \mathbb{Z}$ is commutative, the 3 notions of ideals coincide, and these are the subsets $I = a\mathbb{Z}$, with $a \in \mathbb{Z}$.

(2) The matrices over the integers form a ring, $R = M_k(\mathbb{Z})$, which is noncommutative at $k > 1$. As obvious module over $M_k(\mathbb{Z})$, we have the lattice $V = \mathbb{Z}^k$. As for the ideals, things here are a bit more complicated, but since at $k = 2$ the matrices of type $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ form a left ideal which is not a right ideal, and the matrices of type $\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ form a right ideal which is not a left ideal, at least we know that our 3 types of ideals make sense.

(3) In relation with arithmetic, the integers modulo $N \in \mathbb{N}$ form a ring $R = \mathbb{Z}_N$, which is commutative. Also, the matrices over these integers modulo N form a ring $R' = M_k(\mathbb{Z}_N)$, which is noncommutative at $k > 1$. As for a preliminary study of the modules and ideals, for these two rings, I will leave it to you, and enjoy.

The question that you surely have in mind is, what are ideals good for? Answer:

PROPOSITION 12.5. *For a subgroup $I \subset R$, the following are equivalent:*

(1) *I is a two-sided ideal.*
(2) *R/I is a ring.*

PROOF. This is something which requires some thinking, as follows:

(1) Since the additive group $(R, +)$ is abelian, given an additive subgroup $I \subset R$ we can form the quotient group R/I, which is abelian too, with addition as follows:

$$(a + I) + (b + I) = (a + b + I)$$

Observe that the unit is $(0 + I) = I$, and that inverses are given by $(-a + I)$.

(2) The question is now, can we turn this abelian group R/I into a ring? Normally the multiplication can only be as follows, and with this clarifying our statement, with the condition "R/I is a ring" there meaning, with respect to this precise multiplication:

$$(a + I)(b + I) = (ab + I)$$

(3) But, will this work. As a first observation, there is a bit of analogy here with group theory, where $H \subset G$ must be normal in order for G/H to be a group. Thus, our claim is that the ideal condition is somehow the "analogue of normality, in the ring setting".

(4) In practice now, it is quite clear, exactly as in the group theory setting, that everything will be fine, provided that our multiplication is well-defined. And for this multiplication to be well-defined, the following condition must be satisfied:

$$(a+I)=(a'+I)\ ,\ (b+I)=(b'+I)\quad\implies\quad(ab+I)=(a'b'+I)$$

But this amounts in the following condition to be satisfied:

$$a-a'\in I\ ,\ b-b'\in I\quad\implies\quad ab-a'b'\in I$$

(5) Now comes the math. We have the following identity, which shows that if $I\subset R$ is a two-sided ideal, then the above condition is satisfied, and so done:

$$ab-a'b'=a(b-b')+(a-a')b'$$

(6) Conversely now, if the condition in (4) is satisfied, we have in particular:

$$i-0\in I\ ,\ r-r\in I\quad\implies\quad ir-0r\in I$$

$$r-r\in I\ ,\ i-0\in I\quad\implies\quad ri-r0\in I$$

Thus $I\subset R$ must be a two-sided ideal, and this finishes the proof. □

Many things can be said about rings, modules and ideals, especially in the commutative case. For formulating a theorem on the subject, we have:

THEOREM 12.6. *Assuming that R is commutative and $I\subset R$ is a maximal ideal, in the sense that it is a proper ideal, $I\neq R$, and there is no bigger proper ideal*

$$I\subset J\subset R$$

the quotient ring $F=R/I$ is a field.

PROOF. Here is the proof, and with this being guaranteed to be useful learning:

(1) Before starting, a quick example. We know that over $R=\mathbb{Z}$, the ideals are the subsets $I=p\mathbb{Z}$ with $p\in\mathbb{N}$. But such an ideal is maximal precisely when p is prime, and this is the same as asking for the quotient ring $R/I=\mathbb{Z}_p$ to be a field.

(2) In general now, assume first that R/I is a field. This means that any nonzero element of R/I is invertible, and with our usual conventions for R/I, this reads:

$$\forall a\notin I\ ,\ \exists b\in R\ ,\ (ab+I)=(1+I)$$

Now assume by contradiction that $I\subset R$ is not maximal, so that we have a bigger ideal $I\subset J\subset R$. If we pick $a\in J-I$, we obtain, by the above, the following:

$$a\in J-I\ ,\ b\in R\ ,\ ab=1+i\ ,\ i\in I$$

But this is contradictory, because since J is an ideal, containing I, we must have $ab,i\in J$, so we conclude that we have $1\in J$, and so $J=R$, contradiction.

(3) Conversely, assume now that I is maximal, and assume too, by contradiction, that R/I is not a field. Then we can find a zero divisor in R/I, which reads:

$$(a+I)(b+I)=(I)\ ,\ a,b\notin I$$

In other words, we can find $ab\in I$ with $a,b\notin I$. But then, let us look at:

$$I\subset I+aR\subset R$$

(4) What we have in the middle is an ideal, and it is also clear, from $a\notin I$, that the inclusion on the left is proper. As for the inclusion on the right, our claim is that this is proper too. Indeed, assuming otherwise, we would have a formula as follows:

$$i+ac=1\ ,\ i\in I$$

Now by multiplying everything by b, we obtain from this:

$$ib+acb=b\ ,\ i\in I$$

But this is contradictory, because on the left we have $ib\in I$ and $acb=(ab)c\in I$, which gives $b\in I$, contradicting the condition $b\notin I$. Thus, our claim is proved.

(5) But this is the end of the story, because what we just proved is that what we have in (3) is indeed a proper ideal, contradicting the maximality of I, as desired. □

Still with me, I hope, after all these abstractions, and please believe me, Theorem 12.6 is something of key importance, be that for algebraic geometry, or for arithmetic, as we will soon discover, or even for analysis, in the context of the Banach algebras.

Going ahead now with our general abstract algebra program, let us formulate:

DEFINITION 12.7. *We have notions of fields, vector spaces and algebras, as follows:*

(1) *A field F is a field F as we know them, with in algebra parlance these being the commutative rings R with each nonzero element being invertible.*
(2) *A vector space V over a field F is a vector space as we know them, in algebra parlance these being the modules V over a field F.*
(3) *An algebra A over a field F is a vector space over F, with a ring multiplication operation $\times$, compatible with the vector space structure.*

In relation with this, we already know of course about fields, and in what regards the vector spaces, we know about them since ever, and finally, regarding algebras, we know many algebras of functions from analysis. But, thinking well, from a purely algebraic perspective, all these objects have many operations, and this is why they came at last.

As basic examples now, passed the fields F and the vector spaces V that we know well, we are left with finding interesting examples of algebras A. And here the examples abound, with all being related to geometry or analysis of some sort, as follows:

(1) First we have algebra of polynomials $A = F[X]$. This is a very basic algebra, important for geometry, and with the extra feature that it is commutative, $PQ = QP$.

(2) More generally, we have the algebra of polynomials $A = F[X_1, \ldots, X_N]$. Again, this algebra is important for algebraic geometry, and is commutative, $PQ = QP$.

(3) Still talking commutative algebras, we have many of them coming from analysis, the general principle being that "functions form algebras". More on this in a moment.

(4) We have as well the algebra of matrices $A = M_N(F)$. Again this is a very basic example, that we know well, which this time is not commutative, $ST \neq TS$.

More in detail now, getting to the various algebras of functions, we have here the following key result, bringing among others some further light on Theorem 12.6 too:

THEOREM 12.8. *Given a compact space X, the following happen:*

(1) *The continuous functions $f : X \to \mathbb{C}$ form a complex algebra $C(X)$.*
(2) *Given $x \in X$, the functions satisfying $f(x) = 0$, form an ideal $I \subset C(X)$.*
(3) *This ideal is maximal, and any maximal ideal $I \subset C(X)$ appears in this way.*
(4) *In this picture, the fact that the quotient is a field, $C(X)/I = \mathbb{C}$, is clear.*

PROOF. All this is self-explanatory, the idea being as follows:

(1) This is clear. Observe that our algebra is commutative, $fg = gf$.

(2) This is again clear, because $f(x) = 0$ implies $(fg)(x) = 0$.

(3) This follows from basic topology, via a suitable open cover for X.

(4) This is clear, because $C(X) \to C(X)/I$ maps $f \to f(x) \in \mathbb{C}$. □

There are many other examples of algebras of functions, along these lines. In fact, we can even trick, and view certain algebras, which are certainly not algebras of functions, as algebras of functions too. As an example, here is a wild physics speculation:

SPECULATION 12.9. *We can view the matrix algebra $M_2(\mathbb{C})$ as being the algebra of functions on some sort of quantum space M_2, according to the following formula:*

$$M_2(\mathbb{C}) = C(M_2)$$

This quantum space M_2 formally has $|M_2| = 4$ points, and appears as a sort of twist of $\{1, 2, 3, 4\}$. Moreover, we can integrate over M_2, according to the formula

$$\int_{M_2} T = \frac{T_{11} + T_{22}}{2}$$

with the underlying measure being positive and of mass 1.

To be more precise here, let us be crazy, and define M_2 according to the formula $C(M_2) = M_2(\mathbb{C})$, without really knowing what we are doing. Then, we have:

$$|M_2| = \dim_{\mathbb{C}} C(M_2) = \dim_{\mathbb{C}} M_2(\mathbb{C}) = 4$$

Next, since we have $M_2(\mathbb{C}) \simeq \mathbb{C}^4$ as vector spaces, which reads $C(M_2) \simeq C(1,2,3,4)$, this suggests that we should have $M_2 \sim \{1,2,3,4\}$, as some sort of twisting operation. But this can be given a mathematical formulation too, the idea being that at the level of standard bases of $C(M_2) \simeq C(1,2,3,4)$, the multiplication gets twisted as follows:

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad \longleftrightarrow \quad e_j e_k = \delta_{jk}e_j$$

Finally, in what regards the last assertion, this expresses the standard fact that the normalized trace of 2×2 matrices $tr = Tr/2$ is unital and positive, in the sense that:

$$tr(1) = 1 \quad , \quad T \geq 0 \implies tr(T) \geq 0$$

Excited about this? Such things come from quantum mechanics, as developed by Heisenberg, and the above space M_2 can be given a precise mathematical sense, and is the entry point to "noncommutative geometry". For more on this, see Connes [17].

And good news, that is all. Done with abstract algebra, and good learning that was, that we can use later when needed, and we can now turn to more concrete things.

12b. Galois theory

Good news, we are now ready for Galois theory. In order to discuss this, let us start with a reminder of what we know about fields, from the previous chapters:

THEOREM 12.10. *Given a field F, define its characteristic $p = char(F)$ as being the smallest $p \in \mathbb{N}$ such that the following happens, and as $p = 0$, if this never happens:*

$$\underbrace{1 + \ldots + 1}_{p \text{ times}} = 0$$

Then, assuming $p > 0$, this number p must be prime, we have a field embedding $\mathbb{F}_p \subset F$, and $q = |F|$ must be of the form $q = p^k$, with $k \in \mathbb{N}$. Also, we have the formulae

$$(a+b)^p = a^p + b^p \quad , \quad a^q = a$$

valid for any $a, b \in F$, and the Fermat poynomial $X^q - X$ factorizes as:

$$X^q - X = \prod_{a \in F} (X - a)$$

Also, regardless of p, any finite multiplicative subgroup $G \subset F - \{0\}$ must be cyclic.

PROOF. This is a very crowded statement, containing all sorts of things that we know from chapter 3 and chapter 7, the idea with all this being as follows:

(1) The fact that $p > 0$ must be prime comes by contradiction, by using:

$$(\underbrace{1 + \ldots + 1}_{a\ times}) \times (\underbrace{1 + \ldots + 1}_{b\ times}) = \underbrace{1 + \ldots + 1}_{ab\ times}$$

Indeed, assuming that we have $p = ab$ with $a, b > 1$, the above formula corresponds to an equality of type $AB = 0$ with $A, B \neq 0$ inside F, which is impossible.

(2) Back to the general case, F has a smallest subfield $E \subset F$, called prime field, consisting of the various sums $1 + \ldots + 1$, and their quotients. In the case $p = 0$ we obviously have $E = \mathbb{Q}$. In the case $p > 0$ now, the multiplication formula in (1) shows that the set $S = \{1 + \ldots + 1\}$ is stable under taking quotients, and so $E = S$.

(3) Now with $E = S$ in hand, we obviously have $(E, +) = \mathbb{Z}_p$, and since the multiplication is given by the formula in (1), we conclude that we have $E = \mathbb{F}_p$, as a field. Thus, in the case $p > 0$, we have constructed an embedding $\mathbb{F}_p \subset F$, as claimed.

(4) In the context of the above embedding $\mathbb{F}_p \subset F$, we can say that F is a vector space over $\mathbb{F}_p$, and so we have $|F| = p^k$, with $k \in \mathbb{N}$ being the dimension of this space.

(5) The baby Fermat formula $(a + b)^p = a^p + b^p$, which reminds the Fermat little theorem, $a^p = a(p)$ over $\mathbb{Z}$, follows in the same way, namely from the binomial formula, because all the non-trivial binomial coefficients $\binom{p}{s}$ are multiples of p:

$$(a + b)^p = \sum_{k=0}^{p} \binom{p}{k} a^k b^{p-k} = a^p + b^p$$

(6) As for the Fermat formula $a^q = a$ itself, which implies the assertion about $X^q - X$, this follows from the last assertion, which can be proved via some basic arithmetic inside F, and which for $G = F - \{0\}$ itself, with $|F| = q$, gives $a^{q-1} = 1$, for any $a \neq 0$.

(7) Let us pick indeed an element $g \in G$ of highest order, $n = ord(g)$. Our claim, which will prove the results, is that the order $m = ord(h)$ of any $h \in G$ satisfies $m|n$.

(8) In order to prove this claim, let $d = (m, n)$, write $d = am + bn$ with $a, b \in \mathbb{Z}$, and set $k = g^a h^b$. We have then the following computation:

$$k^m = g^{am} h^{bm} = g^{am} = g^{d-bn} = g^d$$

Similarly, $k^n = h^d$. By using $k^m = g^d$, we obtain the following formula:

$$k^{[m,n]} = k^{mn/d} = (k^m)^{n/d} = (g^d)^{n/d} = g^n = 1$$

Thus $ord(k)|[m, n]$, and our claim is that we have in fact $ord(k) = [m, n]$.

(9) In order to prove this latter claim, assume first that we are in the case $d = 1$. But here the result is clear, because the formulae $k^m = g^d$ and $k^n = h^d$ in (8) read:

$$g = k^m \quad , \quad h = g^n$$

Indeed, since the numbers $n = ord(g)$ and $m = ord(g)$ are prime to each other, we conclude from this that we have $ord(k) = mn$, as desired. Thus, result proved for $d = 1$, and in what regards the general case, where d is arbitrary, this follows from this.

(10) Summarizing, we have proved our claim in (8). Now since the order $n = ord(g)$ was assumed to be maximal, we must have $[m, n]|n$, and so $m|n$. Thus, we have proved our claim in (7), namely that the order $m = ord(h)$ of any $h \in G$ satisfies $m|n$.

(11) But with this claim in hand, the result follows. Indeed, since the polynomial $x^n - 1$ has all the elements $h \in G$ as roots, its degree must satisfy $n \geq |G|$. On the other hand, from $n = ord(g)$ with $g \in G$, we have $n||G|$. We therefore conclude that we have $n = |G|$, which shows that G is indeed cyclic, generated by the element $g \in G$.

(12) Finally, assuming $|F| = q < \infty$, we know that the multiplicative group $F - \{0\}$ is cyclic, of order $q - 1$. Thus, the following formula is satisfied, for any $a \in F - \{0\}$:

$$a^{q-1} = 1$$

Now by multiplying by a, this gives the Fermat formula $a^q = a$, with of course the remark that this formula trivially holds as well for $a = 0$. □

The above result raises many questions. Since most of these questions seem to have something to do with field extensions, let us start by discussing this. We first have:

THEOREM 12.11. *Given a field extension $E \subset F$, we can talk about its Galois group G, as the group of automorphisms of F fixing E. The intermediate fields*

$$E \subset K \subset F$$

are then in correspondence with the subgroups $H \subset G$, with such a field K corresponding to the subgroup H consisting of automorphisms $g \in G$ fixing K.

PROOF. This is something quite self-explanatory, and follows indeed from some algebra, under suitable assumptions, in order for that algebra to properly apply. There are many good books, that you can learn this technology from, such as Lang [63]. □

Getting now towards polynomials and their roots, we have here:

THEOREM 12.12. *Given a field F and a polynomial $P \in F[X]$, we can talk about the abstract splitting field of P, where this polynomial decomposes as:*

$$P(X) = c\prod_i (X - a_i)$$

In particular, any field F has a certain algebraic closure $\bar{F}$, where all the polynomials $P \in F[X]$, and in fact all polynomials $P \in \bar{F}[X]$ too, have roots.

PROOF. This is again something self-explanatory, which follows from Theorem 12.11 and from some extra algebra, under suitable assumptions, in order for that extra algebra to properly apply. Regarding the construction at the end, as main example here we have $\bar{\mathbb{R}} = \mathbb{C}$. However, as an interesting fact, $\bar{\mathbb{Q}} \subset \mathbb{C}$ is a proper subfield. See [63]. □

Good news, with this in hand, we can now elucidate the structure of finite fields:

THEOREM 12.13. *For any prime power $q = p^k$ there is a unique field $\mathbb{F}_q$ having q elements. At $k = 1$ this is the usual $\mathbb{F}_p$. In general, this is the splitting field of:*

$$P = X^q - X$$

Moreover, we can construct an explicit model for $\mathbb{F}_q$, at $q = p^2$ or higher, as

$$\mathbb{F}_q = \mathbb{F}_p[X]/(Q)$$

with $Q \in \mathbb{F}_p[X]$ being a suitable irreducible polynomial, of degree k.

PROOF. There are several assertions here, the idea being as follows:

(1) The first assertion, regarding the existence and uniqueness of $\mathbb{F}_q$, follows from Theorem 12.10 and Theorem 12.12. Indeed, we know from Theorem 12.10 that given a finite field, $|F| = q$ with $k \in \mathbb{N}$, the Fermat polynomial $P = X^q - X$ factorizes as:

$$X^q - X = \prod_{a \in F} (X - a)$$

But this shows, via the general theory from Theorem 12.12, that our field F must be the splitting field of P, and so is unique. As for the existence, this follows again from Theorem 12.12, telling us that the splitting field always exists.

(2) In what regards now the modeling of $\mathbb{F}_q$, at $q = p$ there is nothing to do, because we have our usual $\mathbb{F}_p$ here. At $q = p^2$ and higher, we know from commutative algebra that we have an isomorphism as follows, whenever $Q \in \mathbb{F}_p[X]$ is taken irreducible:

$$\mathbb{F}_q = \mathbb{F}_p[X]/(Q)$$

(3) Regarding now the best choice of the irreducible polynomial $Q \in \mathbb{F}_p[X]$, providing us with a good model for the finite field $\mathbb{F}_q$, that we can use in practice, this question depends on the value of $q = p^k$, and many things can be said here. All in all, our models are quite similar to $\mathbb{C} = \mathbb{R}[i]$, with i being a formal number satisfying $i^2 = -1$.

(4) To be more precise, at the simplest exponent, $q = 4$, to start with, we can use $Q = X^2 + X + 1$, with this being actually the unique possible choice of a degree 2 irreducible polynomial $Q \in \mathbb{F}_2[X]$, and this leads to a model as follows:

$$\mathbb{F}_4 = \left\{ 0, 1, a, a + 1 \,\middle|\, a^2 = a + 1 \right\}$$

To be more precise here, we assume of course that the characteristic of our model is $p = 2$, which reads $x + x = 0$ for any x, and so determines the addition table. As for the multiplication table, this is uniquely determined by $a^2 = -a - 1 = a + 1$.

(5) Next, at exponents of type $q = p^2$ with $p \geq 3$ prime, we can use $Q = X^2 - r$, with r being a non-square modulo p, and with $(p-1)/2$ choices here. We are led to:

$$\mathbb{F}_{p^2} = \left\{ a + b\gamma \,\middle|\, \gamma^2 = r \right\}$$

Here, as before with $\mathbb{F}_4$, our formula is something self-explanatory. Observe the analogy with $\mathbb{C} = \mathbb{R}[i]$, with i being a formal number satisfying $i^2 = -1$.

(6) Finally, at $q = p^k$ with $k \geq 3$ things become more complicated, but the main idea remains the same. We have for instance models for $\mathbb{F}_8$, $\mathbb{F}_{27}$ using $Q = X^3 - X - 1$, and a model for $\mathbb{F}_{16}$ using $Q = X^4 + X + 1$. Many other things can be said here. □

As another application of the above, which motivated Galois, we have:

THEOREM 12.14. *Unlike in degree $N \leq 4$, there is no formula for the roots of polynomials of degree $N = 5$ and higher, with the reason for this, coming from Galois theory, being that S_5 is not solvable. The simplest numeric example is $P = X^5 - X - 1$.*

PROOF. This is something quite tricky, the idea being as follows:

(1) The first assertion, for generic polynomials, is due to Abel-Ruffini, but Galois theory helps in better understanding this, and comes with a number of bonus points too, namely the possibility of formulating a finer result, with Abel-Ruffini's original "generic", which was something algebraic, being now replaced by an analytic "generic", and also with the possibility of dealing with concrete polynomials, such as $P = X^5 - X - 1$.

(2) Regarding now the details of the Galois proof of the Abel-Ruffini theorem, assume that the roots of a polynomial $P \in F[X]$ can be computed by using iterated roots, a bit as for the degree 2 equation, or for the degree 3 and 4 equations, via Cardano. Then, algebrically speaking, this gives rise to a tower of fields as folows, with $F_0 = F$, and each F_{i+1} being obtained from F_i by adding a root, $F_{i+1} = F_i(x_i)$, with $x_i^{n_i} \in F_i$:

$$F_0 \subset F_1 \subset \ldots \subset F_k$$

(3) In order for Galois theory to apply well to this situation, we must make all the extensions normal, which amounts in replacing each $F_{i+1} = F_i(x_i)$ by its extension $K_i(x_i)$, with K_i extending F_i by adding a n_i-th root of unity. Thus, with this replacement, we can assume that the tower in (2) in normal, meaning that all Galois groups are cyclic.

(4) Now by Galois theory, at the level of the corresponding Galois groups we obtain a tower of groups as follows as follows, which is a resolution of the last group G_k, the Galois group of P, in the sense of group theory, in the sense that all quotients are cyclic:

$$G_1 \subset G_2 \subset \ldots \subset G_k$$

As a conclusion, Galois theory tells us that if the roots of a polynomial $P \in F[X]$ can be computed by using iterated roots, then its Galois group $G = G_k$ must be solvable.

(5) In the generic case, the conclusion is that Galois theory tells us that, in order for all polynomials of degree 5 to be solvable, via square roots, the group S_5, which appears there as Galois group, must be solvable, in the sense of group theory. But this is wrong, because the alternating subgroup $A_5 \subset S_5$ is simple, and therefore not solvable.

(6) Finally, regarding the polynomial $P = X^5 - X - 1$, some elementary computations here, based on arithmetic over $\mathbb{F}_2, \mathbb{F}_3$, and involving various cycles of length $2, 3, 5$, show that its Galois group is S_5. Thus, we have our counterexample.

(7) To be more precise, our polynomial factorizes over $\mathbb{F}_2$ as follows:

$$X^5 - X - 1 = (X^2 + X + 1)(X^3 + X^2 + 1)$$

We deduce from this the existence of an element $\tau\sigma \in G \subset S_5$, with $\tau \in S_5$ being a transposition, and with $\sigma \in S_5$ being a 3-cycle, disjoint from it. Thus, we have:

$$\tau = (\tau\sigma)^3 \in G$$

(8) On the other hand since $P = X^5 - X - 1$ is irreducible over $\mathbb{F}_5$, we have as well available a certain 5-cycle $\rho \in G$. Now since $< \tau, \rho >= S_5$, we conclude that the Galois group of P is full, $G = S_5$, and by (4) and (5) we have our counterexample.

(9) Finally, as mentioned in (1), all this shows as well that a random polynomial of degree 5 or higher is not solvable by square roots, and with this being an elementary consequence of the main result from (5), via some standard analysis arguments. □

So long for the main ideas of Galois theory. In practice, what we said in the above was of course something very quick, and for further learning such things, which normally takes some time, we recommend any solid abstract algebra book, such as Lang [63].

Finally, since this was our last pure algebra section in this book, with most of what is to follow being analysis, let us recommend as well some general algebra reading:

(1) Speaking algebra in general, first comes linear algebra, no doubt about this. There are many texts available here, and if you enjoy the present read, you can go with [9].

(2) Then, as mentioned above, for foundations and generalities, and everything you need to know, comes a solid abstract algebra book, such as Lang [63].

(3) Then comes commutative algebra, say learned from Atiyah and MacDonald [7], and algebraic geometry, from Harris [46], Hartshorne [47] or Shafarevich [82].

(4) And then, you are free to learn what you want, from all the algebra greats who wrote great books, such as Atiyah, Connes, Grothendieck, Serre and others.

12c. Transcendence of e

Now that we are done with advanced algebra, time for some tough calculus, hopefully of the same technical level. We first have the following result, about e:

THEOREM 12.15. *The number e from analysis, given by*

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

which numerically means $e = 2.7182818284\ldots$*, is irrational.*

PROOF. Many things can be said here, as follows:

(1) To start with, there are several possible definitions for e, with the old style one, which is quite cool, and that we used in this book too, being as follows:

$$\left(1 + \frac{1}{n}\right)^n \to e$$

The definition in the statement is the modern one. Indeed, imagine that you are looking for a function exp, satisfying $\exp' = \exp$, and $\exp(0) = 1$. With $\exp(x) = \sum c_k x^k$, you must have $c_0 = 1$, then $c_1 = 1$, $c_2 = 1/2$, $c_3 = 1/6$ and so on, meaning:

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

But now, it is an easy exercice to show that $\exp(x + y) = \exp(x)\exp(y)$, which gives $\exp(x) = e^x$, for a certain number $e > 0$. Which number e can only be $e = \exp(1)$.

(2) Getting now to numerics, the series of e converges very fast, when compared to the old style sequence in (1), so if you are in a hurry, this series is for you. We have:

$$\begin{aligned}
e &= \sum_{k=0}^{N-1} \frac{1}{k!} + \frac{1}{N!}\left(1 + \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \ldots\right) \\
&< \sum_{k=0}^{N-1} \frac{1}{k!} + \frac{1}{N!}\left(1 + \frac{1}{N+1} + \frac{1}{(N+1)^2} + \ldots\right) \\
&= \sum_{k=0}^{N-1} \frac{1}{k!} + \frac{1}{N!}\left(1 + \frac{1}{N}\right) \\
&= \sum_{k=0}^{N} \frac{1}{k!} + \frac{1}{N \cdot N!}
\end{aligned}$$

Thus, the error term in the approximation is really tiny, the estimate being:

$$\sum_{k=0}^{N}\frac{1}{k!}<e<\sum_{k=0}^{N}\frac{1}{k!}+\frac{1}{N\cdot N!}$$

(3) Now by using this, you can easily compute the decimals of e. Actually, you can't call yourself mathematician, or scientist, if you haven't done this by hand, just for the fun, but just in case, here is how the approximation goes, for small values of N:

$$N=2\implies 2.5<e<2.75$$

$$N=3\implies 2.666\ldots<e<2.722\ldots$$

$$N=4\implies 2.70833\ldots<e<2.71875\ldots$$

$$N=5\implies 2.71666\ldots<e<2.71833\ldots$$

$$N=6\implies 2.71805\ldots<e<2.71828\ldots$$

$$N=7\implies 2.71825\ldots<e<2.71828\ldots$$

Thus, first 4 decimals computed, $e=2.7182\ldots$, and I would leave the continuation to you. With the remark that, when carefully looking at the above, the estimate on the right works much better than the one on the left, so before getting into more serious numerics, try to find a better lower estimate for e, that can help you in your work.

(4) Getting now to irrationality, a look at $e=2.7182818284\ldots$ might suggest that the $81,82,84\ldots$ values might eventually, after some internal fight, decide for a winner, and so that e might be rational. However, this is wrong, and e is in fact irrational.

(5) So, let us prove now this, that e is irrational. Following Fourier, we will do this by contradiction. So, assume $e=m/n$, and let us look at the following number:

$$x=n!\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)$$

As a first observation, x is an integer, as shown by the following computation:

$$\begin{aligned}x&=n!\left(\frac{m}{n}-\sum_{k=0}^{n}\frac{1}{k!}\right)\\&=m(n-1)!-\sum_{k=0}^{n}n(n-1)\ldots(n-k+1)\\&\in\mathbb{Z}\end{aligned}$$

On the other hand $x > 0$, and we have as well the following estimate:

$$\begin{aligned} x &= n! \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots \\ &= \frac{1}{n} \end{aligned}$$

Thus $x \in (0,1)$, which contradicts our previous finding $x \in \mathbb{Z}$, as desired. □

As a continuation, we have the following result, which is substantially harder:

THEOREM 12.16. *The number e is transcendental.*

PROOF. Assume by contradiction that e is algebraic, with this meaning that it is a root of a polynomial with integer coefficients, $c_i \in \mathbb{Z}$, as follows:

$$c_0 + c_1 e + \ldots + c_n e^n = 0$$

(1) Following Hermite, consider the following polynomials, and we will see later why:

$$f_k(x) = x^k \left[(x-1)\ldots(x-n)\right]^{k+1}$$

Consider also the following quantities, that we will study more in detail later:

$$A_k = \int_0^{\infty} f_k(x) e^{-x} dx$$

By multiplying our equation for e by this quantity A_k, we obtain:

$$c_0 \int_0^{\infty} f_k(x) e^{-x} dx + c_1 \int_0^{\infty} f_k(x) e^{1-x} dx + \ldots c_n \int_0^{\infty} f_k(x) e^{n-x} dx = 0$$

(2) Here comes the trick. Consider the following two quantities:

$$P = c_0 \int_0^{\infty} f_k(x) e^{-x} dx + c_1 \int_1^{\infty} f_k(x) e^{1-x} dx + \ldots + c_n \int_n^{\infty} f_k(x) e^{n-x} dx$$

$$Q = c_1 \int_0^{1} f_k(x) e^{-x} dx + \ldots + c_n \int_0^{n} f_k(x) e^{n-x} dx$$

In terms of these quantities, the formula that we found in (1) reads:

$$P + Q = 0$$

(3) Now let us look at P. Our claim is that this is an integer, $P \in \mathbb{Z}$, and that there are arbitrarily large numbers $k >> 0$ for which the following holds:

$$\frac{P}{k!} \in \mathbb{Z} - \{0\}$$

Indeed, according to our formula above defining P, we have:

$$\begin{aligned} P &= \sum_{r=0}^{n} c_r \int_r^\infty f_k(x) e^{r-x} dx \\ &= \sum_{r=0}^{n} c_r \int_0^\infty f_k(x+r) e^{-x} dx \\ &= \int_0^\infty \left(\sum_{r=0}^{n} c_r f_k(x+r) \right) e^{-x} dx \end{aligned}$$

On the other hand, integrating such functions is easy, according to:

$$\begin{aligned} \int_0^\infty x^s e^{-x} dx &= \int_0^\infty \left(\frac{x^{s+1}}{s+1} \right)' e^{-x} dx \\ &= \int_0^\infty \frac{x^{s+1}}{s+1} e^{-x} dx \\ &= \frac{1}{s+1} \int_0^\infty x^{s+1} e^{-x} dx \end{aligned}$$

Thus, we are led by recurrence on $s \in \mathbb{N}$ to the following formula:

$$\int_0^\infty x^s e^{-x} dx = s!$$

For a linear combination now, we are led to the following formula:

$$g(x) = \sum_s a_s x^s \implies \int_0^\infty g(x) e^{-x} dx = \sum_s a_s s!$$

But this shows that we have indeed $P \in \mathbb{Z}$, and also, via a bit of study based on the exact formula of f_k, from the beginning of (1), that we have in fact:

$$\frac{P}{k!} \in \mathbb{Z}$$

Finally, we still have to prove that we have $P \neq 0$, for arbitrarily large numbers $k >> 0$. But the point here is that for $k+1 > n, |c_0|$, chosen prime, a detailed study of our integral shows that we have $(k+1) \not| P$, and so $P \neq 0$ indeed, as desired.

(4) With this done, let us look now at Q. Our claim is that for $k >> 0$ we have:

$$\left|\frac{Q}{k!}\right| < 1$$

Indeed, by using the exact formula of f_k, from the beginning of (1), we have:

$$\begin{aligned} f_k(x)e^{-x} &= x^k\left[(x-1)\ldots(x-n)\right]^{k+1}e^{-x} \\ &= \left[x(x-1)\ldots(x-n)\right]^k\times(x-1)\ldots(x-n)e^{-x} \end{aligned}$$

We conclude that for $x\in[0,n]$ we have an estimate as follows, with $G,H>0$ being certain constants, appearing as maxima of the two components appearing above:

$$|f_k(x)e^{-x}| < G^kH$$

Now by integrating, we obtain from this the following estimate for Q itself:

$$\begin{aligned} |Q| &= \left|c_1\int_0^1 f_k(x)e^{-x}dx+\ldots+c_ne^n\int_0^n f_k(x)e^{-x}dx\right| \\ &\leq |c_1|\int_0^1|f_k(x)e^{-x}|dx+\ldots+|c_n|e^n\int_0^n|f_k(x)e^{-x}|dx \\ &\leq |c_1|\cdot G^kH+\ldots+|c_n|e^n\cdot nG^kH \\ &= (|c_1|e+\ldots+|c_n|e^n)\,\frac{n(n+1)}{2}\,G^kH \end{aligned}$$

But in this estimate the only term depending on k is the power G^k, and since since $k!$ grows much faster than this power G^k, this proves our claim:

$$\left|\frac{Q}{k!}\right| \approx \frac{G^k}{k!} \to 0$$

(5) And with this, done, because what we found in (3,4) contradicts the formula $P+Q=0$ from (2). Thus e is indeed transcendental, as claimed. □

12d. Transcendence of pi

As a continuation of the above, we would like to prove now, a bit as for e before, that π is irrational, and even transcendental. Before that, however, again as before for e, let us start with some approximation work. We first have the following result:

PROPOSITION 12.17. *We have the following approximations of π,*

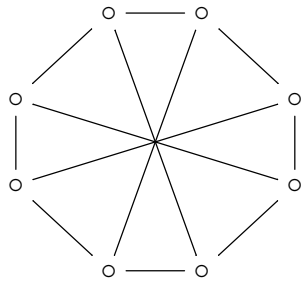
$$2.828 < \pi < 4$$

$$3 < \pi < 3.464$$

$$3.061 < \pi < 3.314$$

obtained respectively by using squares, hexagons and octagons.

PROOF. Leaving the squares and hexagons for you, respectively chess and badminton, let me get directly to MMA questions, in relation with the octagon:



The octagons inscribed and circumscribed to the unit circle have edges as follows, coming by looking at the respective "pizza slices", which are both $45-67.5-67.5$ triangles, the inner ones having radial edge 1, and the outer ones having radial altitude 1:

$$e = 2\sin(22.5^\circ) \quad , \quad E = 2\tan(22.5^\circ)$$

Thus by looking at half perimeters, we obtain the following estimate for π:

$$4e < \pi < 4E \quad \implies \quad 8\sin(22.5^\circ) < \pi < 8\tan(22.5^\circ)$$

We have as well an estimate for π coming by looking at areas, which is as follows, using the fact that the area of a $45-67.5-67.5$ triangle is $\cot(22.5^\circ) \times \text{edge}^2/4$:

$$\begin{aligned} & 2\cot(22.5^\circ)e^2 < \pi < 2\cot(22.5^\circ)E^2 \\ \implies \quad & 8\sin(22.5^\circ)\cos(22.5^\circ) < \pi < 8\tan(22.5^\circ) \\ \implies \quad & 4\sin(45^\circ) < \pi < 8\tan(22.5^\circ) \end{aligned}$$

In order to reach now to some concrete numeric estimates, we first have:

$$\cos(22.5^\circ) = \sqrt{\frac{1+\cos 45^\circ}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

For the sine we can use Pythagoras, $\sin^2 + \cos^2 = 1$, and we obtain:

$$\sin(22.5^\circ) = \sqrt{1 - \frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

Finally, by taking the quotient we obtain a formula for the tangent, as follows:

$$\tan(22.5^\circ) = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \sqrt{2} - 1$$

Time now for the final numerics. The half perimeter estimate for π reads:

$$\begin{aligned} 8\sin(22.5^\circ) < \pi < 8\tan(22.5^\circ) \quad & \implies \quad 4\sqrt{2-\sqrt{2}} < \pi < 8(\sqrt{2}-1) \\ & \implies \quad 3.061 < \pi < 3.314 \end{aligned}$$

As for the area estimate for π, also found above, this reads:

$$\begin{aligned} 4\sin(45^\circ) < \pi < 8\tan(22.5^\circ) &\implies 4/\sqrt{2} < \pi < 8(\sqrt{2}-1) \\ &\implies 2.828 < \pi < 3.314 \end{aligned}$$

Thus, we are led to the conclusions in the statement, and with the remark that the half perimeter estimate beasts the area estimate, which is good to know. □

The above is quite encouraging, and getting now to the general case, that of the arbitrary polygons, we have the following result, further building on what we have:

THEOREM 12.18. *We have the following estimates for π, obtained by computing the half perimeter of an inscribed and circumscribed regular N-gon:*

$$N\sin\left(\frac{180^\circ}{N}\right) < \pi < N\tan\left(\frac{180^\circ}{N}\right)$$

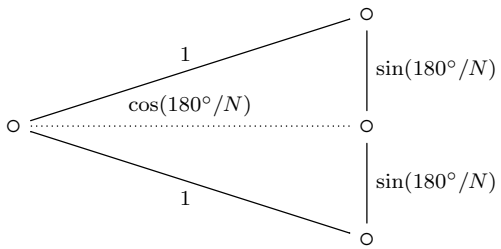
By using the inscribed and circumscribed regular hexadecagon, these estimates give

$$3.121 < \pi < 3.183$$

and with a bit more work, with $N = 2^n$ with $n >> 0$, we obtain $\pi = 3.14159\ldots$

PROOF. We use the same method as before, the idea being as follows:

(1) In order to compute the edge of the inscribed N-gon, let us look at the corresponding "pizza slices". These are isosceles triangles with angle $360^\circ/N$ and edge 1, so when drawing an altitude and computing the lengths, the picture is as follows:

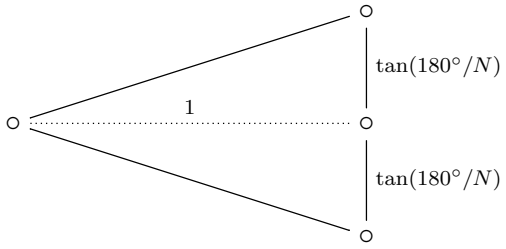


Thus, the edge and half perimeter of the inscribed N-gon are as follows:

$$e = 2\sin\left(\frac{180^\circ}{N}\right) \quad , \quad p = N\sin\left(\frac{180^\circ}{N}\right)$$

(2) Getting now to the circumscribed N-gon, the "pizza slices" here are again isosceles triangles with angle $360^\circ/N$, but this time with altitude 1. Thus when drawing these

altitudes and computing the relevant lengths, the picture is as follows:



Thus, the edge and half perimeter of the circumscribed N-gon are as follows:

$$E = 2\tan\left(\frac{180^\circ}{N}\right) \quad , \quad P = N\tan\left(\frac{180^\circ}{N}\right)$$

(3) Time now to derive some conclusions, from our study. With our half perimeter method, we obtain the estimate from the statement, namely:

$$N\sin\left(\frac{180^\circ}{N}\right) < \pi < N\tan\left(\frac{180^\circ}{N}\right)$$

Note in passing that, as an alternative idea, we can try to use instead areas. But this gives an estimate which is weaker than the one above, as follows:

$$\frac{N}{2}\sin\left(\frac{360^\circ}{N}\right) < \pi < N\tan\left(\frac{180^\circ}{N}\right)$$

(4) In view of numerics, let us begin with some general trigonometry, regarding the sines and tangents of the halves of angles. For the sines, we have:

$$\begin{aligned}
\cos(2t) = 1 - 2\sin^2 t \quad &\implies \quad (1 - 2\sin^2 t)^2 = 1 - \sin^2(2t)\\
&\implies \quad 1 - 2\sin^2 t = \sqrt{1 - \sin^2(2t)}\\
&\implies \quad 2\sin^2 t = 1 - \sqrt{1 - \sin^2(2t)}\\
&\implies \quad \sin t = \sqrt{\frac{1 - \sqrt{1 - \sin^2(2t)}}{2}}
\end{aligned}$$

As for the tangents, we have here the following computation, and with exercise here for you to figure out why we choose the sign plus, for the square root:

$$\tan(2t)=\frac{2\tan t}{1-\tan^2 t}\quad\implies\quad\tan(2t)\tan^2 t+2\tan t-\tan(2t)=0$$

$$\implies\quad\tan t=\frac{-2+\sqrt{4+4\tan^2(2t)}}{2\tan(2t)}$$

$$\implies\quad\tan t=\frac{\sqrt{1+\tan^2(2t)}-1}{\tan(2t)}$$

(5) Now let us see what we get at $N=16$. With $t=11.25^\circ$, and with trigonometric input concerning $2t=22.5^\circ$ from the proof of Proposition 12.17, we obtain:

$$\sin(11.25^\circ)=\sqrt{\frac{1-\sqrt{1-\frac{2-\sqrt{2}}{4}}}{2}}=\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

As for the tangent, again with $t=11.25^\circ$, and with trigonometric input concerning $2t=22.5^\circ$ from the proof of Proposition 12.17, we obtain:

$$\tan(11.25^\circ)=\frac{\sqrt{1+(\sqrt{2}-1)^2}-1}{\sqrt{2}-1}=\frac{\sqrt{4-2\sqrt{2}}-1}{\sqrt{2}-1}$$

(6) At the level of the numerics, the formulae that we obtain are as follows:

$$\sin(11.25^\circ)=0.915\ldots\quad,\quad\tan(11.25^\circ)=0.198\ldots$$

By multiplying by 16, as required by the estimate that we found, we have:

$$16\sin(11.25^\circ)=3.121\ldots\quad,\quad\tan(11.25^\circ)=3.182\ldots$$

Thus, we are led to the hexadecagon estimate in the statement, namely:

$$3.121<\pi<3.183$$

(7) Regarding now better estimates, the method indicated in the statement, namely using $N=2^n$ with $n>>0$, normally works, with the remark however that the convergence is very slow, and with the extra remark, which is concerning too, that the numeric extraction of square roots, required by our formulae in (4), is no easy business.

(8) In short, and contrary to what we saw before for e, approximating π is no easy business. However, it is possible to come up with some better methods, inspired by the above, and one way or another we are led to the known figure for π, namely:

$$\pi=3.14159\ldots$$

By the way, as a young nerd, make sure that you get to know more decimals of π than I do. So, exercise for you to further explore the subject, and learn more about this. □

With this discussed, let us get now to algebraic questions. The situation for π is a bit similar to that for e, discussed before in this chapter, with however a few notable differences appearing, and at the level of the main results, these are as follows:

THEOREM 12.19. *The number $\pi = 3.14159\ldots$ has the following properties:*

(1) *It is irrational.*
(2) *It is transcendental.*

PROOF. This is indeed something quite routine, by using the same ideas as before for e, but with everything being now a bit more technical, the idea being as follows:

(1) In what regards the irrationality of π, no simple argument as for e is available, so we rather have to take our inspiration from the Hermite proof of the transcendence of e, given above. With this idea in mind, consider the following quantities:

$$I_n(t) = \int_{-1}^{1} (1-x^2)^n \cos(xt)dx$$

By double partial integration we obtain the following formula:

$$\begin{aligned}
I_n(t) &= \int_{-1}^{1} (1-x^2)^n \cos(xt)dx \\
&= \int_{-1}^{1} 2nx(1-x^2)^{n-1}\frac{\sin(xt)}{t}\,dx \\
&= \frac{2n}{t}\int_{-1}^{1} x(1-x^2)^{n-1}\sin(xt)dx \\
&= \frac{2n}{t}\int_{-1}^{1} \left[(1-x^2)^{n-1} - 2(n-1)x^2(1-x^2)^{n-2}\right]\frac{\cos(xt)}{t}\,dx \\
&= \frac{2n}{t^2}\int_{-1}^{1} (1-x^2)^{n-2}\left[1-x^2-2(n-1)x^2\right]\cos(xt)dx \\
&= \frac{2n}{t^2}\int_{-1}^{1} (1-x^2)^{n-2}\left[1-(2n-1)x^2\right]\cos(xt)dx \\
&= \frac{2n}{t^2}\int_{-1}^{1} (1-x^2)^{n-2}\left[(2n-1)(1-x^2)-(2n-2)\right]\cos(xt)dx \\
&= \frac{2n}{t^2}\left[(2n-1)I_{n-1}(t)-(2n-2)I_{n-2}(t)\right]
\end{aligned}$$

Thus, we have the following recurrence relation for our quantities:

$$t^2 I_n(t) = 2n(2n-1)I_{n-1}(t) - 4n(n-1)I_{n-2}(t)$$

In terms of $J_n(t) = t^{2n+1}I_n(t)$, this recurrence formula becomes:

$$J_n(t) = 2n(2n-1)J_{n-1}(t) - 4n(n-1)t^2 J_{n-2}(t)$$

Regarding now the initial data, for this latter recurrence, this is as follows:

$$J_0(t) = 2\sin t \quad , \quad J_1(t) = -4t\cos t + 4\sin t$$

We conclude from this that we must have a formula as follows, with P_n, Q_n being certain polynomials of degree $\leq n$, with integer coefficients:

$$J_n(t) = n!(P_n(t)\sin t + Q_n(t)\cos t)$$

Now observe that with $t = \pi/2$, we obtain from this the following formula:

$$\left(\frac{\pi}{2}\right)^{2n+1} I_n\left(\frac{\pi}{2}\right) = n!P_n\left(\frac{\pi}{2}\right)$$

Assume now by contradiction that π is rational, so that $\pi/2 = a/b$ with $a, b \in \mathbb{N}$. We can rewrite the formula found above in the following more convenient way:

$$\frac{a^{2n+1}}{n!} I_n\left(\frac{a}{b}\right) = b^{2n+1}P_n\left(\frac{a}{b}\right)$$

But, by definition of the integrals I_n, we have $I_n(a/b) = I_n(\pi/2) \in (0, 2)$. Thus with $n >> 0$ the number on the left belongs to $(0, 1)$, which is contradictory, because the number on the right is an integer. We conclude that π is irrational, as claimed.

(2) Regarding now the transcendence of π, again it is possible to adapt the ideas of Hermite for e, from the proof of Theorem 12.16, but this remains something quite technical. Instead, it is better to have an algebraic look at this, by using the Lindemann-Weierstrass theorem, which states that if $a_1, \ldots, a_n \in \mathbb{C}$ are algebrically independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebrically independent too over $\mathbb{Q}$. To be more precise:

– To start with, observe that the Lindemann-Weierstrass theorem shows with $n = 1$ that $e = e^1$ is transcendental. Thus, this is definitely something non-trivial.

– However, this is something that can be proved, with some knowledge of algebra and field theory, in the spirit of what we did in the beginning of this chapter.

– And, in relation with π, we can use again $n = 1$, but this time in conjunction with the Euler formula $e^{\pi i} = -1$, and we obtain that π is transcendental. □

As a last piece of mathematics, in relation with trigonometry and approximations, we have the following remarkable result, concerning π itself, due to Buffon:

THEOREM 12.20. *The probability for a needle of length* 1*, when trown on a grid of parallel* 1*-spaced lines, to intersect one line, is:*

$$P = \frac{2}{\pi}$$

Moreover, we have generalizations of this result, with needles of arbitrary length, thrown over a grid of parallel lines, with arbitrary spacing.

PROOF. This is something quite tricky, and mandatory for properly learning probability theory, and science in general, because there are several possible modelings of the problem, leading, quite surprisingly, to different values of P. And, obviously, only one such modeling can be the correct one. We will leave this as an exercise, and enjoy. □

12e. Exercises

This was a quite technical and exciting chapter, and as exercises on this, we have:

EXERCISE 12.21. *Read about rational functions, and what can be done with them.*

EXERCISE 12.22. *Review as well the other basic examples of characteristic* 0 *fields.*

EXERCISE 12.23. *Learn more about ideals, and their use in algebraic geometry.*

EXERCISE 12.24. *Learn more about the algebraic aspects of the compact spaces.*

EXERCISE 12.25. *Learn full Galois theory, including all the needed preliminaries.*

EXERCISE 12.26. *Experiment a bit, with the modeling of various finite fields.*

EXERCISE 12.27. *Clarify what we said, in relation with the transcendence of* e.

EXERCISE 12.28. *Clarify also what we said in relation with the transcendence of* π.

As bonus exercise, read about the Nullstellensatz. This is what algebra is about.

Part IV

Number theory

Because the night belongs to lovers
Because the night belongs to lust
Because the night belongs to lovers
Because the night belongs to us

CHAPTER 13

Primes, revised

13a. Euler estimates

Welcome back to analytic number theory, in relation with the Euler formula, and its ramifications. Let us start with a summary of what we know so far. We first have:

THEOREM 13.1 (Euler). *We have the following formula, P being the primes,*

$$\sum_{n=1}^{\infty}\frac{1}{n}=\prod_{p\in P}\left(1-\frac{1}{p}\right)^{-1}$$

which by standard analysis gives the following estimate,

$$\sum_{p\in P}\frac{1}{p}+\frac{1}{2}>\log\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)=\infty$$

which in turn implies $|P|=\infty$.

PROOF. This is something that we know from chapter 7, but always good to talk about it again. The Euler formula comes from $n=p_1^{a_1}\ldots p_k^{a_k}$, as follows:

$$\sum_{n=1}^{\infty}\frac{1}{n}=\prod_{p\in P}\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots\right)=\prod_{p\in P}\left(1-\frac{1}{p}\right)^{-1}$$

Now observe that the log of the product on the right can be estimated as follows:

$$\begin{aligned}-\sum_{p\in P}\log\left(1-\frac{1}{p}\right)&=\sum_{p\in P}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\\&<\sum_{p\in P}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{2p^3}+\frac{1}{2p^4}+\ldots\\&=\sum_{p\in P}\frac{1}{p}+\frac{1}{2}\sum_{p\in P}\left(\frac{1}{p-1}-\frac{1}{p}\right)\\&<\sum_{p\in P}\frac{1}{p}+\frac{1}{2}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The Euler formula and its proof are something of utter beauty, suggesting doing an enormous amount of things, and yes indeed, doing such things has been one of the favorite pastimes of mathematicians, since. Here is a brief account, of all this:

(1) The Euler formula $\sum_{p\in P} 1/p = \infty$ basically tells us that there are "many primes", but what about the opposite, trying now to prove that there are "few primes"? Well, this comes too from the Euler formula, but in its refined version, which is as follows:

$$\sum_{p<N} \frac{1}{p} \simeq \log\log N$$

Many things can be done here, one of the conclusions being that the N-th prime $\pi(N)$ satisfies $\pi(N) \sim N/\log N$. We will be back to this later in this book.

(2) Still talking analysis, an interesting observation, by Erdős, coming from his own work on the Euler formula, regards the sets $S \subset \mathbb{N}$ satisfying the following condition:

$$\sum_{s\in S} \frac{1}{s} = \infty$$

Based on this, Erdős conjectured that such sets S contain arbitrarily long arithmetic progessions. And the point is that this is a very difficult and fascinating problem, with the case $S = P$ being settled only recently, in the mid 2000s, by Green and Tao.

(3) Leaving aside now estimates and analysis, and going back to the beginning of Euler's proof, let us look more in detail at the formula there, namely:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p\in P} \left(1 - \frac{1}{p}\right)^{-1}$$

This formula is something really beautiful, and the more you look at it, thinking at versions and so on, the more you are lost into the mysteries of number theory.

(4) To be more precise, the above formula suggests introducing the following function, depending on a parameter s, which can be integer, real, or even complex:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

And this is the famous Riemann zeta function, which obsesses all number theorists, be them algebraists, analysts, geometers, physicists, or amateurs. We will be talking about this magical function later in this book, after learning some more analysis.

So, this will be the plan for the present Part IV of this book, talking about all this. In practice now, "shut up and compute" as Dirac used to say, and as a good entry point to all this, we can just try to improve Theorem 13.1, with (1) above in mind.

But this is something that we already studied in chapter 7, with the summary of our conclusions there, obtained via some substantial work, being as follows:

THEOREM 13.2. *We have the following formula, with sum over primes,*

$$\sum_{p<N}\frac{1}{p}>\log\log N-C$$

with the constant on the right being as follows, depending on your skills:

(1) *Basic level:* $C=1/2$.
(2) *Intermediate:* $C=\log 2=0.69314..$
(3) *Advanced:* $C=\log(\pi^2/6)=0.49770..$

PROOF. We know all this from chapter 7, the idea being as follows:

(1) This comes by using the prime factorization $n=p_1^{a_1}\ldots p_k^{a_k}$ and truncating all the computations in proof of Theorem 13.1, as follows:

$$\begin{aligned}\sum_{p<N}\frac{1}{p}+\frac{1}{2}&>\sum_{p<N}\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\ldots\\&=\log\left[\prod_{p<N}\left(1-\frac{1}{p}\right)^{-1}\right]\\&>\log\left(\sum_{n=1}^{N-1}\frac{1}{n}\right)\\&>\log\log N\end{aligned}$$

(2) This comes by using the rival factorization $n=p_1\ldots p_km^2$, with p_i distinct primes, which gives, via a telescoping argument for $\sum 1/m^2$, the following estimate:

$$\begin{aligned}\sum_{p<N}\frac{1}{p}+\log 2&>\log\left[\prod_{p<N}\left(1+\frac{1}{p}\right)\right]+\log 2\\&>\log\left[\prod_{p<N}\left(1+\frac{1}{p}\right)\prod_{m=1}^{N}\frac{1}{m^2}\right]\\&>\log\left(\sum_{n=1}^{N-1}\frac{1}{n}\right)\\&>\log\log N\end{aligned}$$

Observe that is estimate, supposedly at the intermediate level, is weaker than (1).

(3) This follows by further building on (2), by replacing the telescoping estimate $\sum 1/m^2 < 2$ used there by the following exact formula, also due to Euler:

$$\sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}$$

As for the proof of this latter formula, called Basel formula, this intuitively comes from the following manipulation on the Taylor series of $\sin x/x$, based on the fact that the zeroes of this function appear precisely at the points $x=k\pi$, with $k\in\mathbb{Z}$:

$$\begin{aligned}\frac{\sin x}{x}&=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\\&=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\\&=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots\\&=1-\frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\,x^2+\ldots\end{aligned}$$

In practice, all this needs of course a bit more justification. We will be back to this. □

Before moving ahead with the Mertens theorems, substantially improving the above, several comments are in order, with respect to all this. Let us introduce:

DEFINITION 13.3. *Associated to any $s>1$ is the function*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

called Riemann zeta function.

Observe that the above series converges indeed, as a Riemann sum approximation, by usual rectangles, of the following convergent integral:

$$\begin{aligned}\int_1^{\infty}\frac{1}{x^s}\,dx&=\left[\frac{x^{1-s}}{1-s}\right]_1^{\infty}\\&=0-\frac{1}{1-s}\\&=\frac{1}{s-1}\\&<\infty\end{aligned}$$

Based on this, we can further say that, more generally, the series converges for any exponent $s\in\mathbb{C}$ satisfying $Re(s)>1$. But more on this, later in this book.

As a first observation, the Basel formula of Euler reformulates as follows:

THEOREM 13.4. *We have the following formula, coming from the Basel problem:*

$$\zeta(2)=\frac{\pi^2}{6}$$

More generally, any value $\zeta(2k)$ with $k\in\mathbb{N}$ is a rational multiple of π^{2k}.

PROOF. Here the formula of $\zeta(2)$ is from the proof of Theorem 13.2, and the generalization to $\zeta(2k)$ with $k\in\mathbb{N}$ comes by further studying the argument there, namely:

$$\begin{aligned}\frac{\sin x}{x}&=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\\&=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots\\&=1-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,x^2+\ldots\end{aligned}$$

To be more precise, after some combinatorial work, that we will not get into here, right now, we are led to the following formula, with B_n being the Bernoulli numbers:

$$\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}$$

In practice, this gives the following fomulae for the first few values $\zeta(2k)$:

$$\zeta(2)=\frac{\pi^2}{6}\quad,\quad\zeta(4)=\frac{\pi^4}{90}\quad,\quad\zeta(6)=\frac{\pi^6}{945}\quad,\quad\zeta(8)=\frac{\pi^8}{9450}$$

We will come back to this later in this book, with full details, the idea being that such questions are best discussed by using complex analysis. More later. □

Many other things can be said about zeta, along the same lines, but it is not about this that we want to talk, in this chapter, with all this zeta material being deferred to chapter 15 below. What we want to discuss here is what happens to the Euler estimate from Theorem 13.2, when adding an exponent $s\in\mathbb{R}$ there. Let us start with:

PROPOSITION 13.5. *The Euler estimate can be generalized into*

$$\sum_{p<N}\frac{1}{p^s}>\log\left(\int_1^N\frac{1}{x^s}\,dx\right)-\frac{1}{2}\sum_{n=2}^{N-1}\frac{1}{n^s(n^s-1)}$$

with the above integral given by the formula

$$\int_1^N\frac{1}{x^s}\,dx=\begin{cases}\frac{N^{1-s}-1}{1-s}&\text{if }s\neq1\\\log N&\text{if }s=1\end{cases}$$

involving now a real parameter $s\in\mathbb{R}$, with exactly the same proof.

PROOF. By using the unique factorization $n = p_1^{a_1} \dots p_k^{a_k}$, as before, we have:

$$\begin{aligned}\prod_{p<N}\left(1-\frac{1}{p^s}\right)^{-1} &= \prod_{p<N}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\dots\right)\\ &> \sum_{n=1}^{N-1}\frac{1}{n^s}\\ &> \int_1^N \frac{1}{x^s}\,dx\end{aligned}$$

But the product on the left can be estimated by using log, as follows:

$$\begin{aligned}\log\left[\prod_{p<N}\left(1-\frac{1}{p^s}\right)^{-1}\right] &= -\sum_{p<N}\log\left(1-\frac{1}{p^s}\right)\\ &= \sum_{p<N}\frac{1}{p^s}+\frac{1}{2p^{2s}}+\frac{1}{3p^{3s}}+\frac{1}{4p^{4s}}+\dots\\ &< \sum_{p<N}\frac{1}{p^s}+\frac{1}{2p^{2s}}+\frac{1}{2p^{3s}}+\frac{1}{2p^{4s}}+\dots\\ &= \sum_{p<N}\frac{1}{p^s}+\frac{1}{2}\sum_{p<N}\frac{1}{p^s}\cdot\frac{1}{1-1/p^s}\\ &= \sum_{p<N}\frac{1}{p^s}+\frac{1}{2}\sum_{p<N}\frac{1}{p^s(p^s-1)}\\ &< \sum_{p<N}\frac{1}{p^s}+\frac{1}{2}\sum_{n=2}^{N-1}\frac{1}{n^s(n^s-1)}\end{aligned}$$

Thus, we are led to the estimate in the statement. □

In the case $s > 1$, which is the one of main interest, we obtain in this way:

THEOREM 13.6. *We have the following Euler type estimate*

$$\sum_{p<N}\frac{1}{p^s} > \log\left(\frac{1-N^{1-s}}{s-1}\right)-\frac{\zeta(2s)}{2}$$

valid for any value of the parameter $s > 1$.

PROOF. In the case $s > 1$ the estimate that we found in Proposition 13.5 gives:

$$\begin{aligned}
\sum_{p<N}\frac{1}{p^s} &> \log\left(\frac{1-N^{1-s}}{s-1}\right)-\frac{1}{2}\sum_{n=2}^{N-1}\frac{1}{n^s(n^s-1)}\\
&> \log\left(\frac{1-N^{1-s}}{s-1}\right)-\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n^s(n^s-1)}\\
&> \log\left(\frac{1-N^{1-s}}{s-1}\right)-\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{(n-1)^{2s}}\\
&> \log\left(\frac{1-N^{1-s}}{s-1}\right)-\frac{\zeta(2s)}{2}
\end{aligned}$$

Here we have used the following inequality, with $\varepsilon = 1/n < 1$, which is true:

$$\begin{aligned}
\frac{1}{n^s(n^s-1)} < \frac{1}{(n-1)^{2s}} &\iff (n-1)^{2s} < n^s(n^s-1)\\
&\iff \left(1-\frac{1}{n}\right)^{2s} < 1-\frac{1}{n^s}\\
&\iff (1-\varepsilon)^{2s} < 1-\varepsilon^s\\
&\iff (1-\varepsilon)^{2s-1} < \frac{1-\varepsilon^s}{1-\varepsilon}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

It is possible to futher build along the above lines, but we will leave this discussion for later, in chapter 15, when talking more in detail about the Riemann zeta function.

13b. Stirling formula

In order to advance with our estimates, we are in need of more calculus input. There are several possible paths that can be taken here, and perhaps the most straightforward one is via the Stirling formula. This formula, which is obviously related to arithmetic, and is related to many other things too, as we will soon discover, is as follows:

THEOREM 13.7. *We have the Stirling formula*

$$N! \simeq \left(\frac{N}{e}\right)^N\sqrt{2\pi N}$$

valid in the $N \to \infty$ limit.

PROOF. This is something quite tricky, the idea being as follows:

(1) Let us first see what we can get with Riemann sums. We have:

$$\begin{aligned}\log(N!) &= \sum_{k=1}^{N}\log k\\ &\approx \int_1^N \log x\,dx\\ &= N\log N-N+1\end{aligned}$$

By exponentiating, this gives the following estimate, which is not bad:

$$N!\approx\left(\frac{N}{e}\right)^N\cdot e$$

(2) We can improve our estimate by replacing the rectangles from the Riemann sum approach to the integrals by trapezoids. In practice, this gives the following estimate:

$$\begin{aligned}\log(N!) &= \sum_{k=1}^{N}\log k\\ &\approx \int_1^N \log x\,dx+\frac{\log 1+\log N}{2}\\ &= N\log N-N+1+\frac{\log N}{2}\end{aligned}$$

By exponentiating, this gives the following estimate, which gets us closer:

$$N!\approx\left(\frac{N}{e}\right)^N\cdot e\cdot\sqrt{N}$$

(3) In order to conclude, we must take some kind of mathematical magnifier, and carefully estimate the error made in (2). Fortunately, this mathematical magnifier exists, called Euler-Maclaurin formula, and after some tough computations, we get to:

$$N!\simeq\left(\frac{N}{e}\right)^N\sqrt{2\pi N}$$

(4) However, all this remains a bit complicated, so we would like to present now an alternative approach to (3), which also misses some details, but better does the job, explaining where the $\sqrt{2\pi}$ factor comes from. First, by partial integration we have:

$$N!=\int_0^\infty x^Ne^{-x}dx$$

(5) Since the integrand is sharply peaked at $x = N$, as you can see by computing the derivative of $\log(x^N e^{-x})$, this suggests writing $x = N + y$, and we obtain:

$$\begin{aligned}
\log(x^N e^{-x}) &= N \log x - x \\
&= N \log(N + y) - (N + y) \\
&= N \log N + N \log\left(1 + \frac{y}{N}\right) - (N + y) \\
&\simeq N \log N + N\left(\frac{y}{N} - \frac{y^2}{2N^2}\right) - (N + y) \\
&= N \log N - N - \frac{y^2}{2N}
\end{aligned}$$

(6) By exponentiating, we obtain from this the following estimate:

$$x^N e^{-x} \simeq \left(\frac{N}{e}\right)^N e^{-y^2/2N}$$

(7) Now by integrating, we obtain from this the following estimate:

$$\begin{aligned}
N! &= \int_0^\infty x^N e^{-x} dx \\
&\simeq \int_{-N}^N \left(\frac{N}{e}\right)^N e^{-y^2/2N}\, dy \\
&\simeq \left(\frac{N}{e}\right)^N \int_{\mathbb{R}} e^{-y^2/2N}\, dy \\
&= \left(\frac{N}{e}\right)^N \sqrt{2N} \int_{\mathbb{R}} e^{-z^2}\, dz \\
&= \left(\frac{N}{e}\right)^N \sqrt{2\pi N}
\end{aligned}$$

(8) Here we have used at the end the following key formula, due to Gauss:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-z^2}dz\right)^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2}dxdy\\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrdt\\ &= 2\pi\int_0^{\infty}\left(-\frac{e^{-r^2}}{2}\right)'dr\\ &= 2\pi\left[0-\left(-\frac{1}{2}\right)\right]\\ &= \pi\end{aligned}$$

Thus, we have proved the Stirling formula, as formulated in the statement.

(9) Finally, as already mentioned, there are several other proofs for the Stirling formula, with some of them telling us something about the error term too. All this is very interesting mathematics, and exercise for you to learn all this, as much as you can. □

As a basic application of the Stirling formula, which is something very useful, in practice, let us record the following result, dealing with binomial coefficients:

THEOREM 13.8. *We have the following estimate for binomial coefficients,*

$$\binom{N}{K} \simeq \left(\frac{1}{t^t(1-t)^{1-t}}\right)^N \frac{1}{\sqrt{2\pi t(1-t)N}}$$

in the $K \simeq tN \to \infty$ limit, with $t \in (0,1]$. In particular we have

$$\binom{2N}{N} \simeq \frac{4^N}{\sqrt{\pi N}}$$

in the $N \to \infty$ limit, for the central binomial coefficients.

PROOF. The first formula follows indeed from Stirling, as follows:

$$\begin{aligned}\binom{N}{K} &= \frac{N!}{K!(N-K)!}\\ &\simeq \left(\frac{N}{e}\right)^N\sqrt{2\pi N}\left(\frac{e}{K}\right)^K\frac{1}{\sqrt{2\pi K}}\left(\frac{e}{N-K}\right)^{N-K}\frac{1}{\sqrt{2\pi(N-K)}}\\ &= \frac{N^N}{K^K(N-K)^{N-K}}\sqrt{\frac{N}{2\pi K(N-K)}}\\ &\simeq \frac{N^N}{(tN)^{tN}((1-t)N)^{(1-t)N}}\sqrt{\frac{N}{2\pi tN(1-t)N}}\\ &= \left(\frac{1}{t^t(1-t)^{1-t}}\right)^N\frac{1}{\sqrt{2\pi t(1-t)N}}\end{aligned}$$

As for the second formula, this follows from this, at $t=1/2$. □

As a second application now of the Stirling formula, which reveals its true geometric nature, and key importance, we can estimate the volumes of the spheres, as follows:

THEOREM 13.9. *The volume of the unit sphere in $\mathbb{R}^N$ is given by*

$$V\simeq\left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}}$$

in the $N\to\infty$ limit.

PROOF. This is something very standard, the idea being as follows:

(1) The volume of the unit sphere in $\mathbb{R}^N$ can be computed by recurrence on N, by cutting slices, in the obvious way, and we are led in this way to the following formula, with the convention $(N+1)!!=N(N-2)(N-4)\ldots$ for the double factorials:

$$V=\left(\frac{\pi}{2}\right)^{[N/2]}\frac{2^N}{(N+1)!!}$$

(2) But the double factorials can be estimated by using the Stirling formula. Indeed, in the case where $N=2K$ is even, we have the following computation:

$$\begin{aligned}(N+1)!! &= 2^KK!\\ &\simeq \left(\frac{2K}{e}\right)^K\sqrt{2\pi K}\\ &= \left(\frac{N}{e}\right)^{N/2}\sqrt{\pi N}\end{aligned}$$

As for the case where $N = 2K - 1$ is odd, here the estimate goes as follows:

$$\begin{aligned}
(N+1)!! &= \frac{(2K)!}{2^K K!}\\
&\simeq \frac{1}{2^K}\left(\frac{2K}{e}\right)^{2K}\sqrt{4\pi K}\left(\frac{e}{K}\right)^K\frac{1}{\sqrt{2\pi K}}\\
&= \left(\frac{2K}{e}\right)^K\sqrt{2}\\
&= \left(\frac{N+1}{e}\right)^{(N+1)/2}\sqrt{2}\\
&= \left(\frac{N}{e}\right)^{N/2}\left(\frac{N+1}{N}\right)^{N/2}\sqrt{\frac{N+1}{e}}\cdot\sqrt{2}\\
&\simeq \left(\frac{N}{e}\right)^{N/2}\sqrt{e}\cdot\sqrt{\frac{N}{e}}\cdot\sqrt{2}\\
&= \left(\frac{N}{e}\right)^{N/2}\sqrt{2N}
\end{aligned}$$

(3) Now back to the spheres, when N is even, the estimate goes as follows:

$$\begin{aligned}
V &= \left(\frac{\pi}{2}\right)^{N/2}\frac{2^N}{(N+1)!!}\\
&\simeq \left(\frac{\pi}{2}\right)^{N/2}2^N\left(\frac{e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}}\\
&= \left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}}
\end{aligned}$$

As for the case where N is odd, here the estimate goes as follows:

$$\begin{aligned}
V &= \left(\frac{\pi}{2}\right)^{(N-1)/2}\frac{2^N}{(N+1)!!}\\
&\simeq \left(\frac{\pi}{2}\right)^{(N-1)/2}2^N\left(\frac{e}{N}\right)^{N/2}\frac{1}{\sqrt{2N}}\\
&= \sqrt{\frac{2}{\pi}}\left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{2N}}\\
&= \left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}}
\end{aligned}$$

Thus, we are led to the uniform formula in the statement. □

13c. Mertens theorems

Back to Euler and primes, the continuation of the story involves the work of Mertens, that we would like to discuss now. Let us start with some analysis conventions:

DEFINITION 13.10. *We use the following notations:*

(1) *We write $f \simeq g$ when $f - g \to 0$.*
(2) *We write $f \cong g$ when $f - g$ is bounded.*
(3) *We write $f \sim g$ when $f/g \to 1$.*
(4) *We write $f \approx g$ when f/g is bounded.*

Occasionaly, we will use as well the Landau $O(f), o(f)$ symbols, making it for just 2 notations, instead of 4. With these conventions, the formula of Mertens is as follows:

THEOREM 13.11. *We have the following formula, with sum over primes,*

$$\sum_{p\leq N}\frac{1}{p}\simeq\log\log N+M$$

and with $M = 0.26149..$ being a constant, called Mertens constant.

PROOF. This is something quite tricky, the idea being as follows:

(1) As a first comment, observe that we have switched in the statement from sums over primes $p < N$, to sums over primes $p \leq N$. The point is that sums of type $p < N$ were best adapted to the Euler summation, which eventually leads to an integral of $1/x$, that we want to be $\log N$ instead of $\log(N + 1)$. However, as we will see in a moment, the Mertens summation is best written with $p \leq N$. Of course, at the level of the final results, Theorem 13.2 and the present theorem, this does not matter, because:

$$\log\log N\simeq\log\log(N+1)$$

(2) Getting now to the proof, this is based on the following formula, which comes as usual from the unique factorization of integers, $n = p_1^{a_1} \ldots p_k^{a_k}$, with the sum being over prime powers p^k, and with the exponent $[N/p^k]$ being an integer part:

$$N!=\prod_{p^k\leq N}p^{[N/p^k]}$$

(3) By talking the logarithm, we obtain from this the following estimate:

$$\begin{aligned}\log N! &= \sum_{p^k\leq N}\left[\frac{N}{p^k}\right]\log p\\ &= \sum_{p^k\leq N}\left(\frac{N}{p^k}+o(1)\right)\log p\\ &= N\sum_{p^k\leq N}\frac{\log p}{p^k}+o(1)\sum_{p^k\leq N}\log p\end{aligned}$$

(4) By using $\log N!=N\log N+O(N)$, coming from Stirling, this gives:

$$\begin{aligned}\sum_{p^k\leq N}\frac{\log p}{p^k} &= \frac{\log N!}{N}+o\left(\frac{1}{N}\right)\sum_{p^k\leq N}\log p\\ &= \log N+o(1)+o\left(\frac{1}{N}\right)\sum_{p^k\leq N}\log p\end{aligned}$$

(5) Now let us analyze the sum on the right. We have:

$$\begin{aligned}\sum_{p^k\leq N}\log p &\leq \sum_{p\in(N,2N]}\log p\\ &\leq \log\binom{2N}{N}\\ &= O(N)\end{aligned}$$

(6) We conclude that the estimate in (4) can be written as follows:

$$\sum_{p^k\leq N}\frac{\log p}{p^k}=\log N+o(1)$$

(7) Now since the sum of reciprocals of squares is finite, $\sum_{k\geq 1}1/k^2<\infty$, we can remove all the squares from the sum on the left, and we are left with:

$$\sum_{p\leq N}\frac{\log p}{p}=\log N+o(1)$$

(8) But now by doing a partial summation, in the obvious way, this gives a formula as follows, with $M\in\mathbb{R}$ being a certain constant:

$$\sum_{p\leq N}\frac{1}{p}\simeq\log\log N+M+O\left(\frac{1}{\log N}\right)$$

Thus, we are led to the convergence conclusion in the statement, and of course with the precise numerics for the Mertens constant M remaining to be justified. □

The continuation of the story, involving Mertens, Meissel and others, is quite long. The above proof can be improved, with some technical bounds for M, and for the rate of convergence too. Also, and getting now to the point, the Mertens constant M itself, there are several interesting formulae for it. In order to discuss this, we will need:

THEOREM 13.12. *The following limit converges,*

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log N$$

the result being the Euler-Mascheroni constant $\gamma = 0.57721..$

PROOF. This is indeed something very standard, which is best viewed via basic calculus. Consider indeed the following integral, with [.] being the integer part:

$$I = \int_1^\infty \left(\frac{1}{[x]} - \frac{1}{x}\right) dx$$

As a first observation, this integral is clearly positive, $I > 0$, and it converges too, $I < \infty$, with an easy upper bound for it being obtained as follows:

$$\begin{aligned} I &= \int_1^2 \left(\frac{1}{[x]} - \frac{1}{x}\right) dx + \int_2^\infty \left(\frac{1}{[x]} - \frac{1}{x}\right) dx \\ &= 1 - \log 2 + \int_2^\infty \left(\frac{1}{[x]} - \frac{1}{x}\right) dx \\ &< 1 - \log 2 + \int_2^\infty \left(\frac{1}{x-1} - \frac{1}{x}\right) dx \\ &= 1 - \log 2 + \log 2 \\ &= 1 \end{aligned}$$

On the other hand, the truncations of the above integral I are given by:

$$\begin{aligned} I_N &= \int_1^N \left(\frac{1}{[x]} - \frac{1}{x}\right) dx \\ &= \sum_{n=1}^{N-1} \int_n^{n+1} \left(\frac{1}{[x]} - \frac{1}{x}\right) dx \\ &= \sum_{n=1}^{N-1} \frac{1}{n} + \log n - \log(n+1) \\ &= \sum_{n=1}^{N-1} \frac{1}{n} - \log N \end{aligned}$$

Thus we have in fact $I = \gamma$, leading to the conclusion in the statement. □

Getting back to Mertens, he proved in fact three theorems regarding the prime numbers, with Theorem 13.11, the most famous one, being his second theorem. So, time now to discuss this. Here are the three formulae established by Mertens:

THEOREM 13.13. *We have the following Mertens estimates, in the $N \to \infty$ limit,*

$$\sum_{p<N} \frac{\log p}{p} \cong \log N$$

$$\sum_{p<N} \frac{1}{p} \simeq \log\log N + M$$

$$\sum_{p<N} \log\left(1 - \frac{1}{p}\right) \simeq -\log\log N - \gamma$$

$M = 0.26149..$ and $\gamma = 0.57721..$ being the Mertens and Euler-Mascheroni constants.

PROOF. This is something more advanced, that we will not prove here, among others requiring a better knowledge of Euler-Mascheroni constant, the idea being as follows:

(1) The first Mertens formula is something quite standard, in the spirit of Theorem 13.11, and with the precise upper bound obtained by Mertens being as follows:

$$\sum_{p<N} \frac{\log p}{p} < \log N + 2$$

As usual, exercise for you, to read more about this. Alternatively, you can come back here after reading chapter 16, with the Prime Number theorem there implying this.

(2) The second Mertens formula, which is the main one, is something that we know, from Theorem 13.11. However, as already mentioned, the story here is not over yet, because we still have to comment on the numerics of M. More on this in a moment.

(3) The third Mertens formula is equivalent, by exponentiating, to:

$$\prod_{p<N} \left(1 - \frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\log N}$$

In relation with this, observe that we have the following formula:

$$\begin{aligned}\prod_{p<N}\left(1-\frac{1}{p}\right)\prod_{p<N}\left(1+\frac{1}{p}\right) &= \prod_{p<N}\left(1-\frac{1}{p^2}\right)\\ &= \left[\prod_{p<N}\left(1+\frac{1}{p^2}+\frac{1}{p^4}+\ldots\right)\right]^{-1}\\ &\simeq \left[\sum_{n<N}\frac{1}{n^2}\right]^{-1}\\ &\approx 1\end{aligned}$$

Thus, another equivalent formulation of the third Mertens formula is as follows:

$$\prod_{p<N}\left(1+\frac{1}{p}\right)\approx e^{-\gamma}\log N$$

As for the proof of this, as before, exercise for you to try, or read more about this, from any classical number theory book, such as Hardy and Wright [45]. □

Now back to the Mertens constant, we have the following formula for it:

THEOREM 13.14. *The Mertens constant is given by the formula*

$$M=\gamma+\sum_p\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$$

with $\gamma=0.57721..$ *being the Euler-Mascheroni constant.*

PROOF. We know that the Mertens constant appears by definition as follows:

$$\sum_{p<N}\frac{1}{p}\simeq\log\log N+M$$

But, according to Theorem 13.13, we have as well the following formula:

$$\sum_{p<N}\log\left(1-\frac{1}{p}\right)\simeq-\log\log N-\gamma$$

Thus, we are led to the conclusion in the statement. □

The point now is that above considerations eventually lead, via some more work, to the precise numeric figure from Theorem 13.11, namely $M=0.26149...$

13d. Chebycheff estimates

Let us investigate now some related questions, again regarding the primes and their distribution, which look more intuitive and appealing, but which in the end, require more complicated techniques. We would like to estimate the following number:

DEFINITION 13.15. *We define the function* $\pi : \mathbb{N} \to \mathbb{N}$ *by*

$$\pi(N) = \#\Big\{p \leq N \text{ prime}\Big\}$$

the first few values being $0, 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, 6, \ldots$

Many things can be said here, especially now that we are already quite seriously into prime numbers, with the Euler estimates, and the theorems of Mertens, which can be converted into results about $\pi(N)$. However, according to our general policy for this opening chapter on analysis, let us do things slowly. To start with, we have:

PROPOSITION 13.16. *We have the following estimate,*

$$\pi(N) \geq \log\log N$$

coming from the unique factorization of integers, $n = p_1^{a_1} \ldots p_k^{a_k}$.

PROOF. This is something that I learned from my pure algebra colleagues. If we denote by p_n the n-th prime number, according to the unique factorization of integers, and more specifically to the related proof of the infinity of primes, we have:

$$p_{n+1} \leq p_1 \ldots p_n + 1$$

But this gives, by recurrence on n, the following estimate:

$$p_n \leq 2^{2^n}$$

In terms of the function π from Definition 13.15, this estimate reads:

$$\pi(2^{2^n}) \geq n$$

Thus, we obtain an estimate as in the statement, but shifted by 1, and with $\log_2$ instead of log. However, $\log_2$ being for computer scientists, $\log_{10}$ for social science, and $\log = \log_e$ for mathematics, let us stick with log. By using $e^{n-1} > 2^n$ for $n > 3$ we can pass from $\log_2$ to log, and we obtain the formula in the statement. □

Next in line, we have the following estimate, heavily improving Proposition 13.16:

PROPOSITION 13.17. *We have the following estimate,*

$$\pi(N) \geq \frac{\log N}{\log 4}$$

coming from the unique factorization $n = p_1 \ldots p_k m^2$*, with* p_i *distinct.*

PROOF. This is again something that I learned from my algebra colleagues. Consider the first n primes, denoted $p_1, \ldots, p_n$, and let us try to compute the number $f(N)$ of integers $K \leq N$ all whose prime factors are among $\{p_1, \ldots, p_n\}$. By using the factorization in the statement, that we can write as $K = SM^2$ with S square-free, we get:

$$f(N) \leq 2^n \sqrt{N}$$

On the other hand we obviously have $f(N) \geq N$, and we obtain from this:

$$N \leq 4^n \leq 4^{\pi(N)}$$

Thus, we are led to the conclusion in the statement. □

Getting now to a more systematic study of the problem, by using more advanced techniques, following Chebycheff, let us introduce the following related function:

DEFINITION 13.18. *The Chebycheff theta function is given by*

$$\theta(N) = \sum_{p \leq N} \log p$$

with the sum being over primes.

In what follows, the idea will be that of estimating θ, and then converting our results in terms of π. Indeed, in what regards θ, we have a nice estimate for it, as follows:

THEOREM 13.19. *We have the following estimate,*

$$\theta(N) \leq \log 16 \cdot N$$

for the Chebycheff theta function introduced above.

PROOF. This is something quite tricky, using the central binomial coefficients, that we already met in the proof of the Mertens theorem. These coefficients are as follows:

$$\binom{2n}{n} = \frac{(2n)(2n-1)\ldots(n+1)}{n!}$$

Since this coefficient is obviously divisible by all primes $n < p \leq 2n$, we have:

$$\prod_{n < p \leq 2n} p < \binom{2n}{n} < (1+1)^{2n} = 4^n$$

Now in terms of the Chebycheff theta function from Definition 13.18, this gives:

$$\theta(2n) - \theta(n) < \log 4 \cdot n$$

Now by summing, we are led to the formula in the statement. □

We can now formulate a first key theorem of Chebycheff, as follows:

THEOREM 13.20. *We have an estimate as follows,*

$$\pi(N) < C \cdot \frac{N}{\log N}$$

with C being a certain constant, $C < \log 32 + 2$.

PROOF. We have the following estimate, relating the functions θ and π:

$$\begin{aligned}\theta(n) &= \sum_{p\leq n} \log p\\ &\geq \sum_{\sqrt{n}<p\leq n} \log p\\ &\geq \log\sqrt{n}(\pi(n) - \pi(\sqrt{n}))\end{aligned}$$

Now by taking into account the estimate found in Theorem 13.19, we obtain:

$$\begin{aligned}\pi(n) &\leq \frac{2\theta(n)}{\log n} + \sqrt{n}\\ &\leq \log 32 \cdot \frac{n}{\log n} + 2 \cdot \frac{n}{\log n}\\ &= (\log 32 + 2)\frac{n}{\log n}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a second theorem of Chebycheff, going now in the other sense, we have:

THEOREM 13.21. *We have an estimate as follows,*

$$\pi(N) > c \cdot \frac{N}{\log N}$$

with c being a certain constant.

PROOF. This is something more tricky, the idea being as follows:

(1) As before in the previous proof, we use the central binomial coefficients, but written this time, and estimated, in a different way, as follows:

$$\binom{2n}{n} = \frac{n+1}{1} \cdot \frac{n+2}{2} \cdots \frac{n+1}{n} \geq 2^n$$

If we denote by v_p the exponent of each p inside this coefficient, we obtain:

$$\prod_p p^{v_p} \geq 2^n$$

Equivalently, by taking the logarithm, this gives the following formula:

$$\sum_p v_p \log p \geq n \log 2$$

(2) On the other hand, the above exponents v_p are given by the following formula, with m_p standing for the highest number such that $p^{m_p} \leq 2n$:

$$\begin{aligned} v_p &= \sum_{k=1}^{m_p} \left[\frac{2n}{p^k}\right] - \left[\frac{n}{p^k}\right] \\ &\leq m_p \\ &= \left[\frac{\log 2n}{\log p}\right] \end{aligned}$$

(3) Now by putting the estimates in (1) and (2) together, we obtain:

$$\sum_{p<2n} \left[\frac{\log 2n}{\log p}\right] \cdot \log p \geq n \log 2$$

(4) It is convenient now to split the sum into two parts, as follows:

$$\begin{aligned} n \log 2 &\leq \sum_{p<2n} \left[\frac{\log 2n}{\log p}\right] \cdot \log p \\ &= \sum_{p<\sqrt{2n}} \left[\frac{\log 2n}{\log p}\right] \cdot \log p + \sum_{p>\sqrt{2n}} \left[\frac{\log 2n}{\log p}\right] \cdot \log p \\ &\leq \sqrt{2n} \log 2n + \theta(2n) \end{aligned}$$

(5) We conclude from this that we have the following estimate:

$$\theta(2n) \geq n \log 2 - \sqrt{2n} \log 2n$$

But this gives a constant c such that the following happens:

$$\theta(n) > cn$$

(6) In order to conclude now, observe that we have:

$$\theta(n) = \sum_{p \leq n} \log p \leq \pi(n) \log n$$

Thus, we obtain the following estimate, for the function π itself:

$$\pi(n) \geq \frac{\theta(n)}{\log n} \geq c \cdot \frac{n}{\log n}$$

Thus, we are led to the conclusion in the statement. □

We can now put the two Chebycheff theorems together, as follows:

THEOREM 13.22. *We have the following estimate for the π function,*

$$\pi(N) \approx \frac{N}{\log N}$$

in the sense that the quotient of these quantities is bounded from above, and below.

PROOF. According to Theorem 13.20 and Theorem 13.21, we have:

$$c \cdot \frac{N}{\log N} \leq \pi(N) \leq C \cdot \frac{N}{\log N}$$

Thus, we are led to the conclusion in the statement. □

In practice, the Chebycheff estimates are strong enough in order to prove the Bertrand postulate, stating that we should have a prime number as follows:

$$N < p < 2N$$

However, the story is not over here, because we have the following conjecture:

$$\pi(N) \sim \frac{N}{\log N}$$

And here, things become fairly complicated, with this formula being known to hold indeed, as the Prime Number Theorem, but with the proofs being all complicated. We will come back to this later, towards the end of the present book.

13e. Exercises

Welcome to advanced number theory, and as exercises on this, we have:

EXERCISE 13.23. *Read the full story, including proofs too, for $\sum_n 1/n^2 = \pi^2/6$.*

EXERCISE 13.24. *Learn more about the Bernoulli numbers, and their properties.*

EXERCISE 13.25. *Read some other available proofs of the Stirling formula.*

EXERCISE 13.26. *Clarify what we said above about the Mertens constant.*

EXERCISE 13.27. *Read more about the Euler-Mascheroni constant.*

EXERCISE 13.28. *Read the proof of the first and third Mertens theorems.*

EXERCISE 13.29. *Work out some further elementary estimates for $\pi(N)$.*

EXERCISE 13.30. *Learn about the various applications of the Chebycheff estimates.*

As bonus exercise, try computing some other particular values of ζ.

CHAPTER 14

Complex analysis

14a. Complex functions

As mentioned on several occasions in chapter 13, in order to further advance on our various questions regarding the primes, we are in need of more calculus input. And here, passed many ad-hoc things that can be said, and technologies that can be developed, we are mostly in need of complex analysis. For instance, as we will soon discover, the zeta function naturally lives over $\mathbb{C}$, so the ground field for analytic number theory is $\mathbb{C}$.

Be said in passing, as a remarkable coincidence, $\mathbb{C}$ is also the ground field for many other interesting theories, such as quantum mechanics, and advanced physics in general. So, with complex analysis, we will be learning here extremely useful mathematics.

Getting started now, we have already met complex functions $f : \mathbb{C} \to \mathbb{C}$ in chapter 9, when talking complex numbers, so let us begin with a review of that material. The simplest such functions are the polynomials, and to be known about them is:

THEOREM 14.1. *Each polynomial $P \in \mathbb{C}[X]$ can be regarded as a complex function $P : \mathbb{C} \to \mathbb{C}$, which is continuous. Also, we can write*

$$P(x) = a(x - r_1) \ldots (x - r_n)$$

with $a \in \mathbb{C}$, and with the numbers $r_1, \ldots, r_n \in \mathbb{C}$ being the roots of P.

PROOF. Assume indeed that P has no roots at all, and pick $z \in \mathbb{C}$ such that:

$$|P(z)| = \min_{x \in \mathbb{C}} |P(x)| > 0$$

Since $Q(t) = P(z + t) - P(z)$ vanishes at $t = 0$, we have the following estimate:

$$P(z + t) \simeq P(z) + ct^k$$

With $t = rw$, with $r > 0$ small, and with $|w| = 1$, this estimate becomes:

$$P(z + rw) \simeq P(z) + cr^k w^k$$

Now recall that we assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$|P(z + rw)| < |P(z)|$$

Thus, contradiction, P has a root, and by recurrence it has N roots, as stated. □

Next in line, we have the rational functions, which are defined as follows:

THEOREM 14.2. *The quotients of complex polynomials $f = P/Q$ are called rational funtions. When written in reduced form, with P, Q prime to each other,*

$$f = \frac{P}{Q}$$

is well-defined and continuous outside the zeroes $P_f \subset \mathbb{C}$ of Q, called poles of f:

$$f : \mathbb{C} - P_f \to \mathbb{C}$$

In addition, the rational functions, regarded as algebraic expressions, are stable under summing, making products and taking inverses, so they form a field $\mathbb{C}(X)$.

PROOF. There are several things going on here, the idea being as follows:

(1) First of all, we can surely talk about quotients of polynomials, $f = P/Q$, regarded as abstract algebraic expressions. Also, the last assertion is clear, because we can indeed perform sums, products, and take inverses, by using the following formulae:

$$\frac{P}{Q} + \frac{R}{S} = \frac{PS + QR}{QS} \quad , \quad \frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS} \quad , \quad \left(\frac{P}{Q}\right)^{-1} = \frac{Q}{P}$$

(2) The question is now, given a rational function f, can we regard it as a complex function? In general, we cannot say that we have $f : \mathbb{C} \to \mathbb{C}$, for instance because $f(x) = x^{-1}$ is not defined at $x = 0$. More generally, assuming $f = P/Q$ with $P, Q \in \mathbb{C}$, we cannot talk about $f(x)$ when x is a root of Q, unless of course we are in the special situation where x is a root of P too, and we can simplify the fraction.

(3) In view of this discussion, in order to solve our question, we must avoid the situation where the polynomials P, Q have common roots. But this can be done by writing our rational function f in reduced form, as follows, with $P, Q \in \mathbb{C}[X]$ prime to each other:

$$f = \frac{P}{Q}$$

(4) Now with this convention made, it is clear that f is well-defined, and continuous too, outside of the zeroes of f. Now since these zeroes can be obviously recovered from the knowledge of f itself, as being the points where "f explodes", we can call them poles of f, and so we have a function $f : \mathbb{C} - P_f \to \mathbb{C}$, as in the statement.

(5) Finally, the fact that the rational functions are stable under summing, making products and taking inverses, so that they form a field $\mathbb{C}(X)$, is clear. In fact, we have already talked about this in chapter 12, but always good to talk about it again. □

Still talking rational functions, in the complex plane these are in fact quite mysterious objects, as shown by the following informal result, discussed in chapter 9:

PROPOSITION 14.3. *We have the following formula, valid for any $|x| < 1$,*

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

but, for $x \in \mathbb{C} - \mathbb{R}$, the geometric meaning of this formula is quite unclear.

PROOF. Here the formula in the statement holds indeed, by multiplying and cancelling terms, and with the convergence being justified by the following estimate:

$$\left| \sum_{n=0}^{\infty} x^n \right| \leq \sum_{n=0}^{\infty} |x|^n = \frac{1}{1-|x|}$$

As for the last assertion, this is something quite informal. Indeed, for $x = 1/2$ our formula is something clear, by cutting the interval $[0, 2]$ into half, and so on:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

More generally, such things work for $x \in (-1, 1)$. However, for $x \in \mathbb{C} - \mathbb{R}$ we are led to a mysterious spiral, and the only case where the formula is "obvious", geometrically speaking, is when $x = rw$, with $r \in [0, 1)$, and w being a root of unity. Mystery. □

Getting now to more complicated functions, such as sin, cos, exp, log, again many things extend well from real to complex, the basic theory here being as follows:

THEOREM 14.4. *The functions* $\sin, \cos, \exp, \log$ *have complex extensions, given by*

$$\sin x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+1}}{(2l+1)!} \quad , \quad \cos x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l}}{(2l)!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad , \quad \log(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

with $|x| < 1$ needed for log*, which are continuous over their domain, and satisfy the formulae $e^{x+y} = e^x e^y$ and $e^{ix} = \cos x + i \sin x$.*

PROOF. This is a mixture of trivial and non-trivial results, as follows:

(1) We already know about e^x from chapter 9, the idea being that the convergence of the series, and then the continuity of e^x, come from the following estimate:

$$|e^x| \leq \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|} < \infty$$

Regarding $\sin x$, the same method works, with the following estimate:

$$|\sin x| \leq \sum_{l=0}^{\infty} \frac{|x|^{2l+1}}{(2l+1)!} \leq \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|}$$

The same goes for $\cos x$, the estimate here being as follows:

$$|\cos x| \leq \sum_{l=0}^{\infty} \frac{|x|^{2l}}{(2l)!} \leq \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|}$$

Finally, regarding the basic formulae satisfied by $\sin, \cos, \exp$, we already know from chapter 9 that the exponential has indeed the following properties:

$$e^{x+y} = e^x e^y \quad , \quad e^{ix} = \cos x + i \sin x$$

(2) In order to discuss now the complex logarithm function log, let us first study some more the complex exponential function exp. By using $e^{x+y} = e^x e^y$ we obtain $e^x \neq 0$ for any $x \in \mathbb{C}$, so the complex exponential function is as follows:

$$\exp : \mathbb{C} \to \mathbb{C} - \{0\}$$

Now since we have $e^{x+iy} = e^x e^{iy}$ for $x, y \in \mathbb{R}$, with e^x being surjective onto $(0, \infty)$, and with e^{iy} being surjective onto the unit circle $\mathbb{T}$, we deduce that $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ is surjective. Also, again by using $e^{x+iy} = e^x e^{iy}$, we deduce that we have:

$$e^x = e^y \iff x - y \in 2\pi i \mathbb{Z}$$

(3) With these ingredients in hand, we can now talk about log. Indeed, let us fix a horizontal strip in the complex plane, having width 2π:

$$S = \left\{ x + iy \middle| x \in \mathbb{R}, y \in [a, a + 2\pi) \right\}$$

We know from the above that the restriction map $\exp : S \to \mathbb{C} - \{0\}$ is bijective, so we can define log as to be the inverse of this map:

$$\log = \exp^{-1} : \mathbb{C} - \{0\} \to S$$

(4) In practice now, the best is to choose for instance $a = 0$, or $a = -\pi$, as to have the whole real line included in our strip, $\mathbb{R} \subset S$. In this case on $\mathbb{R}_+$ we recover the usual logarithm, while on $\mathbb{R}_-$ we obtain complex values, as for instance $\log(-1) = \pi i$ in the case $a = 0$, or $\log(-1) = -\pi i$ in the case $a = -\pi$, coming from $e^{\pi i} = -1$.

(5) Finally, assuming $|x| < 1$, we can consider the following series, which converges:

$$f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

We have then $e^{f(x)} = 1 + x$, and so $f(x) = \log(1 + x)$, when $1 + x \in S$. □

Moving ahead, Theorem 14.4 leads us into the question on whether the other formulae that we know about $\sin, \cos$, such as the values of these functions on sums $x + y$, or on doubles $2x$, extend to the complex setting. Things are quite tricky here, and in relation with this, we have the following result, which is something of general interest:

THEOREM 14.5. *The following functions, called hyperbolic sine and cosine,*

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

are subject to the following formulae:

(1) $e^x = \cosh x + \sinh x$.
(2) $\sinh(ix) = i \sin x$, $\cosh(ix) = \cos x$, *for* $x \in \mathbb{R}$.
(3) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$.
(4) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.
(5) $\sinh x = \sum_l \frac{x^{2l+1}}{(2l+1)!}$, $\cosh x = \sum_l \frac{x^{2l}}{(2l)!}$.

PROOF. The formula (1) follows from definitions. As for (2), this follows from:

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = \frac{\cos x + i \sin x}{2} - \frac{\cos x - i \sin x}{2} = i \sin x$$

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i \sin x}{2} + \frac{\cos x - i \sin x}{2} = \cos x$$

Regarding now (3,4), observe first that the formula $e^{x+y} = e^x + e^y$ reads:

$$\cosh(x + y) + \sinh(x + y) = (\cosh x + \sinh x)(\cosh y + \sinh y)$$

Thus, we have some good explanation for (3,4), and in practice, these formulae can be checked by direct computation, as follows:

$$\frac{e^{x+y} - e^{-x-y}}{2} = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$\frac{e^{x+y} + e^{-x-y}}{2} = \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

Finally, (5) is clear from the definition of sinh, cosh, and from $e^x = \sum_k \frac{x^k}{k!}$. □

Finally, we can talk as well about powers, in the following way:

FACT 14.6. *Under suitable assumptions, we can talk about* x^y *with* $x, y \in \mathbb{C}$, *and in particular about the complex functions* a^x *and* x^a, *with* $a \in \mathbb{C}$.

To be more precise, in what regards x^y, we already know from basic calculus that things are quite tricky, even in the real case. In the complex case the same problems appear, along with some more, but these questions can be solved by using the above theory of exp, log. To be more precise, in order to solve the first question, we can set:

$$x^y = e^{y \log x}$$

We will be back to these functions later, when we will have more tools for studying them. In fact, all of a sudden, we are now into quite complicated mathematics, and we cannot really deal with such questions, with bare hands. More later.

14b. Holomorphic functions

Let us study now the differentiability of the complex functions $f : \mathbb{C} \to \mathbb{C}$. Things here are quite tricky, but let us start with a straightforward definition, as follows:

DEFINITION 14.7. *We say that a function $f : X \to \mathbb{C}$ is differentiable in the complex sense when the following limit is defined for any $x \in X$:*

$$f'(x) = \lim_{t \to 0} \frac{f(x + t) - f(x)}{t}$$

In this case, we also say that f is holomorphic, and we write $f \in H(X)$.

As basic examples, we have the power functions $f(x) = x^n$. Indeed, the derivative of such a power function can be computed exactly as in the real case, and we get:

$$\begin{aligned}
(x^n)' &= \lim_{t \to 0} \frac{(x + t)^n - x^n}{t} \\
&= \lim_{t \to 0} \frac{nx^{n-1}t + \binom{n}{2} x^{n-2}t^2 + \ldots + t^n}{t} \\
&= \lim_{t \to 0} \frac{nx^{n-1}t}{t} \\
&= nx^{n-1}
\end{aligned}$$

We will see later more computations of this type, similar to those from the real case. To summarize, our definition of differentiability seems to work nicely, so let us start developing some theory. The general results from the real case extend well, as follows:

PROPOSITION 14.8. *We have the following results:*

(1) $(f + g)' = f' + g'$.
(2) $(\lambda f)' = \lambda f'$.
(3) $(fg)' = f'g + fg'$.
(4) $(f \circ g)' = f'(g)g'$.

PROOF. These formulae are all clear from definitions, following exactly as in the real case. Thus, we are led to the conclusions in the statement. □

As an obvious consequence of (1,2) above, any poynomial $P \in \mathbb{C}[X]$ is differentiable, with its derivative being given by the same formula as in the real case, namely:

$$P(x) = \sum_{k=0}^{n} c_k x^k \implies P'(x) = \sum_{k=1}^{n} k c_k x^{k-1}$$

More generally, any rational function $f \in \mathbb{C}(X)$ is differentiable on its domain, that is, outsides its poles, because if we write $f = P/Q$ with $P, Q \in \mathbb{C}[X]$, we have:

$$f' = \left(\frac{P}{Q}\right)' = \frac{P'Q - PQ'}{Q^2}$$

Let us record these conclusions in a statement, as follows:

PROPOSITION 14.9. *The following happen:*

(1) *Any polynomial $P \in \mathbb{C}[X]$ is holomorphic, and in fact infinitely differentiable in the complex sense, with all its derivatives being polynomials.*
(2) *Any rational function $f \in \mathbb{C}(X)$ is holomorphic, and in fact infinitely differentiable, with all its derivatives being rational functions.*

PROOF. This follows indeed from the above discussion. □

Let us look now into more complicated complex functions that we know. And here, surprise, things are quite tricky, the result being as follows:

THEOREM 14.10. *The following happen:*

(1) $\sin, \cos, \exp, \log$ *are holomorphic, and in fact are infinitely differentiable, with their derivatives being given by the same formulae as in the real case.*
(2) *However, functions like $\bar{x}$ or $|x|$ are not holomorphic, and this because the limit defining $f'(x)$ depends on the way we choose $t \to 0$.*

PROOF. There are several things going on here, the idea being as follows:

(1) Here the first assertion is standard, because our functions $\sin, \cos, \exp, \log$ have Taylor series that we know, and the derivative can be therefore computed by using the same rule as in the real case, similar to the one for polynomials, namely:

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \implies f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

(2) Regarding now the function $f(x) = \bar{x}$, the point here is that we have:

$$\frac{f(x+t) - f(x)}{t} = \frac{\bar{x} + \bar{t} - \bar{x}}{t} = \frac{\bar{t}}{t}$$

But this limit does not converge with $t \to 0$, for instance because with $t \in \mathbb{R}$ we obtain 1 as limit, while with $t \in i\mathbb{R}$ we obtain -1 as limit. In fact, with $t = rw$ with $|w| = 1$ fixed and $r \in \mathbb{R}$, $r \to 0$, we can obtain as limit any number on the unit circle:

$$\lim_{r \to 0} \frac{f(x + rw) - f(x)}{rw} = \lim_{r \to 0} \frac{r\bar{w}}{rw} = \bar{w}^2$$

(3) The situation for the function $f(x) = |x|$ is similar. To be more precise, we have:

$$\frac{f(x + rw) - f(x)}{rw} = \frac{|x + rw| - |x|}{r} \cdot \bar{w}$$

Thus with $|w| = 1$ fixed and $r \to 0$ we obtain a certain multiple of $\bar{w}$, with the multiplication factor being computed as follows:

$$\begin{aligned}\frac{|x+rw|-|x|}{r} &= \frac{|x+rw|^2-|x|^2}{(|x+rw|+|x|)r}\\ &\simeq \frac{xr\bar{w}+\bar{x}rw}{2|x|r}\\ &= Re\left(\frac{x\bar{w}}{|x|}\right)\end{aligned}$$

Now by making w vary on the unit circle, as in (2) above, we can obtain in this way limits pointing in all possible directions, so our limit does not converge, as stated. □

The above result is quite surprising, because we are so used, from the real case, to the notion of differentiability to correspond to some form of "smoothness" of the function, and to be more precisely, "smoothness at first order". Or, if you prefer, to correspond to the "non-bumpiness" of the function. So, we are led to the following dilemma:

DILEMMA 14.11. *It's either that $\bar{x}$ and $|x|$ are smooth, as the intuition suggests, and we are wrong with our definition of differentiability. Or that $\bar{x}$ and $|x|$ are bumpy, while this being not very intuitive, and we are right with our definition of differentiability.*

And we won't get discouraged by this. After all, this is just some empty talking, and if there is something to rely upon, mathematics and computations, these are the computations from the proof of Theorem 14.10. So, based on that computations, let us formulate the following definition, coming as a complement to Definition 14.7:

DEFINITION 14.12. *A function $f : X \to \mathbb{C}$ is called differentiable:*

(1) *In the real sense, if the following two limits converge, for any $x \in X$:*

$$f_1'(x) = \lim_{t\in\mathbb{R}\to 0}\frac{f(x+t)-f(x)}{t} \quad , \quad f_i'(x) = \lim_{t\in i\mathbb{R}\to 0}\frac{f(x+t)-f(x)}{t}$$

(2) *In a radial sense, if the following limit converges, for any $x \in X$, and $w \in \mathbb{T}$:*

$$f_w'(x) = \lim_{t\in w\mathbb{R}\to 0}\frac{f(x+t)-f(x)}{t}$$

(3) *In the complex sense, if the following limit converges, for any $x \in X$:*

$$f'(x) = \lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$$

If f is differentiable in the complex sense, we also say that f is holomorphic.

We can see now more clearly what is going on. We have (3) $\implies$ (2) $\implies$ (1) in general, and most of the functions that we know, namely the polynomials, the rational functions, and $\sin, \cos, \exp, \log$, satisfy (3). As for the functions $\bar{x}, |x|$, these do not satisfy

(3), and do not satisfy (2) either, but they satisfy however (1). It is possible to say more about all this, and we will certainly come back to this topic, later in this book.

Back to business now, all the examples of holomorphic functions that we have are infinitely differentiable, and this raises the question of finding a function such that f' exists, while f'' does not exist. Quite surprisingly, we will see that such functions do not exist. In order to get into this latter phenomenon, let us start with:

THEOREM 14.13. *Each power series $f(x) = \sum_n c_n x^n$ has a radius of convergence*

$$R \in [0, \infty]$$

which is such that f converges for $|x| < R$, and diverges for $|x| > R$. We have:

$$R = \frac{1}{C} \quad , \quad C = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$$

Also, in the case $|x| = R$ the function f can either converge, or diverge.

PROOF. This follows from the Cauchy criterion for series, from chapter 5, which says that a series $\sum_n x_n$ converges if $c < 1$, and diverges if $c > 1$, where:

$$c = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$$

Indeed, with $x_n = |c_n x^n|$ we obtain that the convergence radius $R \in [0, \infty]$ exists, and is given by the formula in the statement. Finally, for the examples and counterexamples at the end, when $|x| = R$, the simplest here is to use $f(x) = \sum_n x^n$, where $R = 1$. □

Back now to our questions regarding derivatives, we have:

THEOREM 14.14. *Assuming that a function $f : X \to \mathbb{C}$ is analytic, in the sense that it is a series, around each point $x \in X$,*

$$f(x + t) = \sum_{n=0}^{\infty} c_n t^n$$

it follows that f is infinitely differentiable, in the complex sense. In particular, f' exists, and so f is holomorphic in our sense.

PROOF. Assuming that f is analytic, as in the statement, we have:

$$f'(x + t) = \sum_{n=1}^{\infty} n c_n t^{n-1}$$

Moreover, the radius of convergence is the same, as shown by the following computation, using the Cauchy formula for the convergence radius, and $\sqrt[n]{n} \to 1$:

$$\frac{1}{R'} = \limsup_{n\to\infty} \sqrt[n]{|nc_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{1}{R}$$

Thus f' exists and is analytic, on the same domain, and this gives the result. □

14c. Cauchy formula

Our goal in what follows will be that of proving that any holomorphic function is analytic. This is something quite subtle, which cannot be proved with bare hands, and requires lots of preliminaries. Getting to these preliminaries now, our claim is that a lot of useful knowledge, in order to deal with the holomorphic functions, can be gained by further studying the analytic functions, and even the usual polynomials $P \in \mathbb{C}[X]$.

So, let us further study the polynomials $P \in \mathbb{C}[X]$, and other analytic functions. We already know from before that in the polynomial case, $P \in \mathbb{C}[X]$, some interesting things happen, because any such polynomial has a root, and even $\deg(P)$ roots, after a recurrence. Keeping looking at polynomials, with the same methods, we are led to:

THEOREM 14.15. *Any polynomial $P \in \mathbb{C}[X]$ satisfies the maximum principle, in the sense that given a domain D, with boundary γ, we have:*

$$\exists x \in \gamma \quad , \quad |P(x)| = \max_{y \in D} |P(y)|$$

That is, the maximum of $|P|$ over a domain is attained on its boundary.

PROOF. In order to prove this, we can split D into connected components, and then assume that D is connected. Moreover, we can assume that D has no holes, and so is homeomorphic to a disk, and even assume that D itself is a disk. But with this assumption made, the result follows from by contradiction, by using the same arguments as in the proof of the existence of a root. To be more precise, assume $\deg P \geq 1$, and that the maximum of $|P|$ is attained at the center of a disk $D = D(z, r)$:

$$|P(z)| = \max_{x \in D} |P(x)|$$

We can write then $P(z + t) \simeq P(z) + ct^k$ with $c \neq 0$, for t small, and by suitably choosing the argument of t on the unit circle we conclude, as in the proof for the existence of the roots, that the function $|P|$ cannot have a local maximum at z, as stated. □

A good explanation for the fact that the maximum principle holds for polynomials $P \in \mathbb{C}[X]$ could be that the values of such a polynomial inside a disk can be recovered from its values on the boundary. And fortunately, this is indeed the case, and we have:

THEOREM 14.16. *Given a polynomial $P \in \mathbb{C}[X]$, and a disk D, with boundary γ, we have the following formulae, with the integrations being the normalized, mass 1 ones:*

(1) *P satisfies the plain mean value formula $P(x) = \int_D P(y)dy$.*
(2) *P satisfies the boundary mean value formula $P(x) = \int_\gamma P(y)dy$.*

PROOF. As a first observation, the two mean value formulae in the statement are equivalent, by restricting the attention to disks D, having as boundaries circles γ, and using annuli and polar coordinates for the proof of the equivalence. As for the formulae

themselves, these can be checked by direct computation for a disk D, with the formulation in (2) being the most convenient. Indeed, for a monomial $P(x) = x^n$ we have:

$$\begin{aligned}
\int_\gamma y^n dy &= \frac{1}{2\pi}\int_0^{2\pi}(x+re^{it})^n dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n}\binom{n}{k}x^k(re^{it})^{n-k}dt\\
&= \sum_{k=0}^{n}\binom{n}{k}x^k r^{n-k}\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-k)t}dt\\
&= \sum_{k=0}^{n}\binom{n}{k}x^k r^{n-k}\delta_{kn}\\
&= x^n
\end{aligned}$$

Here we have used the following key identity, valid for any exponent $m \in \mathbb{Z}$:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}e^{imt}dt &= \frac{1}{2\pi}\int_0^{2\pi}\cos(mt)+i\sin(mt)dt\\
&= \delta_{m0}+i\cdot 0\\
&= \delta_{m0}
\end{aligned}$$

Thus, we have the result for monomials, and the general case follows by linearity. □

All the above is very nice, but we can in fact do even better, with a more powerful integration formula. Let us start with some preliminaries. We first have:

PROPOSITION 14.17. *We can integrate functions f over curves γ by setting*

$$\int_\gamma f(x)dx = \int_a^b f(\gamma(t))\gamma'(t)dt$$

with this quantity being independent on the parametrization $\gamma : [a,b] \to \mathbb{C}$.

PROOF. We must prove that the quantity in the statement is independent on the parametrization. In other words, we must prove that if we pick two different parametrizations $\gamma, \eta : [a,b] \to \mathbb{C}$ of our curve, then we have the following formula:

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\eta(t))\eta'(t)dt$$

But for this purpose, let us write $\gamma = \eta\phi$, with $\phi : [a,b] \to [a,b]$ being a certain function, that we can assume to be bijective, via an elementary cut-and-paste argument.

By using the chain rule for derivatives, and the change of variable formula, we have:

$$\begin{aligned}\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b f(\eta\phi(t))(\eta\phi)'(t)dt\\ &= \int_a^b f(\eta\phi(t))\eta'(\phi(t))\phi'(t)dt\\ &= \int_a^b f(\eta(t))\eta'(t)dt\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The main properties of the above integration method are as follows:

PROPOSITION 14.18. *We have the following formula, for a union of paths:*

$$\int_{\gamma\cup\eta} f(x)dx = \int_\gamma f(x)dx + \int_\eta f(x)dx$$

Also, when reversing the path, the integral changes its sign.

PROOF. Here the first assertion is clear from definitions, and the second assertion comes from the change of variable formula, by using Proposition 14.17. □

Now by getting back to polynomials, we have the following result:

THEOREM 14.19. *Any polynomial $P \in \mathbb{C}[X]$ satisfies the Cauchy formula*

$$P(x) = \frac{1}{2\pi i}\int_\gamma \frac{P(y)}{y-x}\,dy$$

with the integration over γ being constructed as above.

PROOF. This follows by using abstract arguments and computations similar to those in the proof of Theorem 14.16. Indeed, by linearity we can assume $P(x) = x^n$. Also, by using a cut-and-paste argument, we can assume that we are on a circle:

$$\gamma : [0, 2\pi] \to \mathbb{C} \quad , \quad \gamma(t) = x + re^{it}$$

By using now the computation from the proof of Theorem 14.16, we obtain:

$$\begin{aligned}\int_\gamma \frac{y^n}{y-x}\,dy &= \int_0^{2\pi} \frac{(x+re^{it})^n}{re^{it}}\,rie^{it}dt\\ &= i\int_0^{2\pi}(x+re^{it})^n dt\\ &= i\cdot 2\pi x^n\end{aligned}$$

Thus, we are led to the formula in the statement. □

All this is quite interesting, and obviously, we are now into some serious mathematics. Importantly, Theorem 14.15, Theorem 14.16 and Theorem 14.19 provide us with a path for proving the converse of Theorem 14.14. Indeed, if we manage to prove the Cauchy formula for any holomorphic function $f : X \to \mathbb{C}$, then it will follow that our function is analytic, and so infinitely differentiable. So, let us start with the following result:

THEOREM 14.20. *The Cauchy formula, namely*

$$f(x) = \frac{1}{2\pi i}\int_\gamma \frac{f(y)}{y-x}\,dy$$

holds for any holomorphic function $f : X \to \mathbb{C}$.

PROOF. This is something standard, which can be proved as follows:

(1) Our first claim is that given $f \in H(X)$, with $f' \in C(X)$, the integral of f' vanishes on any path. Indeed, by using the change of variable formula, we have:

$$\begin{aligned}\int_\gamma f'(x)dx &= \int_a^b f'(\gamma(t))\gamma'(t)dt\\ &= f(\gamma(b)) - f(\gamma(a))\\ &= 0\end{aligned}$$

(2) Our second claim is that given $f \in H(X)$ and a triangle $\Delta \subset X$, we have:

$$\int_\Delta f(x)dx = 0$$

Indeed, let us call $\Delta = ABC$ our triangle. Now consider the midpoints A', B', C' of the edges BC, CA, AB, and then consider the following smaller triangles:

$$\Delta_1 = AC'B' \quad , \quad \Delta_2 = BA'C' \quad , \quad \Delta_3 = CB'A' \quad , \quad \Delta_4 = A'B'C'$$

These smaller triangles partition then Δ, and due to our above conventions for the vertex ordering, which produce cancellations when integrating over them, we have:

$$\int_\Delta f(x)dx = \sum_{i=1}^4 \int_{\Delta_i} f(x)dx$$

Thus we can pick, among the triangles Δ_i, a triangle $\Delta^{(1)}$ such that:

$$\left|\int_\Delta f(x)dx\right| \leq 4\left|\int_{\Delta^{(1)}} f(x)dx\right|$$

In fact, by repeating the procedure, we obtain triangles $\Delta^{(n)}$ such that:

$$\left|\int_\Delta f(x)dx\right| \leq 4^n\left|\int_{\Delta^{(n)}} f(x)dx\right|$$

(3) Now let z be the limiting point of these triangles $\Delta^{(n)}$, and fix $\varepsilon > 0$. By using the fact that the functions $1, x$ integrate over paths up to 0, coming from (1), we obtain the following estimate, with $n \in \mathbb{N}$ being big enough, and L being the perimeter of Δ:

$$\begin{aligned}\left|\int_{\Delta^{(n)}} f(x)dx\right| &= \left|\int_{\Delta^{(n)}} f(x) - f(z) - f'(z)(x-z)dx\right| \\ &\leq \int_{\Delta^{(n)}} |f(x) - f(z) - f'(z)(x-z)|\,dx \\ &\leq \int_{\Delta^{(n)}} \varepsilon|x-z|dx \\ &\leq \varepsilon(2^{-n}L)^2\end{aligned}$$

Now by combining this with the estimate in (2), this proves our claim.

(4) The rest is quite routine. First, we can pass from triangles to boundaries of convex sets, in a straightforward way, with the same conclusion as in (2), namely:

$$\int_\gamma f(x)dx = 0$$

Getting back to what we want to prove, namely the Cauchy formula for an arbitrary holomorphic function $f \in H(X)$, let $x \in X$, and consider the following function:

$$g(y) = \begin{cases} \frac{f(y)-f(x)}{y-x} & (y \neq x) \\ f'(x) & (y = x) \end{cases}$$

Now assuming that γ encloses a convex set, we can apply what we found, namely vanishing of the integral, to this function g, and we obtain the Cauchy formula for f.

(5) Finally, the extension to general curves is standard, and standard as well is the discussion of what exactly happens at x, in the above proof. See Rudin [74]. □

As a main application of the Cauchy formula, we have:

THEOREM 14.21. *The following conditions are equivalent, for a function $f : X \to \mathbb{C}$:*

(1) *f is holomorphic.*
(2) *f is infinitely differentiable.*
(3) *f is analytic.*
(4) *The Cauchy formula holds for f.*

PROOF. This is routine from what we have, the idea being as follows:

(1) $\implies$ (4) is non-trivial, but we know this from Theorem 14.20.

(4) $\implies$ (3) is something trivial, because we can expand the series in the Cauchy formula, and we conclude that our function is indeed analytic.

(3) $\implies$ (2) $\implies$ (1) are both elementary, known from Theorem 14.14. □

As another application of the Cauchy formula, we have:

THEOREM 14.22. *Any holomorphic function $f : X \to \mathbb{C}$ satisfies the maximum principle, in the sense that given a domain D, with boundary γ, we have:*

$$\exists x \in \gamma \quad , \quad |f(x)| = \max_{y \in D} |f(y)|$$

That is, the maximum of $|f|$ over a domain is attained on its boundary.

PROOF. This follows indeed from the Cauchy formula. Observe that the converse is not true, for instance because functions like $\bar{x}$ satisfy too the maximum principle. We will be back to this later, when talking about harmonic functions. □

As before with polynomials, a good explanation for the fact that the maximum principle holds could be that the values of our function inside a disk can be recovered from its values on the boundary. And fortunately, this is indeed the case, and we have:

THEOREM 14.23. *Given an holomorphic function $f : X \to \mathbb{C}$, and a disk D, with boundary γ, the following happen:*

(1) *f satisfies the plain mean value formula $f(x) = \int_D f(y)dy$.*
(2) *f satisfies the boundary mean value formula $f(x) = \int_\gamma f(y)dy$.*

PROOF. As usual, this follows from the Cauchy formula, with of course some care in passing from integrals constructed as in Proposition 14.17 to integrals viewed as averages, which are those that we refer to, in the present statement. □

Finally, as yet another application of the Cauchy formula, which is something nice-looking and conceptual, we have the following statement, called Liouville theorem:

THEOREM 14.24. *An entire, bounded holomorphic function*

$$f : \mathbb{C} \to \mathbb{C} \quad , \quad |f| \leq M$$

must be constant. In particular, knowing $f \to 0$ with $z \to \infty$ gives $f = 0$.

PROOF. This follows as usual from the Cauchy formula, namely:

$$f(x) = \frac{1}{2\pi i} \int_\gamma \frac{f(y)}{y - x}\, dy$$

Alternatively, we can view this as a consequence of Theorem 14.23, because given two points $x \neq y$, we can view the values of f at these points as averages over big disks centered at these points, say $D = D_x(R)$ and $E = D_y(R)$, with $R >> 0$:

$$f(x) = \int_D f(z)dz \quad , \quad f(y) = \int_E f(z)dz$$

Indeed, the point is that when the radius goes to ∞, these averages tend to be equal, and so we have $f(x) \simeq f(y)$, which gives $f(x) = f(y)$ in the limit. □

14d. Harmonic functions

With the Cauchy formula proved, and applied, done with complex analysis, you might say? You must be kidding, that was just the tip of the iceberg, and many things remain to be discussed. We will have a brief look at some of them in this section.

In order to reach to a continuation of the above, let us formulate:

DEFINITION 14.25. *The Laplace operator in 2 dimensions is:*

$$\Delta f = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2}$$

A function $f : \mathbb{R}^2 \to \mathbb{C}$ satisfying $\Delta f = 0$ will be called harmonic.

Here the Laplace operator is something very standard, coming from virtually all branches of physics, and its presence here remains to be explained, yes I know. So, let us try to find the functions $f : \mathbb{R}^2 \to \mathbb{C}$ which are harmonic. And here, as a good surprise, we have an interesting link with the holomorphic functions:

THEOREM 14.26. *Any holomorphic function $f : \mathbb{C} \to \mathbb{C}$, when regarded as function*

$$f : \mathbb{R}^2 \to \mathbb{C}$$

is harmonic. Moreover, the conjugates $\bar{f}$ of holomorphic functions are harmonic too.

PROOF. The first assertion follows from the following computation, for the power functions $f(z) = z^n$, with the usual notation $z = x + iy$:

$$\begin{aligned}
\Delta z^n &= \frac{d^2 z^n}{dx^2} + \frac{d^2 z^n}{dy^2} \\
&= \frac{d(nz^{n-1})}{dx} + \frac{d(inz^{n-1})}{dy} \\
&= n(n-1)z^{n-2} - n(n-1)z^{n-2} \\
&= 0
\end{aligned}$$

As for the second assertion, this follows from $\Delta \bar{f} = \overline{\Delta f}$, which is clear from definitions, and which shows that if f is harmonic, then so is its conjugate $\bar{f}$. □

All this is quite interesting, and the idea in what follows will be that of developing the theory of harmonic functions, as a generalization of the theory that we know for the holomorphic functions, but covering as well functions of type $\bar{z}$.

As a first goal, in order to understand the harmonic functions, we can try to find the homogeneous polynomials $P \in \mathbb{R}[x, y]$ which are harmonic. In order to do so, the most convenient is to use the variable $z = x + iy$, and think of these polynomials as being homogeneous polynomials $P \in \mathbb{R}[z, \bar{z}]$. With this convention, the result is as follows:

THEOREM 14.27. *The degree n homogeneous polynomials $P \in \mathbb{R}[x,y]$ which are harmonic are precisely the linear combinations of*

$$P = z^n \quad , \quad P = \bar{z}^n$$

with the usual identification $z = x + iy$.

PROOF. As explained above, any homogeneous polynomial $P \in \mathbb{R}[x,y]$ can be regarded as an homogeneous polynomial $P \in \mathbb{R}[z,\bar{z}]$, with the change of variables $z = x+iy$, and in this picture, the degree n homogeneous polynomials are as follows:

$$P(z) = \sum_{k+l=n} c_{kl} z^k \bar{z}^l$$

In oder to solve now the Laplace equation $\Delta P = 0$, we must compute the quantities $\Delta(z^k\bar{z}^l)$, for any k, l. But the computation here is routine. We first have the following formula, with the derivatives being computed with respect to the variable x:

$$\begin{aligned}\frac{d(z^k\bar{z}^l)}{dx} &= (z^k)'\bar{z}^l + z^k(\bar{z}^l)' \\ &= kz^{k-1}\bar{z}^l + lz^k\bar{z}^{l-1}\end{aligned}$$

By taking one more time the derivative with respect to x, we obtain:

$$\begin{aligned}\frac{d^2(z^k\bar{z}^l)}{dx^2} &= k(z^{k-1}\bar{z}^l)' + l(z^k\bar{z}^{l-1})' \\ &= k\left[(z^{k-1})'\bar{z}^l + z^{k-1}(\bar{z}^l)'\right] + l\left[(z^k)'\bar{z}^{l-1} + z^k(\bar{z}^{l-1})'\right] \\ &= k\left[(k-1)z^{k-2}\bar{z}^l + lz^{k-1}\bar{z}^{l-1}\right] + l\left[kz^{k-1}\bar{z}^{l-1} + (l-1)z^k\bar{z}^{l-2}\right] \\ &= k(k-1)z^{k-2}\bar{z}^l + 2klz^{k-1}\bar{z}^{l-1} + l(l-1)z^k\bar{z}^{l-2}\end{aligned}$$

With respect to the variable y, the computations are similar, but some $\pm i$ factors appear, due to $z' = i$ and $\bar{z}' = -i$, coming from $z = x + iy$. We first have:

$$\begin{aligned}\frac{d(z^k\bar{z}^l)}{dy} &= (z^k)'\bar{z}^l + z^k(\bar{z}^l)' \\ &= ikz^{k-1}\bar{z}^l - ilz^k\bar{z}^{l-1}\end{aligned}$$

By taking one more time the derivative with respect to y, we obtain:

$$\begin{aligned}\frac{d^2(z^k\bar{z}^l)}{dy^2} &= ik(z^{k-1}\bar{z}^l)' - il(z^k\bar{z}^{l-1})' \\ &= ik\left[(z^{k-1})'\bar{z}^l + z^{k-1}(\bar{z}^l)'\right] - il\left[(z^k)'\bar{z}^{l-1} + z^k(\bar{z}^{l-1})'\right] \\ &= ik\left[i(k-1)z^{k-2}\bar{z}^l - ilz^{k-1}\bar{z}^{l-1}\right] - il\left[ikz^{k-1}\bar{z}^{l-1} - i(l-1)z^k\bar{z}^{l-2}\right] \\ &= -k(k-1)z^{k-2}\bar{z}^l + 2klz^{k-1}\bar{z}^{l-1} - l(l-1)z^k\bar{z}^{l-2}\end{aligned}$$

We can now sum the formulae that we found, and we obtain:

$$\begin{aligned}\Delta(z^k\bar{z}^l) &= \frac{d^2(z^k\bar{z}^l)}{dx^2}+\frac{d^2(z^k\bar{z}^l)}{dy^2}\\ &= k(k-1)z^{k-2}\bar{z}^l+2klz^{k-1}\bar{z}^{l-1}+l(l-1)z^k\bar{z}^{l-2}\\ &\quad -k(k-1)z^{k-2}\bar{z}^l+2klz^{k-1}\bar{z}^{l-1}-l(l-1)z^k\bar{z}^{l-2}\\ &= 4klz^{k-1}\bar{z}^{l-1}\end{aligned}$$

In other words, we have reached to the following nice formula:

$$f=z^k\bar{z}^l\implies\Delta f=\frac{4klf}{|z|^2}$$

Now let us get back to our homogeneous polynomial P, written as follows:

$$P(z)=\sum_{k+l=n}c_{kl}z^k\bar{z}^l$$

By using the above formula, it follows that the Laplacian of P is given by:

$$\Delta P(z)=\frac{4}{|z|^2}\sum_{k+l=n}klc_{kl}z^k\bar{z}^l$$

We conclude that the Laplace equation for P takes the following form:

$$\begin{aligned}\Delta P=0 &\iff klc_{kl}=0,\forall k,l\\ &\iff [k,l\neq 0\implies c_{kl}=0]\\ &\iff P=c_{n0}z^n+c_{0n}\bar{z}^n\end{aligned}$$

Thus, we are led to the conclusion in the statement. And with the observation that the real formulation of the final result is something quite complicated, and so, for one more time, the use of the complex variable $z=x+iy$ is something very useful. □

As a conclusion to what we have so far, we know that the holomorphic functions, and so their real and imaginary parts too, are harmonic. That is, if f is holomorphic, then the following function is harmonic, for any values of the parameters $\alpha,\beta\in\mathbb{C}$:

$$f_{\alpha\beta}=\alpha Re(f)+\beta Im(f)$$

Observe that this result covers all the examples that we have so far, for instance with the function $\bar{z}$, that we know to be harmonic, appearing as follows:

$$\bar{z}=Re(z)-iIm(z)$$

Moreover, Theorem 14.27 seems to suggest that the converse of this is true. We will see later that this is something which happens indeed, at least locally.

In order to further comment on this, let us formulate the following definition:

DEFINITION 14.28. *The Cauchy-Riemann operators are*

$$\partial = \frac{1}{2}\left(\frac{d}{dx} - i\frac{d}{dy}\right) \quad , \quad \bar{\partial} = \frac{1}{2}\left(\frac{d}{dx} + i\frac{d}{dy}\right)$$

where $\frac{d}{dx}$ *and* $\frac{d}{dy}$ *are the usual partial derivatives for complex functions.*

There are many things that can be said about these Cauchy-Riemann operators $\partial, \bar{\partial}$, the idea being that in many contexts, these are better to use than the usual partial derivatives $\frac{d}{dx}, \frac{d}{dy}$, and with this being a bit like the usage of the variables $z, \bar{z}$, instead of the decomposition $z = x + iy$, for many questions regarding the complex numbers. At the general level, the main properties of $\partial, \bar{\partial}$ can be summarized as follows:

THEOREM 14.29. *Assume that* $f : X \to \mathbb{C}$ *is differentiable in the real sense.*

(1) f *is holomorphic precisely when* $\bar{\partial} f = 0$.
(2) *In this case, its derivative is* $f' = \partial f$.
(3) *The Laplace operator is given by* $\Delta = 4\partial\bar{\partial}$.
(4) f *is harmonic precisely when* $\partial\bar{\partial} f = 0$.

PROOF. We can assume by linearity that we are dealing with differentiability questions at 0. Since our function $f : X \to \mathbb{C}$ is differentiable in the real sense, we have a formula as follows, with $z = x + iy$, and with $a, b \in \mathbb{C}$ being the partial derivatives at 0:

$$f(z) = ax + by + o(z)$$

Now observe that we can write this formula in the following way:

$$\begin{aligned} f(z) &= a \cdot \frac{z + \bar{z}}{2} + b \cdot \frac{z - \bar{z}}{2i} + o(z) \\ &= a \cdot \frac{z + \bar{z}}{2} + b \cdot \frac{i\bar{z} - iz}{2} + o(z) \\ &= \frac{a - ib}{2} \cdot z + \frac{a + ib}{2} \cdot \bar{z} + o(z) \end{aligned}$$

Now by dividing by z, we obtain from this the following formula:

$$\begin{aligned} \frac{f(z)}{z} &= \frac{a - ib}{2} + \frac{a + ib}{2} \cdot \frac{\bar{z}}{z} + o(1) \\ &= \partial f(0) + \bar{\partial} f(0) \cdot \frac{\bar{z}}{z} + o(1) \end{aligned}$$

But this gives the first two assertions, because in order for the derivative $f'(0)$ to exist, appearing as the $z \to 0$ limit of the above quantity, the coefficient of $\bar{z}/z$, which does not

converge, must vanish. Regarding now the third assertion, this follows from:

$$\begin{aligned}\Delta &= \frac{d^2}{dx^2}+\frac{d^2}{dy^2}\\ &= \left(\frac{d}{dx}-i\frac{d}{dy}\right)\left(\frac{d}{dx}+i\frac{d}{dy}\right)\\ &= 4\partial\bar{\partial}\end{aligned}$$

As for the last assertion, this is clear from this latter formula of Δ. □

With this discussed, as a next objective, and getting back now to more concrete things, let us try to find the harmonic functions which are radial, in the following sense:

$$f(z)=\varphi(|z|)$$

However, things are quite tricky here, involving a blowup phenomenon at the dimension value $N=2$, which is precisely the one that we are interested in. In view of this phenomenon, it makes sense to move now to arbitrary $N\in\mathbb{N}$ dimensions. So, let us introduce the Laplace operator, acting on the functions $f:\mathbb{R}^N\to\mathbb{C}$, as follows:

$$\Delta f=\sum_{i=1}^{N}\frac{d^2f}{dx_i^2}$$

As before in 2 dimensions, we say that a function $f:\mathbb{R}^N\to\mathbb{C}$ is harmonic when $\Delta f=0$. With these conventions, the result about radial harmonics is as follows:

THEOREM 14.30. *The fundamental radial solutions of $\Delta f=0$ are*

$$f(x)=\begin{cases}||x||^{2-N} & (N\neq 2)\\ \log||x|| & (N=2)\end{cases}$$

with the log *at $N=2$ basically coming from $\log'=1/x$.*

PROOF. Consider indeed a radial function, defined outside the origin $x=0$. This function can be written as follows, with $\varphi:(0,\infty)\to\mathbb{C}$ being a certain function:

$$f:\mathbb{R}^N-\{0\}\to\mathbb{C}\quad,\quad f(x)=\varphi(||x||)$$

Our first goal will be that of reformulating the Laplace equation $\Delta f = 0$ in terms of the one-variable function $\varphi : (0, \infty) \to \mathbb{C}$. For this purpose, observe that we have:

$$\begin{aligned}
\frac{d||x||}{dx_i} &= \frac{d\sqrt{\sum_{i=1}^N x_i^2}}{dx_i} \\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{\sum_{i=1}^N x_i^2}} \cdot \frac{d\left(\sum_{i=1}^N x_i^2\right)}{dx_i} \\
&= \frac{1}{2} \cdot \frac{1}{||x||} \cdot 2x_i \\
&= \frac{x_i}{||x||}
\end{aligned}$$

By using this formula, we have the following computation:

$$\begin{aligned}
\frac{df}{dx_i} &= \frac{d\varphi(||x||)}{dx_i} \\
&= \varphi'(||x||) \cdot \frac{d||x||}{dx_i} \\
&= \varphi'(||x||) \cdot \frac{x_i}{||x||}
\end{aligned}$$

By differentiating one more time, we obtain the following formula:

$$\begin{aligned}
\frac{d^2 f}{dx_i^2} &= \frac{d}{dx_i}\left(\varphi'(||x||) \cdot \frac{x_i}{||x||}\right) \\
&= \frac{d\varphi'(||x||)}{dx_i} \cdot \frac{x_i}{||x||} + \varphi'(||x||) \cdot \frac{d}{dx_i}\left(\frac{x_i}{||x||}\right) \\
&= \left(\varphi''(||x||) \cdot \frac{x_i}{||x||}\right) \cdot \frac{x_i}{||x||} + \varphi'(||x||) \cdot \frac{||x|| - x_i \cdot x_i/||x||}{||x||^2} \\
&= \varphi''(||x||) \cdot \frac{x_i^2}{||x||^2} + \varphi'(||x||) \cdot \frac{||x||^2 - x_i^2}{||x||^3}
\end{aligned}$$

Now by summing over $i \in \{1, \ldots, N\}$, this gives the following formula:

$$\begin{aligned}
\Delta f &= \sum_{i=1}^N \varphi''(||x||) \cdot \frac{x_i^2}{||x||^2} + \sum_{i=1}^N \varphi'(||x||) \cdot \frac{||x||^2 - x_i^2}{||x||^3} \\
&= \varphi''(||x||) \cdot \frac{||x||^2}{||x||^2} + \varphi'(||x||) \cdot \frac{(N-1)||x||^2}{||x||^3} \\
&= \varphi''(||x||) + \varphi'(||x||) \cdot \frac{N-1}{||x||}
\end{aligned}$$

Thus, with $r = ||x||$, the Laplace equation $\Delta f = 0$ can be reformulated as follows:

$$\varphi''(r) + \frac{(N-1)\varphi'(r)}{r} = 0$$

Equivalently, the equation that we want to solve is as follows:

$$r\varphi'' + (N-1)\varphi' = 0$$

Now observe that we have the following formula:

$$\begin{aligned}(r^{N-1}\varphi')' &= (N-1)r^{N-2}\varphi' + r^{N-1}\varphi'' \\ &= r^{N-2}((N-1)\varphi' + r\varphi'')\end{aligned}$$

Thus, the equation to be solved can be simply written as follows:

$$(r^{N-1}\varphi')' = 0$$

We conclude that $r^{N-1}\varphi'$ must be a constant K, and so, that we must have:

$$\varphi' = Kr^{1-N}$$

But the fundamental solutions of this latter equation are as follows:

$$\varphi(r) = \begin{cases} r^{2-N} & (N \neq 2) \\ \log r & (N = 2) \end{cases}$$

Thus, we are led to the conclusion in the statement. □

Back now to the general theory of harmonic functions, the above passage to arbitrary $N \in \mathbb{N}$ dimensions, that we will adopt, proves to be something fruitful, allowing us to see many things obscured by various $N = 2$ phenomena. Among others, in analogy with the usual theory of the holomorphic functions, we can formulate the following statement:

THEOREM 14.31. *The harmonic functions in N dimensions obey to the same general principles as the holomorphic functions, namely:*

(1) *The plain mean value formula.*
(2) *The boundary mean value formula.*
(3) *The maximum modulus principle.*
(4) *The Liouville theorem.*

PROOF. This is something quite straightforward, the idea being as follows:

(1) Regarding the plain mean value formula, here the statement is that given an harmonic function $f : X \to \mathbb{C}$, and a ball B, the following happens:

$$f(x) = \int_B f(y)dy$$

In order to prove this formula, we can assume that our ball B is centered at 0, say of radius $r > 0$. Now if we denote by χ_r the characteristic function of this ball, nomalized as to integrate up to 1, we want to prove that we have the following formula:

$$f = f * \chi_r$$

To be more precise, here $*$ is the standard convolution operation, given by:

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x - y)g(y)dy$$

For proving the above formula, let us pick a number $0 < s < r$, and a solution w of the following equation, supported on B, which can be constructed explicitly:

$$\Delta w = \chi_r - \chi_s$$

By using the basic properties of the convolution operation $*$, we have:

$$\begin{aligned} f * \chi_r - f * \chi_s &= f * (\chi_r - \chi_s) \\ &= f * \Delta w \\ &= \Delta f * w \\ &= 0 \end{aligned}$$

Thus $f * \chi_r = f * \chi_s$, and by letting now $s \to 0$, we get $f * \chi_r = f$, as desired.

(2) Regarding the boundary mean value formula, here the statement is that given an harmonic function $f : X \to \mathbb{C}$, and a ball B, with boundary γ, the following happens:

$$f(x) = \int_\gamma f(y)dy$$

But this follows as a consequence of the plain mean value formula in (1), with our two mean value formulae, the one there and the one here, being in fact equivalent, by using annuli and radial integration for the proof of the equivalence, in the obvious way.

(3) Regarding the maximum modulus principle, the statement here is that any holomorphic function $f : X \to \mathbb{C}$ has the property that the maximum of $|f|$ over a domain is attained on its boundary. That is, given a domain D, with boundary γ, we have:

$$\exists x \in \gamma \quad , \quad |f(x)| = \max_{y \in D} |f(y)|$$

But this is something which follows again from the mean value formula in (1), first for the balls, and then in general, by using a standard division argument.

(4) Finally, regarding the Liouville theorem, the statement here is that an entire, bounded harmonic function must be constant:

$$f : \mathbb{R}^N \to \mathbb{C} \quad , \quad \Delta f = 0 \quad , \quad |f| \leq M \quad \implies \quad f = \text{constant}$$

As a slightly weaker statement, again called Liouville theorem, we have the fact that an entire harmonic function which vanishes at ∞ must vanish globally:

$$f:\mathbb{R}^N\to\mathbb{C}\quad,\quad \Delta f=0\quad,\quad \lim_{x\to\infty}f(x)=0\quad\implies\quad f=0$$

But can view these as a consequence of the mean value formula in (1), because given two points $x\neq y$, we can view the values of f at these points as averages over big balls centered at these points, say $B=B_x(R)$ and $C=B_y(R)$, with $R>>0$:

$$f(x)=\int_B f(z)dz\quad,\quad f(y)=\int_C f(z)dz$$

Indeed, the point is that when the radius goes to ∞, these averages tend to be equal, and so we have $f(x)\simeq f(y)$, which gives $f(x)=f(y)$ in the limit, as desired. □

So long for harmonic functions in arbitrary N dimensions. Getting back now to 2 dimensions, as a useful complement to what we have in Theorem 14.31, we have:

THEOREM 14.32. *The real harmonic functions $f:X\to\mathbb{R}$ with $X\subset\mathbb{C}$ are locally the real parts of the holomorphic functions $g:X\to\mathbb{C}$.*

PROOF. This is something that we announced before, and which is a bit more technical, the idea being that this can be indeed established by using the technology from Theorem 14.31, at $N=2$. We refer to Rudin [74] for the proof of this. □

14e. Exercises

This was a quite standard calculus chapter, and as exercises on this, we have:

EXERCISE 14.33. *Experiment some more with the geometric series, over $\mathbb{C}$.*

EXERCISE 14.34. *Learn more about the logarithm, as a complex function.*

EXERCISE 14.35. *Learn as well about a^x and x^a, as complex functions.*

EXERCISE 14.36. *Learn more about hyperbolic functions, and the need for them.*

EXERCISE 14.37. *Clarify the relation between the various notions of differentiability.*

EXERCISE 14.38. *Fill in the various details, in the proof of the Cauchy formula.*

EXERCISE 14.39. *Learn more about harmonic functions, and their properties.*

EXERCISE 14.40. *Read the proof of the last theorem, $f=Re(g)$, locally.*

As bonus exercise, read some quantum mechanics, the scalars there being $\mathbb{C}$ too.

CHAPTER 15

Zeta function

15a. Real zeta

We have already met the Riemann zeta function on several occasions, in the above, at values $s > 1$ of the parameter, with the conclusion every time that this function is intimately related to the primes. In this chapter we discuss a systematic approach to this phenomenon, by using complex analysis. As a first observation, we can talk without much pain about zeta at complex values of s as well, in the following way:

THEOREM 15.1. *We can talk about the Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at any $s \in \mathbb{C}$ with $Re(z) > 1$.

PROOF. We have the following computation, assuming $s = r + it$ with $r > 1$:

$$\begin{aligned}
|\zeta(s)| &= \left|\sum_{n=1}^{\infty} \frac{1}{n^s}\right| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{|n^s|} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^r} \\
&< 1 + \int_1^{\infty} \frac{1}{x^r}\, dx \\
&= 1 + \left[\frac{x^{1-r}}{1-r}\right]_1^{\infty} \\
&= 1 + \frac{1}{r-1}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a first result, we can write zeta as an Euler product, as follows:

THEOREM 15.2. *We have the following formula, with product over the primes,*

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}$$

valid for any exponent $s\in\mathbb{C}$ with $Re(s)>1$.

PROOF. We have the following computation, with everything converging:

$$\begin{aligned}\zeta(s)&=\sum_{n=1}^{\infty}\frac{1}{n^s}\\&=\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\ldots\right)\\&=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Some further useful formulae for zeta come in terms of the Möbius function. Let us first recall that the Möbius function is defined, in general, as follows:

DEFINITION 15.3. *The Möbius function of a lattice L is given by*

$$\mu(a,b)=\begin{cases}1&\text{if } a=b\\-\sum_{a\leq c<b}\mu(a,c)&\text{if } a<b\\0&\text{if } a\not\leq b\end{cases}$$

with this construction being performed by recurrence.

As an illustration for this, consider the lattice of positive integers $L=\mathbb{N}$, with the order $a\leq b$ when $a|b$. Let us try to compute the following function:

$$\mu(n)=\mu(1,n)$$

For this function, the recurrence relation from Definition 15.3 reads:

$$\mu(n)=\begin{cases}1&\text{if } n=1\\-\sum_{m|n,m<n}\mu(m)&\text{if } n>1\end{cases}$$

We have the following computations, to start with, for any prime number p:

$$\mu(1)=1$$

$$\mu(p)=-\mu(1)=-1$$

$$\mu(p^2)=-\mu(1)-\mu(p)=-1+1=0$$

$$\mu(p^3)=-\mu(1)-\mu(p)-\mu(p^2)=-1+1-0=0$$

In general, by recurrence, for any prime p, we have the following formula:

$$\mu(p^k)=\begin{cases}1 & \text{if } k=0\\ -1 & \text{if } k=1\\ 0 & \text{if } k\geq 2\end{cases}$$

Which does not look very good, but let us do some more computations. When coming with a second prime number q, different from p, we have the following computation:

$$\mu(pq)=-\mu(1)-\mu(p)-\mu(q)=-1+1+1=1$$

Which looks quite nice, at least we don't get a 0. So let us add now to the picture a third prime r, distinct from p,q. We have the following computation:

$$\begin{aligned}\mu(pqr)&=-\mu(1)-\mu(p)-\mu(q)-\mu(r)-\mu(pq)-\mu(pr)-\mu(qr)\\ &=-1+1+1+1-1-1-1\\ &=-1\end{aligned}$$

When adding a fourth prime number s to be picture, distinct from p,q,r, we have:

$$\begin{aligned}\mu(pqrs)&=-\mu(1)-\mu(p)-\mu(q)-\mu(r)-\mu(s)\\ &\quad-\mu(pq)-\mu(pr)-\mu(ps)-\mu(qr)-\mu(qs)-\mu(rs)\\ &\quad-\mu(pqr)-\mu(pqs)-\mu(prs)-\mu(qrs)\\ &=-1+4-6+4\\ &=1\end{aligned}$$

In general now, by recurrence, for any $p_1,\ldots,p_k$ distinct primes, we have:

$$\mu(1,p_1\ldots p_k)=(-1)^k$$

Which looks good, and the next question is, what is the formula unifying what we have, for $n=p^k$, and $n=p_1\ldots p_k$? And here the answer is quite obvious, as follows:

THEOREM 15.4. *The Möbius function $\mu(n)=\mu(1,n)$ is given by*

$$\mu(n)=\begin{cases}(-1)^k & \text{if } n=p_1\ldots p_k\\ 0 & \text{if } n \text{ is not square}-\text{free}\end{cases}$$

and its two-variable version is given by $\mu(a,b)=\delta_{a|b}\mu(b/a)$.

PROOF. We already have some good evidence for the first formula in the statement, but let us do some more computations. For p,q distinct primes, we have:

$$\begin{aligned}\mu(pq^2)&=-\mu(1)-\mu(p)-\mu(pq)-\mu(q)\\ &=-1+1-1+1\\ &=0\end{aligned}$$

Similarly, we have as well the following computation:

$$\begin{aligned}\mu(pq^3) &= -\mu(1)-\mu(p)-\mu(pq)-\mu(pq^2)-\mu(q)-\mu(q^2)\\ &= -1+1-1-0+1-0\\ &= 0\end{aligned}$$

Which makes it pretty clear that we have $\mu(pq^k)=0$ for any $k\geq 2$, by recurrence. Next, when adding a third prime r to the picture, distinct from p,q, we have:

$$\begin{aligned}\mu(pqr^2) &= -\mu(1)-\mu(p)-\mu(pq)-\mu(pr)-\mu(pqr)-\mu(pr^2)\\ &\quad -\mu(q)-\mu(qr)-\mu(qr^2)-\mu(r)-\mu(r^2)\\ &= -1+1-1-1+1-0\\ &\quad +1-1-0+1-0\\ &= 0\end{aligned}$$

And so on, we certainly have very solid evidence now for our formula, and we will leave its proof in general, by recurrence, as an instructive exercise. As for the second formula in the statement, regarding the general function $\mu(a,b)$, this follows from this. □

Back to theory, the main interest in the Möbius function comes from the Möbius inversion formula, which in linear algebra terms can be stated and proved as follows:

THEOREM 15.5. *We have the following implication,*

$$f(b)=\sum_{a\leq b}g(a)\quad\implies\quad g(b)=\sum_{a\leq b}\mu(a,b)f(a)$$

valid for any two functions $f,g:L\to\mathbb{C}$.

PROOF. Consider the adjacency matrix of L, given by the following formula:

$$K_{ab}=\begin{cases}1 & \text{if } a\leq b\\ 0 & \text{if } a\not\leq b\end{cases}$$

Our claim is that the inverse of this matrix is the Möbius matrix of L, given by:

$$M_{ab}=\mu(a,b)$$

Indeed, the above matrix K is upper triangular, and when trying to invert it, we are led to the recurrence in Definition 15.3, so to the Möbius matrix M. Thus we have:

$$M=K^{-1}$$

Thus, in practice, we are led to the inversion formula in the statement. □

Getting back now to the Möbius function for the integers, from Theorem 15.4, many interesting things can be said about it. First, we have the following formula:

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n\geq 2\end{cases}$$

Next, we have the following formula, which is elementary too:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$$

We have as well the following formula, which is elementary too:

$$\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{n}=-1$$

Next, we have the following formula, with γ being the Euler constant:

$$\sum_{n=1}^{\infty}\frac{\mu(n)\log^2 n}{n}=-2\gamma$$

Finally, we have the following related formula, valid for any $|q|<1$:

$$\sum_{n=1}^{\infty}\frac{\mu(n)q^n}{1-q^n}=q$$

Getting back now to the zeta function, we have the following formula:

THEOREM 15.6. *We have the following formula,*

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$

with μ being the Möbius function, given by the formula

$$\mu(n)=\begin{cases}(-1)^k & \text{if } n=p_1\ldots p_k\\ 0 & \text{if } n \text{ is not square}-\text{free}\end{cases}$$

valid for any exponent $s\in\mathbb{C}$ with $Re(s)>1$.

PROOF. We have the following computation, with everything converging:

$$\begin{aligned}\frac{1}{\zeta(s)} &= \prod_p\left(1-\frac{1}{p^s}\right)\\ &= \sum_{k=0}^{\infty}(-1)^k\prod_{p_1\ldots p_k}\frac{1}{p_1^s\ldots p_k^s}\\ &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Along the same lines, as another elementary result about zeta, we have:

THEOREM 15.7. *The square of the zeta function is given by*

$$\zeta^2(s)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}$$

with $\tau(n)$ being the number of divisors of n, for any $s\in\mathbb{C}$ with $Re(s)>1$.

PROOF. We have the following computation, with everything converging:

$$\zeta(s)^2=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(kl)^s}=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}$$

Thus, we are led to the conclusion in the statement. □

There are some further formulae of the same type, again involving the zeta function, and the above function $\tau(n)$, counting the number of divisors of n. First, we have:

$$\frac{\zeta^3(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{\tau(n^2)}{n^s}$$

Along the same lines, we have as well the following formula:

$$\frac{\zeta^4(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{\tau(n)^2}{n^s}$$

In order to present now a number of more advanced results regarding the zeta function, which are very useful in practice, we will need the following well-known fact:

THEOREM 15.8. *We can talk about the gamma function*

$$\Gamma(s)=\int_0^{\infty}x^{s-1}e^{-x}\,dx$$

extending the usual factorial of integers, $\Gamma(s)=(s-1)!$.

PROOF. The integral converges indeed, and by partial integration we have:

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty x^s e^{-x}\,dx\\ &= \int_0^\infty sx^{s-1}e^{-x}\,dx\\ &= s\Gamma(s)\end{aligned}$$

Regarding now the case $s \in \mathbb{N}$, for the initial value $s = 1$ we have:

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1$$

Thus, for $s \in \mathbb{N}$ we have indeed $\Gamma(s) = (s-1)!$, as claimed. □

Many interesting things can be said about the gamma function, notably with a computation at the half-integers, and with this being related to geometry, as follows:

THEOREM 15.9. *The gamma function is given at half-integers by*

$$\Gamma(n) = (n-1)! \quad , \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!!}{2^n}\sqrt{\pi}$$

and we have the following formula for it, with $c = \sqrt{2}, \sqrt{\pi}$ for N even, odd:

$$\Gamma\left(\frac{N}{2}\right) = \frac{(N-1)!!}{2^{(N-1)/2}}\,c$$

Moreover, the volume of the unit sphere in $\mathbb{R}^N$, which is given by

$$V = \left(\frac{\pi}{2}\right)^{[N/2]}\frac{2^N}{(N+1)!!}$$

can be expressed in terms of these values of the gamma function.

PROOF. There are several things going on here, the idea being as follows:

(1) We already know, from Theorem 15.8, that for $n \in \mathbb{N}$ we have:

$$\Gamma(n) = (n-1)!$$

(2) Regarding now the half-integers, we first have the following computation, using at the end the formula of the Gauss integral, that we learned in chapter 13:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{-1/2}e^{-x}\,dx\\ &= \int_0^\infty y^{-1}e^{-y^2}2y\,dy\\ &= 2\int_0^\infty e^{-y^2}dy\\ &= \sqrt{\pi}\end{aligned}$$

Next, by using $\Gamma(s+1)=s\Gamma(s)$, we have the following computations:

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\,\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)=\frac{3}{4}\,\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)=\frac{15}{8}\,\sqrt{\pi}$$

$$\vdots$$

Thus, we can solve the problem by recurrence, and we obtain in this way, with our usual convention $N!!=(N-1)(N-3)(N-5)\ldots$ for the double factorials:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5\ldots(2n-1)}{2^n}\,\sqrt{\pi}=\frac{(2n)!!}{2^n}\,\sqrt{\pi}$$

(3) Regarding now the unification of the formulae that we have in (1) and (2), let us first rewrite the formula that we just found in (2), in the following way:

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{(2n)!!}{2^n}\,\sqrt{\pi}$$

Which looks good and in final form, no questions about this, so for a unification we are left with refurbishing the nice formula found in (1), in the following way:

$$\begin{aligned}\Gamma\left(\frac{2n}{2}\right) &= \Gamma(n)\\ &= (n-1)!\\ &= \frac{2.4.6\ldots(2n-2)}{2^{n-1}}\\ &= \frac{(2n-1)!!}{2^{n-1}}\\ &= \frac{(2n-1)!!}{2^{n-1/2}}\sqrt{2}\end{aligned}$$

And with this, job done, we are led to the uniform formula in the statement.

(4) Finally, regarding the spheres, the assertion, which is something quite traditional, is self-explanatory. However, doing so will only result in complicating our formula, so we will not do this. For more on this, and asymptotics too, we refer to chapter 13. □

Getting back now to zeta, we can formulate a key result about it, as follows:

THEOREM 15.10. *We have the following formula,*

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx$$

valid for any $s\in\mathbb{C}$ with $Re(s)>1$.

PROOF. We have indeed the following computation:

$$\begin{aligned}\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx &= \int_0^\infty\frac{x^{s-1}}{e^x}\cdot\frac{1}{1-e^{-x}}\,dx\\ &= \int_0^\infty x^{s-1}(e^{-x}+e^{-2x}+e^{-3x}+\ldots)\\ &= \sum_{n=1}^\infty\int_0^\infty x^{s-1}e^{-nx}\,dx\\ &= \sum_{n=1}^\infty\int_0^\infty\left(\frac{y}{n}\right)^{s-1}e^{-y}\,\frac{dy}{n}\\ &= \sum_{n=1}^\infty\frac{1}{n^s}\int_0^\infty y^{s-1}e^{-y}\,dy\\ &= \zeta(s)\Gamma(s)\end{aligned}$$

Thus, we are led to the formula in the statement. □

15b. Special values

At a more advanced level now, we can try to compute particular values of ζ. Things are quite tricky here, and we have the following result, briefly discussed before:

THEOREM 15.11. *We have the following formula, for the even integers $s = 2k$,*

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}$$

with B_n being the Bernoulli numbers, which in practice gives the formulae

$$\zeta(2) = \frac{\pi^2}{6} \quad , \quad \zeta(4) = \frac{\pi^4}{90} \quad , \quad \zeta(6) = \frac{\pi^6}{945} \quad , \quad \zeta(8) = \frac{\pi^8}{9450} \quad , \quad \ldots$$

generalizing the formula $\zeta(2) = \pi^2/6$ of Euler, solving the Basel problem.

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, at $s = 2$ the Euler computation, from before, was as follows:

$$\begin{aligned}
\frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\\
&= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\ldots\\
&= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\ldots\\
&= 1 - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,x^2 + \ldots
\end{aligned}$$

It is possible to use the same idea for dealing with $\zeta(2k)$ with $k \in \mathbb{N}$, but this is quite complicated, and in addition the above method of Euler needs some justification, that we have not really provided before, so in short, not a path to be followed.

(2) Instead, we have the following luminous computation, using Theorem 15.10:

$$\begin{aligned}
\zeta(2k) &= \frac{1}{\Gamma(2k)}\int_0^\infty \frac{x^{2k-1}}{e^x - 1}\,dx\\
&= \frac{1}{(2k-1)!}\int_0^\infty \frac{x^{2k-1}}{e^x - 1}\,dx\\
&= \frac{1}{(2k-1)!}\int_0^\infty \frac{(2\pi t)^{2k-1}}{e^{2\pi t} - 1}\,2\pi dt\\
&= \frac{(2\pi)^{2k}}{(2k-1)!}\int_0^\infty \frac{t^{2k-1}}{e^{2\pi t} - 1}\,dt
\end{aligned}$$

(3) But, we recognize on the right the integral giving rise to the even Bernoulli numbers, with one of the many definitions of these numbers being as follows:

$$B_{2k} = 4k(-1)^{k+1} \int_0^\infty \frac{t^{2k-1}}{e^{2\pi t} - 1}\, dt$$

Thus, we can finish our computation of the values $\zeta(2k)$ as follows:

$$\begin{aligned} \zeta(2k) &= \frac{(2\pi)^{2k}}{(2k-1)!} \cdot (-1)^{k+1} \frac{B_{2k}}{4k} \\ &= (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2 \cdot (2k)!} \end{aligned}$$

(4) Regarding now the Bernoulli numbers, there is a long story here. At the beginning, we have the following formula of Bernoulli, standing as a definition for them:

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} B_k n^{m+1-k}$$

This leads to the following recurrence relation, which computes them:

$$B_m = -\frac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k} B_k$$

In practice, we can see that the odd Bernoulli numbers all vanish, except for the first one, $B_1 = -1/2$, and that the even Bernoulli numbers are as follows:

$$\frac{1}{6} \quad , \quad -\frac{1}{30} \quad , \quad \frac{1}{42} \quad , \quad -\frac{1}{30} \quad , \quad \frac{5}{66} \quad , \quad -\frac{691}{2730} \quad , \quad \frac{7}{6} \quad , \quad \ldots$$

(5) For analytic purposes, the Bernoulli numbers are best viewed as follows, with this coming from the fact that the coefficients satisfy the above recurrence relation:

$$\begin{aligned} \frac{x}{e^x - 1} &= \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \\ &= 1 - \frac{1}{2} x + \frac{1}{6} \cdot \frac{x^2}{2!} - \frac{1}{30} \cdot \frac{x^4}{4!} + \frac{1}{42} \cdot \frac{x^6}{6!} - \frac{1}{30} \cdot \frac{x^8}{8!} + \ldots \end{aligned}$$

Observe that all this is related as well to the hyperbolic functions, via:

$$\frac{x}{2} \left(\coth \frac{x}{2} - 1 \right) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

The point now is that, in relation with our zeta business, the above analytic formulae give, after some calculus, the formula that we used in (3), namely:

$$B_{2k} = 4k(-1)^{k+1} \int_0^\infty \frac{t^{2k-1}}{e^{2\pi t} - 1}\, dt$$

(6) Finally, no discussion about the Bernoulli numbers would be complete without mentioning the Euler-Maclaurin formula, involving them, which is as follows:

$$\begin{aligned}\sum_{k=0}^{n-1} f(x) &\simeq \int_0^n f(x)dx - \frac{1}{2}(f(n)-f(0)) \\ &+\frac{1}{6}\cdot\frac{f'(n)-f'(0)}{2!} - \frac{1}{30}\cdot\frac{f^{(3)}(n)-f^{(3)}(0)}{4!} \\ &+\frac{1}{42}\cdot\frac{f^{(5)}(n)-f^{(5)}(0)}{6!} - \frac{1}{30}\cdot\frac{f^{(7)}(n)-f^{(7)}(0)}{8!} + \ldots\end{aligned}$$

(7) And there is more coming from the complex extension of the zeta function, by analytic continuation, that we will discuss later. An an announcement here, the values of zeta at the negative integers $0, -1, -2, -3, \ldots$ will not be ∞, but rather given by:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

Alternatively, we have the following formula for the Bernoulli numbers:

$$B_n = (-1)^{n-1} n\zeta(1-n)$$

(8) In any case, we are led to the various conclusions in the statement, both theoretical and numeric. And exercise for you of course to learn more about the Bernoulli numbers, and beware of the freakish notations used by mathematicians there. □

As a more digest form of Theorem 15.11, let us record as well:

THEOREM 15.12. *The generating function of the numbers $\zeta(2k)$ with $k \in \mathbb{N}$ is*

$$\sum_{k=0}^{\infty} \zeta(2k)x^{2k} = -\frac{\pi x}{2}\cot(\pi x)$$

and with this generalizing the formula involving Bernoulli numbers.

PROOF. This is something tricky, again, the idea being as follows:

(1) A version of the recurrence formula for Bernoulli numbers is as follows:

$$B_{2n} = -\frac{1}{n+1/2}\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}$$

Now observe that this formula can be written in the following way:

$$\frac{B_{2n}}{(2n)!} = -\frac{1}{n+1/2}\sum_{k=1}^{n-1}\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2n-2k}}{(2n-2k)!}$$

In view of Theorem 15.11, we obtain the following formula, valid at any $n > 1$:

$$\zeta(2n) = \frac{1}{n + 1/2} \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n - 2k)$$

(2) But this allows the computation of the series in the statement, by squaring that series. Indeed, consider the following modified version of that series:

$$f(x) = 2 \sum_{k=0}^{\infty} \zeta(2k) \left(\frac{x}{\pi}\right)^{2k}$$

By squaring, and using the recurrence formula for the numbers $\zeta(2n)$ found in (1), with some care at the values $n = 0, 1$, not covered by that formula, we obtain:

$$f^2 + f + x^2 = xf'$$

(3) But this is precisely the functional equation satisfied by $g(x) = -x \cot x$. Indeed, by using the well-known formula $\cot' = -\cot^2 - 1$, we have:

$$\begin{aligned} xg' &= x(-\cot x - x \cot' x) \\ &= x(-\cot x + x \cot^2 x + x) \\ &= g + g^2 + x^2 \end{aligned}$$

(4) We conclude that we have $f = g$, which reads:

$$2 \sum_{k=0}^{\infty} \zeta(2k) \left(\frac{x}{\pi}\right)^{2k} = -x \cot x$$

Now by replacing $x \to \pi x$, we obtain the formula in the statement. □

Regarding now the values $\zeta(2k + 1)$ with $k \in \mathbb{N}$, the story here is more complicated, with the first such number being the Apéry constant, given by:

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

There has been a lot of work on this number, by Apéry and others, and on the higher $\xi(2k + 1)$ values as well. Let us record here the following result, a bit of physics flavor:

THEOREM 15.13. *We have the following formula,*

$$\zeta(s) = \int_0^1 \ldots \int_0^1 \frac{dx_1 \ldots dx_s}{1 - x_1 \ldots x_s}$$

valid for any $s \in \mathbb{N}$, $s \geq 2$.

PROOF. This follows as usual from some calculus, the idea being as follows:

(1) At $s=2$ we have indeed the following computation, using Theorem 15.10:

$$\begin{aligned}\int_0^1\int_0^1\frac{1}{1-xy}\,dxdy &= \int_0^1\left[-\frac{\log(1-xy)}{y}\right]_0^1 dy\\ &= -\int_0^1\frac{\log(1-y)}{y}\,dy\\ &= -\int_0^\infty\frac{\log(e^{-t})}{1-e^{-t}}\,e^{-t}dt\\ &= \int_0^\infty\frac{t}{e^t-1}\,dt\\ &= \zeta(2)\Gamma(2)\\ &= \zeta(2)\end{aligned}$$

(2) In the general case, $s\in\mathbb{N}$, the best is to start with the following formula:

$$\frac{1}{1-x_1\ldots x_s}=\sum_{n=0}^\infty(x_1\ldots x_s)^n$$

Thus, the integral in the statement is given by the following formula:

$$\int_0^1\ldots\int_0^1\frac{dx_1\ldots dx_s}{1-x_1\ldots x_s}=\sum_{n=0}^\infty\int_0^1\ldots\int_0^1(x_1\ldots x_s)^n dx_1\ldots dx_s$$

But this leads to the formula in the statement, after some computations.

(3) Before leaving, let us see as well, out of mathematical curiosity, what happens at the exponent $s=1$. Here the integral in the statement is:

$$\begin{aligned}\int_0^1\frac{1}{1-x}\,dx &= [-\log(1-x)]_0^1\\ &= -\log(1-1)+\log(1-0)\\ &= \infty+0\\ &= \zeta(1)\end{aligned}$$

Not a big deal, you would say, but as an interesting remark, since $\log(1-x)\simeq -x$, we are led to the conclusion that ζ, when suitably extended by analytic continuation, should have a simple pole at $s=1$, with residue 1. We will be back to this, in a moment. □

Many other things can be said about ζ and its special values, as a continuation of the above, and check here any advanced number theory book. In what concerns us, we will rather head towards the analytic left half-plane $Re(s)\leq 1$, using complex analysis.

15c. Complex zeta

Quite remarkably, with a bit of complex analysis, we can have the zeta function working in the whole complex plane, via analytic continuation. However, analytic continuation being Devil's business, we will explain this slowly, by gradually going from the analytic right half-plane $Re(s) > 1$, that we understand well, to other parts of $\mathbb{C}$.

Getting started with our exploratory trip West, and make sure that you have enough food, water and weapons, let us first see what happens at $s = 1$. Here we have:

PROPOSITION 15.14. *We have the following formula,*

$$\lim_{s\to 1}(s-1)\zeta(s) = 1$$

showing that the complex zeta has a simple pole at $s = 1$, with residue 1.

PROOF. We have the following computation, using $\Gamma(1) = 1$:

$$\begin{aligned}\lim_{s\to 1}(s-1)\zeta(s) &= \lim_{s\to 1}(s-1)\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx\\ &= \lim_{t\to 0}\int_0^\infty \frac{tx^t}{e^x-1}\,dx\\ &= 1\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a more advanced result now, on the same topic, we have:

THEOREM 15.15. *We have the following formula,*

$$\lim_{s\to 1}\left|\zeta(s) - \frac{1}{s-1}\right| = \gamma$$

with γ being the Euler-Mascheroni constant.

PROOF. This is something more advanced, the idea being as follows:

(1) The Euler-Mascheroni constant is related to the zeta function by:

$$\gamma = \sum_{n=2}^\infty (-1)^n\frac{\zeta(n)}{n}$$

(2) On the other hand, we have we well the following formula:

$$\gamma = \lim_{s\to 1^+}\sum_{n=1}^\infty \frac{1}{n^s} - \frac{1}{s^n}$$

But in terms of the zeta function, this latter formula simply reads:

$$\gamma = \lim_{s\to 1^+}\zeta(s) - \frac{1}{s-1}$$

(3) Thus, we are led to the formula in the statement. Note that we have as well:

$$\gamma=\lim_{s\to 0}\frac{\zeta(1+s)+\zeta(1-s)}{2}$$

Indeed, this follows from the formula in the statement. □

Leaving aside now $s=1$, let us focus on the other points, $s=1+it$ with $t\neq 0$, of the boundary line $Re(s)=1$, between known and unknown. We have here:

THEOREM 15.16. *The Riemann zeta function, namely*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

converges at any $s=1+it$ with $t\neq 0$.

PROOF. We have the following computation, to start with:

$$\begin{aligned}\zeta(1+it)&=\sum_{n=1}^{\infty}\frac{1}{n^{1+it}}\\&=\sum_{n=1}^{\infty}\frac{1}{ne^{it\log n}}\\&=\sum_{n=1}^{\infty}\frac{e^{-it\log n}}{n}\\&=\sum_{n=1}^{\infty}\frac{\cos(t\log n)-i\sin(t\log n)}{n}\end{aligned}$$

And then, the convergence at $t\neq 0$ can be proved via some calculus. □

Let us get now into the true unknown, $Re(s)<1$, with our first objective being that of understanding what happens in the strip $0<Re(s)<1$. We first have here:

PROPOSITION 15.17. *Unlike the standard Riemann series, which diverges,*

$$\zeta(1)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots=\infty$$

the signed version of this series, called standard Dirichlet series, converges,

$$\eta(1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots<\infty$$

and we can even compute its value, $\eta(1)=\log 2$.

PROOF. Here the convergence of the series $\eta(1)$ can be proved in a variety of ways, for instance by grouping terms and comparing to $\zeta(2) < \infty$:

$$\eta(1) = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \ldots < \zeta(2) < \infty$$

As for the exact formula of $\eta(1)$, this follows from the Taylor formula for log:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots$$

Indeed, by plugging in $x = 1$, we obtain the formula in the statement. □

Thus, we have our idea, "forcing" zeta to converge in the strip $0 < Re(s) < 1$, by adding signs, and then recovering zeta, or rather its analytic continuation, in this same strip, by removing the signs. This leads to the following remarkable result:

THEOREM 15.18. *We have the following formula,*

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

which can stand as definition for ζ, in the strip $0 < Re(s) < 1$.

PROOF. This is something elementary, known since Dirichlet and Euler, but of key importance, and with many consequences, the idea being as follows:

(1) To start with, we can define the Dirichlet function η as being the signed version of ζ, exactly as we did in Proposition 15.17 at $s = 1$, as follows:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

Observe that this function converges indeed in the strip $0 < Re(s) < 1$.

(2) We must now connect ζ and η, at $Re(s) > 1$, and this can be done as follows:

$$\begin{aligned}
\zeta(s) + \eta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\
&= 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^s} \\
&= 2^{1-s} \sum_{k=1}^{\infty} \frac{1}{k^s} \\
&= 2^{1-s} \zeta(s)
\end{aligned}$$

(3) But this gives the following formula, valid at any exponent $s \in \mathbb{C}$ satisfying $Re(s) > 1$, and which is the formula in the statement:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\,\eta(s)$$

(4) In order now to conclude, we can invoke the theory of analytic continuation. Skipping some theoretical details here, and we refer for instance to Rudin [74] for all this, what we have in the statement is a formula for ζ in the whole right half-plane, $Re(s) > 0$, which is analytic, and more specifically meromorphic, with a single pole, at $s = 1$, and which coincides with the usual formula of ζ on the usual domain of definition, $Re(s) > 1$. But, in this situation, the theory of analytic continuation tells us that we can redefine ζ all over the right half-plane, $Re(s) > 0$, by the formula in the statement, and with this extension being unique, as per the general properties of the meromorphic functions.

(5) Finally, observe that our present result proves Theorem 15.16 as well. Thinking retrospectively, we were in need there precisely of a Dirichlet type idea. □

15d. Riemann formula

Getting now to the left half-plane, $Re(s) < 0$, many methods are available here, and with the main one, due to Riemann himself, which is something quite tough, but unavoidable for understanding the zeta function as a whole, being as follows:

THEOREM 15.19. *We have the following formula of Riemann, relating the values of zeta at s and $1 - s$,*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s)\zeta(1 - s)$$

which holds on the strip $0 < Re(s) < 1$, and can serve as definition for zeta in the left half-plane, $Re(s) < 0$, by analytic continuation.

PROOF. This is something subtle, with even understanding the statement being non-trivial business, and with the proof being complicated too, the idea being as follows:

(1) To start with, let us check our formula for mistakes. With $Re(s) > 1$ our formula tells us that the familiar $\zeta(s)$ can be expressed in terms of some virtual number $\zeta(1 - s)$, which remains to be defined later, and normally no problem with this.

(2) However, looking more carefully, there might be a problem coming from the sine, which vanishes at $s = 2k$ with $k \in \mathbb{N}$. But, the point is that $\Gamma(1 - s)$ has a pole at $s = 2k$, compensating for this vanishing of the sine. So, as a conclusion here, not only we avoided the contradictory $\zeta(2k) = 0$, but also know that, later when it will come to discuss $\zeta(1 - 2k)$, that will be a usual complex number, with no need for a pole there.

(3) Conversely now, let us plug in numbers with $Re(s) < 0$, so that $Re(1-s) > 1$. Here what our formula tells us is that the familiar $\zeta(1 - s)$, when multiplied by the quantities

in the statement, produces a candidate $\zeta(s)$ for the analytic continuation in the left half-plane $Re(s) < 0$. So, very good, no contradiction whatsoever here, and in addition this tells us, confirming the finding in (2), that zeta will have no poles at $Re(s) < 0$.

(4) Now let us have a look at the strip $0 < Re(s) < 1$. Here our function ζ is already existent, thanks to Theorem 15.18, and we have something to prove, namely that the Riemann formula in the statement holds indeed, in this strip $0 < Re(s) < 1$.

(5) But this is something that can be proved indeed, via some non-trivial calculus, done by Riemann a long time ago, and which has been barely simplified, since. In order to get started, we use the following formula for the gamma function:

$$\Gamma\left(\frac{s}{2}\right) = n^s \pi^{\frac{s}{2}} \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx$$

(6) Thus, we are led to the following formula for the zeta function:

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{\frac{s}{2}} \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx \\ &= \pi^{\frac{s}{2}} \int_0^\infty x^{\frac{s}{2}-1} \sum_{n=1}^\infty e^{-n^2\pi x} dx\end{aligned}$$

(7) Now let us call Ψ the function appearing on the right, namely:

$$\Psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x} dx$$

With this convention, the formula that we found can be written as follows:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\Psi(x)dx$$

(8) Now let us have a look at the function Ψ. By Poisson summation we obtain:

$$\sum_{n=-\infty}^\infty e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^\infty e^{-\frac{n^2\pi}{x}}$$

We conclude that our function Ψ satisfies the following equation:

$$2\Psi(x) + 1 = \frac{1}{\sqrt{x}}\left(2\Psi\left(\frac{1}{x}\right) + 1\right)$$

(9) With this equation in hand, let us go back to the formula for zeta in (7). We can further process that formula, in the following way:

$$\begin{aligned}
\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}\Psi(x)dx\\
&= \int_0^1 x^{\frac{s}{2}-1}\Psi(x)dx+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)dx\\
&= \int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\Psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{2}}-\frac{1}{2}\right)dx+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)dx\\
&= \frac{1}{s-1}+\frac{1}{s}+\int_0^1 x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)dx+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)dx
\end{aligned}$$

(10) We conclude from this that we have the following formula:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty\left(x^{-\frac{s+1}{2}}+x^{\frac{s}{2}-1}\right)\Psi(x)dx$$

Now since the expression on the right is invariant under $s\to 1-s$, we obtain:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

But with this we are done, because this latter formula is equivalent to the Riemann symmetry formula in the statement, namely:

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

(11) Next, there is some discussion at the border of the strip too, with the formula relating the values at $Re(s)=1$, all finite except for a pole at $s=1$, to the values at $Re(s)=0$, which all follow to be finite, thanks to the mechanism explained in (2).

(12) Now with this done, we can take the formula in the statement as a definition for zeta in the left half-plane, $Re(s)<0$, and with the general theory of analytic continuation telling us, a bit like before, at the end of the proof of Theorem 15.18, that this continuation is unique, thanks to the general properties of the meromorphic functions.

(13) So, this was for the idea of the proof, and in practice, there are of course many details still in need to be checked, and we will leave this as an instructive exercise. □

Observe that, in what regards the Riemann formula itself, this remains a key symmetry formula of our newly defined zeta function, as a meromorphic function over $\mathbb{C}$.

All the above starts to be a bit heavy, and as a summary of all this, we have:

THEOREM 15.20. *We can talk about the Riemann zeta, as a meromorphic function $\zeta : \mathbb{C} \to \mathbb{C}$, with a single pole, at $s = 1$ with residue 1. At $Re(s) > 1$ we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and more generally at $Re(s) > 0$ we have the following formula:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

Also, the values of zeta at any s and $1 - s$ are related by the Riemann formula

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s)\zeta(1 - s)$$

with Γ being as usual the gamma function.

PROOF. This is a summary of our various findings from Theorems 15.18 and 15.19 and their proofs, and with the thing to be always kept in mind, when dealing with all this, being that the formula at $Re(s) > 0$ generalizes indeed the formula at $Re(s) > 1$, thanks to a trivial computation, explained in the proof of Theorem 15.18. □

Getting back now to the Riemann formula from Theorem 15.19, passed the technical difficulties for establishing it, this is something very beautiful and useful, with a lot of symmetry in it, making it clear that the strip $0 < Re(s) < 1$ is what matters, and that the vertical axis $Re(s) = 1/2$ is where interesting things should happen.

As a consequence of the Riemann formula, we have the following version of it:

THEOREM 15.21. *We have the following version of the Riemann formula,*

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1 - s)$$

symmetric in $s, 1 - s$, which is in fact equivalent to it.

PROOF. The above formula is indeed equivalent to the one in Theorem 15.19, and is in fact what comes out from computations, when proving Theorem 15.19. □

In practice, the quantity in Theorem 15.21 is best normalized as follows:

THEOREM 15.22. *The following function, called ξ function,*

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

satisfies $\xi(s) = \xi(1 - s)$.

PROOF. Again, the above Riemann formula is equivalent to the previous ones, with the function ξ being what is used in computations, when proving Theorem 15.19. □

We have zeta up and working in the full complex plane $\mathbb{C}$, as a meromorphic function with a single pole at 1, and this gives rise to many interesting questions. To start with, regarding the analytic continuation, by other means, the situation is as follows:

(1) A first formula, due to Hasse, which works at any $s \neq 1$, is as follows:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^s}$$

(2) A second formula, due to Hasse too, which again works at any $s \neq 1$, is:

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^{s-1}}$$

(3) We also have the following version, nicer, but working only at $Re(s) > 0$:

$$\zeta(s) = \frac{1}{s-1} \sum_{n=1}^{\infty} \left(\frac{n}{(n+1)^s} - \frac{n-s}{n^s} \right)$$

(4) But we can modify this latter formula as follows, as to have it at $Re(s) > -1$:

$$\zeta(s) = \frac{1}{s-1} \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \left(\frac{2n+3+s}{(n+1)^{s+2}} - \frac{2n-1-s}{n^{s+2}} \right)$$

(5) And so on, the idea being that we can conquer the whole left half-plane $Re(s) < 0$ in this way, step by step, with at each step a more complicated formula being needed.

Getting now to a second question, other general formulae satisfied by zeta, there are many of them. To start with, we can write a Laurent series expansion, as follows:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} (1-s)^n$$

The Laurent coefficients are the Euler-Mascheroni constant $\gamma_0 = \gamma$, and:

$$\gamma_n = \lim_{m \to \infty} \left[\left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} \right) - \frac{(\log m)^{n+1}}{n+1} \right]$$

We also have the following formula, involving generalized binomial coefficients:

$$\frac{\zeta(s)}{s} = \frac{1}{s-1} - \sum_{n=1}^{\infty} \binom{n+s-1}{n+1} (\zeta(s+n) - 1)$$

Getting now to a third question, special values of zeta, we have already seen the formulae of $\zeta(2k)$ with $k \in \mathbb{N}$, the idea being these can be recaptured from:

$$\sum_{k=0}^{\infty} \zeta(2k) x^{2k} = -\frac{\pi x}{2} \cot(\pi x)$$

In practice, we get the following formula, with B_n being the Bernoulli numbers:

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2 \cdot (2k)!}$$

Now by Riemann reflection, we obtain from this the following formula:

$$\zeta(-2k+1) = -\frac{B_{2k}}{2k}$$

In fact, by Riemann reflection, we have the following formula, for any $n \in \mathbb{N}$:

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

Regarding now the values $\zeta(2k+1)$ with $k \in \mathbb{N}$, things here are quite complicated, starting with the Apéry constant, which is as follows, not computable:

$$\zeta(3) = 1.20205..$$

However, there are many interesting formulae relating the numbers $\zeta(2k+1)$, or more generally the numbers $\zeta(n)$, between themselves. We first have:

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = 1 \qquad , \qquad \sum_{k=1}^{\infty} (\zeta(2k) - 1) = \frac{3}{4}$$

$$\sum_{k=1}^{\infty} (\zeta(2k+1) - 1) = \frac{1}{4} \qquad , \qquad \sum_{k=2}^{\infty} (-1)^k (\zeta(k) - 1) = \frac{1}{2}$$

Along the same lines, a second series of formulae is as follows:

$$\sum_{k=1}^{\infty} (-1)^k \frac{\zeta(k)}{k} = 0 \qquad , \qquad \sum_{k=1}^{\infty} \frac{\zeta(k) - 1}{k} = 0$$

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} = \gamma \qquad , \qquad \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k} = 1 - \gamma$$

And there are many more formulae computing or relating the values of zeta at positive integers, more specialized, quite often Ramanujan-looking.

Getting now to zeroes, as a consequence of Theorem 15.19, we have:

THEOREM 15.23. *We have the following formula, for any integer $k \geq 1$,*

$$\zeta(-2k) = 0$$

with these being called the "trivial zeroes" of ζ.

PROOF. We recall that the Riemann symmetry formula from Theorem 15.19 is as follows, valid all over the complex plane, as an equality of meromorphic functions:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

By plugging in the value $s = -2k$, with $k \geq 1$ integer, we obtain:

$$\begin{aligned} \zeta(-2k) &= 2^{-2k} \pi^{-2k-1} \sin(k\pi)\Gamma(1+2k)\xi(1+2k) \\ &= 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

15e. Exercises

Welcome to the zeta function, and no way back. As exercises about it, we have:

EXERCISE 15.24. *Learn more about the Möbius function, and its various properties.*

EXERCISE 15.25. *Learn also more about the gamma function, and its properties.*

EXERCISE 15.26. *Find, if needed, your own best notation for the Bernoulli numbers.*

EXERCISE 15.27. *Experiment a bit with the multiple integral formula for $\xi(s)$, $s \in \mathbb{N}$.*

EXERCISE 15.28. *Learn more about the Dirichlet function, and its properties.*

EXERCISE 15.29. *Read, with full details, the theory of analytic continuation.*

EXERCISE 15.30. *Clarify all details, in relation with the analytic continuation of ζ.*

EXERCISE 15.31. *Learn as well about the work of Hasse and others, on ζ.*

As standard bonus exercise, spend more time with ζ, the more the better.

CHAPTER 16

Riemann hypothesis

16a. Back to primes

Let us go back to the main result from chapter 13, namely the Chebycheff estimate there, which was as follows, with the function $\pi(x)$ counting the primes $p \leq x$:

$$\pi(x) \approx \frac{x}{\log x}$$

As mentioned in chapter 13, Hadamard and de la Vallée Poussin were able, using the Riemann zeta function, to prove the Prime Number Theorem, which states that:

$$\pi(x) \sim \frac{x}{\log x}$$

We will explain here this result, which is highly-non trivial, even by modern standards, following the original proof of Hadamard and de la Vallée Poussin.

We will comment as well on some other known proofs of the Prime Number Theorem, notably with the Selberg proof, not using zeta. And finally, we will comment on the Riemann hypothesis, which is related to all this, and to many other things.

So, this will be the plan for this chapter, basically with a Theorem coming with two different proofs, which is highly unusual, you might think that we have something against the first proof, or against the zeta function in general. Quite the opposite, we love zeta. But the Selberg proof is instructive as well, revealing some things about prime numbers not necessarily captured by the mighty zeta, and we will discuss it too.

Getting to work now, our tools for proving the Prime Number Theorem, following Hadamard and de la Vallée Poussin, will be, besides the Riemann zeta function ζ, the modified Chebycheff function ψ and the von Mangoldt function Λ. We have:

DEFINITION 16.1. *The modified Chebycheff and von Mangoldt functions are*

$$\psi(x) = \sum_{p^k \leq x} \log p \quad , \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

related by the formulae $\psi(x) = \sum_{n \leq x} \Lambda(n)$ *and* $\Lambda(n) = \psi(n) - \psi(n-)$.

You might of course ask, why using two functions instead of one. Good point, and in answer, we will see a bit later that, in the context of certain delicate questions, the Chebycheff function and the von Mangoldt function are not exactly the same thing.

In relation with the Prime Number Theorem, that we want to prove, we have:

PROPOSITION 16.2. *We have the following equivalence,*

$$\pi(x) \sim \frac{x}{\log x} \iff \psi(x) \sim x$$

with the condition on the left being the Prime Number Theorem one.

PROOF. This is something elementary, coming from two estimates, as follows:

(1) In one sense, we have the following basic estimate:

$$\begin{aligned}
\psi(x) &= \sum_{p^k \leq x} \log p \\
&= \sum_{p \leq x} \log p \left[\frac{\log x}{\log p} \right] \\
&\leq \sum_{p \leq x} \log x \\
&= \pi(x) \log x
\end{aligned}$$

(2) In the other sense, we have the following estimate, valid for any $\varepsilon > 0$:

$$\begin{aligned}
\psi(x) &= \sum_{p^k \leq x} \log p \\
&\geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \\
&\geq \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) \log x \\
&= (1-\varepsilon)(\pi(x) + O(x^{1-\varepsilon})) \log x
\end{aligned}$$

Thus, we are led to the equivalence in the statement. □

In order to estimate now the Chebycheff function ψ, we would need an analytic formula for it. However, finding such a formula is not obvious with bare hands, so let us examine instead the same question for the von Mangoldt function Λ, with the hope that we do have an analytic formula for Λ, that can be translated afterwards in terms of ψ.

And good news, our plan works, with the formula for Λ being as follows:

PROPOSITION 16.3. *The von Mangoldt function satisfies*

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -(\log \zeta(s))'$$

with ζ being the Riemann zeta function.

PROOF. We use the Euler product formula for zeta, namely:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

By taking the logarithm, we obtain from this the following formula:

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right)$$

Now by differentiating, we obtain the following formula:

$$\begin{aligned}
(\log \zeta(s))' &= -\sum_p \left(1 - \frac{1}{p^s}\right)^{-1} \frac{d(1-p^{-s})}{ds} \\
&= \sum_p \left(1 - \frac{1}{p^s}\right)^{-1} \frac{dp^{-s}}{ds} \\
&= -\sum_p \left(1 - \frac{1}{p^s}\right)^{-1} p^{-s} \log p \\
&= -\sum_p \frac{p^s}{p^s - 1} \cdot \frac{1}{p^s} \log p \\
&= -\sum_p \frac{\log p}{p^s - 1}
\end{aligned}$$

On the other hand, the sum on the left in the statement is given by:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} &= \sum_{n=p^k}\frac{\log p}{n^s}\\ &= \sum_p \log p\sum_{k=1}^{\infty}\frac{1}{p^{ks}}\\ &= \sum_p \log p\cdot\frac{1}{p^s}\left(1-\frac{1}{p^s}\right)^{-1}\\ &= \sum_p\frac{\log p}{p^s-1}\end{aligned}$$

Thus, we are led to the equality in the statement. □

Now let us turn to the second part of our plan, namely reformulating the formula for Λ that we found in terms of ψ. This is something more delicate, leading to:

THEOREM 16.4. *The modified Chebycheff function is given by*

$$\psi(x)=x-\log(2\pi)-\sum_{\zeta(s)=0}\frac{x^s}{s}$$

for $x\notin\mathbb{Z}$, with the sum being over all the zeroes of zeta.

PROOF. This follows via some complex analysis and tricks, as follows:

(1) To start with, we know from Definition 16.1 that the functions ψ and Λ are related by the following conversion formulae, which are both trivial:

$$\psi(x)=\sum_{n\leq x}\Lambda(n)\quad,\quad\Lambda(n)=\psi(n)-\psi(n-)$$

The problem now is to use these conversion formulae, in order to reformulate in terms of ψ the formula for Λ that we found in Proposition 16.3, namely:

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}=-(\log\zeta(s))'$$

(2) As a first step, we have the following computation, with at the beginning the $n = 1$ term ignored, and at the end, the $n = 1$ term added, because these vanish anyway:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} &= \sum_{n=2}^{\infty}\frac{\psi(n)-\psi(n-)}{n^s}\\ &= \sum_{n=2}^{\infty}\frac{\psi(n)-\psi(n-1)}{n^s}\\ &= \sum_{n=1}^{\infty}\psi(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\end{aligned}$$

(3) Thus, we have the following equation, in terms of the function ψ:

$$\sum_{n=1}^{\infty}\psi(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right) = -(\log\zeta(s))'$$

(4) The problem is now, how to fine-tune this, into something truly analytical, involving the function $\psi(x)$ with real argument, $x > 1$. For this purpose, it is convenient to further modify the Chebycheff step function ψ, by making it continuous, as follows:

$$\varphi(x) = \int_1^x \psi(t)dt$$

(5) Observe that this latter function can be expressed in terms of Λ, as follows:

$$\varphi(x) = \sum_{n\leq x}(x-n)\Lambda(n)$$

Also, as another remark, in relation with Proposition 16.2, we have:

$$\psi(x)\sim x \iff \varphi(x)\sim\frac{x^2}{2}$$

Thus, we can normally do everything with φ instead of ψ. However, for our purposes here, φ will be a secondary object, with our main function remaining ψ.

(6) The point now is that we have the following formula, as a contour integral, with $r > 1$, coming via some manipulations involving the Cauchy formula:

$$\frac{\varphi(x)}{x^2} = \frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}\sum_{n=1}^{\infty}\psi(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)ds$$

(7) We recognize on the right the sum from (3), and by plugging that in, we get:

$$\begin{aligned}\frac{\varphi(x)}{x^2} &= -\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}(\log\zeta(s))'ds\\ &= -\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}\cdot\frac{\zeta'(s)}{\zeta(s)}\,ds\end{aligned}$$

(8) Now since the function $\zeta'(s)/\zeta(s)$ has a simple pole at 1, with residue -1, we can separate the contribution of that pole, and we get, again with $r>1$:

$$\frac{\varphi(x)}{x^2}=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}\left(\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}\right)ds$$

(9) In order to simplify notation, let us introduce the following function:

$$f(s)=\frac{1}{s(s+1)}\left(\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}\right)$$

In terms of this function, the formula that we found above reads:

$$\begin{aligned}\frac{\varphi(x)}{x^2} &= \frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}x^{s-1}f(x)\,ds\\ &= \frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{1}{2\pi}\int_{-\infty}^{\infty}x^{r+it-1}f(r+it)\,dt\\ &= \frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{x^{r-1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt\end{aligned}$$

(10) Thus, getting back now to the usual Chebycheff function ψ, we have:

$$\frac{1}{x^2}\int_1^x\psi(t)dt=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{x^{r-1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt$$

By multiplying both sides by x^2, we have the following formula:

$$\int_1^x\psi(t)dt=\frac{(x-1)^2}{2}-\frac{x^{r+1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt$$

(11) Now by taking the derivative with respect to x, this formula gives:

$$\begin{aligned}\psi(x) &= \frac{d}{dx}\left[\frac{(x-1)^2}{2}-\frac{x^{r+1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt\right]\\ &= x-1+\frac{d}{dx}\left[\frac{x^{r+1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt\right]\end{aligned}$$

(12) The point now is that, by computing the derivative on the right, we get:

$$\psi(x) = x - \log(2\pi) - \sum_{\zeta(s)=0} \frac{x^s}{s}$$

Thus, we are led to the conclusion in the statement. □

Now remember from Proposition 16.2 that what we want to do is to estimate ψ, with the following estimate, proving the Prime Number theorem, being our goal:

$$\psi(x) \sim x$$

Looking at the formula in Theorem 16.4, the x is already there, $\log(2\pi)$ does not matter, and what is left to prove that the sum over zeroes of ζ does not matter either:

$$\sum_{\zeta(s)=0} \frac{x^s}{s} = o(x)$$

In what regards the trivial zeroes, things are easily settled here, as follows:

PROPOSITION 16.5. *The contribution to the modified Chebycheff function ψ of the trivial zeroes of zeta, namely $-2, -4, -6, \ldots$, is given by*

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} = -\frac{1}{2} \log\left(1 - \frac{1}{x^2}\right)$$

and this quantity vanishes in the $x \to \infty$ limit.

PROOF. We have indeed the following computation:

$$\sum_{k=1}^{\infty} \frac{x^{-2k}}{2k} = \sum_{k=1}^{\infty} \frac{1}{2kx^{2k}} = -\log\left(1 - \frac{1}{x^2}\right)$$

Thus, we are led to the conclusion in the statement. □

16b. Prime distribution

Regarding now the non-trivial zeroes of zeta, we know from chapter 15 that these lie inside the strip $0 \leq Re(s) \leq 1$, and as a first observation, we have:

PROPOSITION 16.6. *The contribution to the modified Chebycheff function ψ of the non-trivial zeroes of zeta lying in the strip $0 \leq Re(s) < 1$ satisfies*

$$\sum_{\zeta(s)=0} \frac{x^s}{s} = o(x)$$

so we are left with studying the zeroes on the line $Re(s) = 1$.

PROOF. This is something quite self-explanatory, with some care needed however when summing all the $o(x)$ quantities associated to the zeroes in question. As for the final conclusion, this comes by combining our finding with Proposition 16.5. □

We are now getting to the core of the proof, with the key ingredient being:

THEOREM 16.7. *The Riemann zeta function has no zero on the line*

$$Re(s) = 1$$

and no zero on the line $Re(s) = 0$ *either.*

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, the $Re(s) = 0$ result, that we will not need here for our current purposes, in view of Proposition 16.6, but which of course has great theoretical interest, follows from the $Re(s) = 1$ result, via the Riemann reflection formula from chapter 15.

(2) In order to study now the zeta function on the line $Re(s) = 1$, we use the Euler product formula for this function, namely:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

By taking the logarithm, we obtain from this the following formula:

$$\begin{aligned}\log\zeta(s) &= -\sum_p \log\left(1 - \frac{1}{p^s}\right)\\ &= \sum_p \sum_{k=0}^{\infty} \frac{1}{kp^{ks}}\end{aligned}$$

(3) Now with $s = r + it$ as usual, this formula reads:

$$\begin{aligned}\log\zeta(s) &= \sum_p \sum_{k=0}^{\infty} \frac{1}{kp^{k(r+it)}}\\ &= \sum_p \sum_{k=0}^{\infty} \frac{p^{-kit}}{kp^{kr}}\\ &= \sum_p \sum_{k=0}^{\infty} \frac{e^{-kit\log p}}{kp^{kr}}\\ &= \sum_p \sum_{k=0}^{\infty} \frac{\cos(kt\log p) - i\sin(kt\log p)}{kp^{kr}}\end{aligned}$$

(4) Now remember the following formula, for the complex exponentials:

$$|e^z|^2 = e^z \cdot \overline{e^z} = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2Re(z)}$$

Thus we have $|e^z| = e^{Re(z)}$, and by using this with $z = \log\zeta(s)$, we get:

$$\begin{aligned} |\zeta(s)| &= |\exp(\log\zeta(s))| \\ &= \exp(Re(\log\zeta(s))) \\ &= \exp\left(\sum_p\sum_{k=0}^{\infty}\frac{\cos(kt\log p)}{kp^{kr}}\right) \end{aligned}$$

(5) In order to get an estimate, we use the following formula, valid for any $\alpha\in\mathbb{R}$:

$$\begin{aligned} 2(1+\cos\alpha)^2 &= 2+4\cos\alpha+2\cos^2\alpha \\ &= 3+4\cos\alpha+\cos(2\alpha) \end{aligned}$$

Indeed, by using this, we obtain from the formula in (4) the following estimate:

$$\begin{aligned} |\zeta(r)^3\zeta(r+it)^4\zeta(r+2it)| &= \exp\left(\sum_p\sum_{k=0}^{\infty}\frac{3+4\cos(kt\log p)+\cos(2kt\log p)}{kp^{kr}}\right) \\ &= \exp\left(\sum_p\sum_{k=0}^{\infty}\frac{2(1+\cos(kt\log p))^2}{kp^{kr}}\right) \\ &\geq 1 \end{aligned}$$

(6) But with this, we can now finish. Assume indeed by contradiction $\zeta(1+it)=0$, for some $t\neq 0$, and let us look at the following quantity, in the $r\to 1^+$ limit:

$$K = \zeta(r)^3\zeta(r+it)^4\zeta(r+2it)$$

What happens then in the $r\to 1^+$ limit is that we have $\zeta(r)^3\to\infty$ with triple pole behavior, $\zeta(r+it)^4\to 0$ with quadruple zero behavior, and $\zeta(r+2it)\to\zeta(2it)$ with analytic behavior. But since $3<4$ the quadruple zero will kill the triple pole, and so:

$$\lim_{r\to 1^+} K = 0$$

But this contradicts the estimate found in (5), and so our theorem is proved. □

By putting now everything together, we obtain:

THEOREM 16.8 (Prime Number Theorem). *We have*

$$\pi(x)\sim\frac{\log x}{x}$$

in the $x\to\infty$ limit.

PROOF. This follows by putting everything together, as follows:

(1) We know from Proposition 16.2 that $\pi(x) \sim x/\log x$ is equivalent to $\psi(x) \sim x$.

(2) We have in Theorem 16.4 a formula for $\psi(x)$, in terms of the zeroes of zeta.

(3) Most of these zeroes are taken care of by Proposition 16.5 and Proposition 16.6.

(4) As for the remaining zeroes, there are none, as shown by Theorem 16.7. □

As already mentioned on several occasions, the Prime Number Theorem is something quite powerful, often beating the other estimates that we have been talking about, in this book, starting from chapter 13, and in fact even before, starting with chapter 7.

It is actually instructive at this point to go back to chapter 13, and review the material there, with simplified proofs for the estimates that we found, and with proofs for the estimates that we did not prove, by using the Prime Number Theorem.

16c. Selberg formula

As mentioned in the beginning of the present chapter, there are as well some other known proofs of the Prime Number Theorem, which are more modern, notably:

(1) The now classical Selberg proof, which does not use the zeta function.

(2) The Newman proof, not using zeta either, and which is a bit shorter.

Both these proofs are instructive, having advantages and disadvantages with respect to the original proof by Hadamard and de la Vallée Poussin, and we will briefy discuss this here. We will be quite short, by leaving many computations as exercises.

Getting started, let us first recall that we have the following result:

PROPOSITION 16.9. *The Prime Number theorem is equivalent to the estimate*

$$\sum_{n\leq x}\Lambda(n)\sim x$$

for the von Mangoldt function Λ.

PROOF. This is something that we already know, from earlier in this chapter, the idea being as follows. Consider the modified Chebycheff and von Mangoldt functions:

$$\psi(x)=\sum_{p^k\leq x}\log p\quad,\quad \Lambda(n)=\begin{cases}\log p & \text{if } n=p^k\\ 0 & \text{otherwise}\end{cases}$$

These functions are related by the following formulae:

$$\psi(x)=\sum_{n\leq x}\Lambda(n)\quad,\quad \Lambda(n)=\psi(n)-\psi(n-)$$

Our claim is that we have the following equivalence, with the condition on the left corresponding by definition to the Prime Number theorem, and with the condition on the right being, in view of the above conversion formulae, the one in the statement:

$$\pi(x) \sim \frac{x}{\log x} \iff \psi(x) \sim x$$

But this is something elementary, coming from two estimates explained earlier in this chapter. In one sense, we have indeed the following basic estimate:

$$\psi(x) \leq \sum_{p \leq x} \log x = \pi(x) \log x$$

As for the other sense, here we have the following estimate, valid for any $\varepsilon > 0$:

$$\psi(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) \log x = (1-\varepsilon)(\pi(x) + O(x^{1-\varepsilon})) \log x$$

Thus, we are led to the equivalence in the statement. □

In order to deal now with the estimate from Proposition 16.9, as a starting point, we will use the following elementary estimate for Λ, with different weights:

THEOREM 16.10. *We have the following estimate,*

$$\sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] = x \log x - x + O(\log x)$$

for any $x \geq 2$.

PROOF. This is something quite standard, the idea being as folows:

(1) To start with, we have the following formula, for the quantity on the left:

$$\begin{aligned} \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] &= \sum_{n \leq x} \sum_{d|n} \Lambda(d) \\ &= \sum_{n \leq x} \log n \\ &= \log[x]! \end{aligned}$$

(2) In order to estimate now the quantity that we found above, we can use the Euler summation formula, which is something elementary, as follows, with the usual convention $x = [x] + \{x\}$ for the integer and fractionary parts of $x \in \mathbb{R}$:

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t)dt + \int_y^x \{t\} f'(t)dt - \{x\} f(x) + \{y\} f(y)$$

To be more precise, in the case $y = a$ and $x = a + b$ with a, b integers, the Euler summation formula can be deduced by integrating by parts, as follows:

$$\begin{aligned}
\int_a^{a+b} \{t\} f'(t) dt &= \sum_{k=0}^{b-1} \int_{a+k}^{a+k+1} \{t\} f'(t) dt \\
&= \sum_{k=0}^{b-1} \int_{a+k}^{a+k+1} (t - a - k) f'(t) dt \\
&= \sum_{k=0}^{b-1} \Big[(t - a - k) f(t) \Big]_{a+k}^{a+k+1} - \int_{a+k}^{a+k+1} f(t) dt \\
&= \sum_{k=0}^{b-1} f(a + k + 1) - \int_{a+k}^{a+k+1} f(t) dt \\
&= \sum_{k=0}^{b-1} f(a + k + 1) - \int_a^{a+b} f(t) dt
\end{aligned}$$

Indeed, this gives the following formula, which is the Euler one, in this case:

$$\sum_{k=0}^{b-1} f(a + k + 1) = \int_a^{a+b} \{t\} f'(t) dt + \int_a^{a+b} f(t) dt$$

As for the general case, where the bounds $x, y \in \mathbb{R}$ are arbitrary, this follows by making some adjustements in the above computations, both at left and at right, which make appear the extra term $-\{x\} f(x) + \{y\} f(y)$ on the right, as stated.

(3) Getting back now to what we wanted to prove, by using the Euler summation formula for the function $f(t) = \log t$, we obtain the following formula:

$$\begin{aligned}
\log[x]! &= \sum_{n \leq x} \log n \\
&= \int_1^x \log t \, dt + \int_1^x \frac{\{t\}}{t} \, dt - \{x\} \log x \\
&= x \log x - x + 1 + \int_1^x \frac{\{t\}}{t} \, dt + O(\log x) \\
&= x \log x - x + O\left(\int_1^x \frac{1}{t} \, dt \right) + O(\log x) \\
&= x \log x - x + O(\log x)
\end{aligned}$$

Thus, we are led to the estimate in the statement. □

Now let us have a look at the estimate that we have, from Theorem 16.10, and at the estimate that we want, from Proposition 16.9. The one that we have is:

$$\sum_{n\leq x}\Lambda(n)\left[\frac{x}{n}\right]=x\log x-x+O(\log x)$$

And the one that we want is a bit similar, but with different weigths, as follows:

$$\sum_{n\leq x}\Lambda(n)\sim x$$

In view of this, the idea will be that of establishing first some general results, connecting such type of estimates, and then applying them to our situation.

Such general results, connecting sums of a series, with different weights, are called Tauberian theorems, and the one that we need here, due to Shapiro, is as follows:

THEOREM 16.11. *Consider a series $a_n\geq 0$ satisfying the condition*

$$\sum_{n\leq x}a_n\left[\frac{x}{n}\right]=x\log x+O(x)$$

for any $x\geq 1$. We have then the following estimate, for any $x\geq 1$:

$$\sum_{n\leq x}\frac{a_n}{n}=\log x+O(1)$$

Also, we can find constants $A,B>0$ such that the following estimates hold,

$$Ax\leq\sum_{n\leq x}a_n\leq Bx$$

on the left for $x>>0$, and on the right for any $x\geq 1$.

PROOF. This is something quite technical, the idea being as follows:

(1) Consider the following sums, with $a_n\geq 0$ being as in the statement:

$$S(x)=\sum_{n\leq x}a_n\quad,\quad T(x)=\sum_{n\leq x}a_n\left[\frac{x}{n}\right]$$

Our first goal will be that of proving the inequality $S(x) \leq Bx$ at the end. For this purpose, observe first that we have the following estimate:

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= \sum_{n\leq x} a_n \left[\frac{x}{n}\right] - 2\sum_{n\leq x/2} a_n \left[\frac{x}{2n}\right] \\ &= \sum_{n\leq x/2} a_n \left(\left[\frac{x}{n}\right] - 2\left[\frac{x}{2n}\right]\right) + \sum_{x/2<n\leq x} a_n \left[\frac{x}{n}\right] \\ &\geq \sum_{x/2<n\leq x} a_n \left[\frac{x}{n}\right] \\ &= \sum_{x/2<n\leq x} a_n \\ &= S(x) - S\left(\frac{x}{2}\right) \end{aligned}$$

Now recall that, in terms of $T(x)$, our assumption on the series a_n was as follows:

$$T(x) = x\log x + O(x)$$

But this gives the following estimate, for the quantity considered above:

$$\begin{aligned} T(x) - 2T\left(\frac{x}{2}\right) &= x\log x + O(x) - 2\left(\frac{x}{2}\log\frac{x}{2} + O\left(\frac{x}{2}\right)\right) \\ &= x\log x + O(x) - x\log x + O(x) \\ &= O(x) \end{aligned}$$

We conclude that, in terms of $S(x)$, we have the following estimate:

$$S(x) - S\left(\frac{x}{2}\right) = O(x)$$

Now this tells us that we can find a constant $C > 0$ such that, for any $x \geq 1$:

$$S(x) - S\left(\frac{x}{2}\right) \leq Cx$$

By replacing $x \to x/2$, we conclude that the following estimate must hold too:

$$S\left(\frac{x}{2}\right) - S\left(\frac{x}{4}\right) \leq \frac{Cx}{2}$$

By replacing again $x \to x/2$, the following estimate must hold too:

$$S\left(\frac{x}{4}\right) - S\left(\frac{x}{8}\right) \leq \frac{Cx}{4}$$

Now by summing all these estimates we obtain, as desired, with $B = 2C$:

$$S(x) \leq Cx + \frac{Cx}{2} + \frac{Cx}{4} + \ldots = 2Cx$$

(2) With this done, let us turn now to the proof of the first assertion in the statement. By using the estimate $S(x) \leq Bx$ that we just proved, we have:

$$\begin{aligned} T(x) &= \sum_{n\leq x} a_n \left[\frac{x}{n}\right] \\ &= \sum_{n\leq x} a_n \left(\frac{x}{n} + O(1)\right) \\ &= x\sum_{n\leq x} \frac{a_n}{n} + O\left(\sum_{n\leq x} a_n\right) \\ &= x\sum_{n\leq x} \frac{a_n}{n} + O(S(x)) \\ &= x\sum_{n\leq x} \frac{a_n}{n} + O(x) \end{aligned}$$

But this gives the first assertion, by using our assumption on the series a_n in the statement, which reads $T(x) = x\log x + O(x)$, in the following way:

$$\begin{aligned} \sum_{n\leq x} \frac{a_n}{n} &= \frac{T(x) - O(x)}{x} \\ &= \frac{T(x)}{x} + O(1) \\ &= \frac{x\log x + O(x)}{x} + O(1) \\ &= \log x + O(1) \end{aligned}$$

(3) It remains to prove now the last assertion in the statement, namely the estimate $S(x) \geq Ax$ for $x >> 0$. For this purpose, consider the following sums:

$$A(x) = \sum_{n\leq x} \frac{a_n}{n}$$

What we just found in (2) can be written as follows, with $E(x) = O(1)$:

$$A(x) = \log x + E(x)$$

Now observe that for any number $c \in (0, 1)$ we have the following estimate, with $M > 0$ being an upper bound for the absolute value of the function $E(x) = O(1)$, and with the variable x being subject to the condition $cx \geq 1$:

$$\begin{aligned} A(x) - A(cx) &= \log x + E(x) - \log(cx) - E(cx) \\ &= E(x) - E(cx) - \log c \\ &\geq -2M - \log c \end{aligned}$$

If we choose the number $c \in (0,1)$ as for the quantity on the right to be 1, we conclude that we have the following estimate, valid for any $x \geq 1/c$:

$$A(x) - A(cx) \geq 1$$

On the other hand, the quantity on the left is subject to the following estimate:

$$\begin{aligned} A(x) - A(cx) &= \sum_{n \leq x} \frac{a_n}{n} - \sum_{n \leq cx} \frac{a_n}{n} \\ &= \sum_{cx < n \leq x} \frac{a_n}{n} \\ &\leq \frac{1}{cx} \sum_{n \leq x} a_n \\ &= \frac{S(x)}{cx} \end{aligned}$$

We conclude that we have the following estimate, for $x \geq 1/c$:

$$\frac{S(x)}{cx} \geq 1$$

Thus, we are led to the estimate in the statement, $S(x) \geq Ax$, with $A = c$, valid for $x >> 0$, and more specifically valid for $x \geq 1/c$, with $c \in (0,1)$ being as above. □

As a first application of Theorem 16.11, in relation with our questions, we have:

THEOREM 16.12. *We have the following estimate, for any $x \geq 1$:*

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$$

Also, we can find constants $A, B > 0$ such that the following estimates hold,

$$Ax \leq \sum_{n \leq x} \psi(x) \leq Bx$$

on the left for $x >> 0$, and on the right for any $x \geq 1$.

PROOF. As a consequence of Theorem 16.10, we have the following estimate:

$$\begin{aligned} \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] &= x \log x - x + O(\log x) \\ &= x \log x + O(x) \end{aligned}$$

Thus Theorem 16.11 applies with $a_n = \Lambda(n)$ and gives the first formula, namely:

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$$

Also by Theorem 16.11, we obtain as well estimates as follows, with $A, B > 0$:

$$Ax \leq \sum_{n\leq x} \Lambda(n) \leq Bx$$

Now since we have $\psi(x) = \sum_{n\leq x} \Lambda(n)$, this gives the second assertion. □

As a second application of Theorem 16.11, also of interest to us, we have:

THEOREM 16.13. *We have the following estimate, for any $x \geq 1$:*

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1)$$

Also, we can find constants $A, B > 0$ such that the following estimates hold,

$$Ax \leq \sum_{n\leq x} \theta(x) \leq Bx$$

on the left for $x >> 0$, and on the right for any $x \geq 1$.

PROOF. This is something a bit more tricky. It is routine to prove that:

$$\sum_{p\leq x} \log p \left[\frac{x}{p}\right] = x \log x + O(x)$$

In view of this, consider the following version of the von Mangoldt function:

$$\Theta(n) = \begin{cases} \log p & \text{if } n = p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

In terms of this function Θ, the above estimate takes the following form:

$$\sum_{n\leq x} \Theta(n) \left[\frac{x}{n}\right] = x \log x + O(x)$$

Thus Theorem 16.11 applies with $a_n = \Theta(n)$ and gives the following formula:

$$\sum_{n\leq x} \frac{\Theta(n)}{n} = \log x + O(1)$$

But this gives the formula in the statement, namely:

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1)$$

Also by Theorem 16.11, we obtain as well estimates as follows, with $A, B > 0$:

$$Ax \leq \sum_{n\leq x} \Theta(n) \leq Bx$$

Now since we have $\theta(x) = \sum_{n\leq x} \Theta(n)$, this gives the second assertion. □

As yet another application of all this, again of interest to us, we have:

THEOREM 16.14. *We have the following estimate,*

$$\sum_{n\leq x}\psi\left(\frac{x}{n}\right)=x\log x-x+O(\log x)$$

as well as the following estimate,

$$\sum_{n\leq x}\theta\left(\frac{x}{n}\right)=x\log x+O(\log x)$$

both valid for any $x\geq 1$.

PROOF. These results are quite elementary, the idea being as follows:

(1) We recall from Theorem 16.10 that we have the following estimate:

$$\sum_{n\leq x}\Lambda(n)\left[\frac{x}{n}\right]=x\log x-x+O(\log x)$$

But this gives the first formula in the statement, by using:

$$\psi(x)=\sum_{n\leq x}\Lambda(n)$$

(2) As explained in the proof of Theorem 16.14, we have the following estimate:

$$\sum_{n\leq x}\Theta(n)\left[\frac{x}{n}\right]=x\log x+O(x)$$

But this gives the first formula in the statement, by using:

$$\theta(x)=\sum_{n\leq x}\Theta(n)$$

Thus, we are led to the conclusions in the statement. □

Getting now to what we wanted to prove, the Prime Number theorem, the continuation of the story, due to Selberg, is more complicated. We first have the following result:

PROPOSITION 16.15. *Given a function $G:(0,\infty)\to\mathbb{C}$, appearing as*

$$G(x)=\log x\sum_{n\leq x}F\left(\frac{x}{n}\right)$$

with $F:(0,\infty)\to\mathbb{C}$, we have the following equality,

$$F(x)\log x+\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)=\sum_{d\leq x}\mu(d)G\left(\frac{x}{d}\right)$$

with μ and Λ being the Möbius and von Mangoldt functions.

PROOF. This is something elementary, coming from the basic properties of μ and Λ. Indeed, we first have the following elementary equality, valid for any function F:

$$F(x)\log x=\sum_{n\leq x}F\left(\frac{x}{n}\right)\log\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)$$

On the other hand, recall that the von Mangoldt function Λ satisfies:

$$\Lambda(n)=\sum_{d|n}\mu(d)\log\frac{n}{d}$$

Thus, we have as well the following equality, again valid for any function F:

$$\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)=\sum_{n\leq x}F\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)\log\left(\frac{n}{d}\right)$$

Now by adding the two equalities that we have, we obtain the following formula:

$$\begin{aligned}
&F(x)\log x+\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)\\
=\;&\sum_{n\leq x}F\left(\frac{x}{n}\right)\log\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)+\sum_{n\leq x}F\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)\log\left(\frac{n}{d}\right)\\
=\;&\sum_{n\leq x}F\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)\left[\log\left(\frac{x}{n}\right)+\log\left(\frac{n}{d}\right)\right]\\
=\;&\sum_{n\leq x}F\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)\log\left(\frac{x}{d}\right)\\
=\;&\sum_{n\leq x}\sum_{d|n}F\left(\frac{x}{n}\right)\mu(d)\log\left(\frac{x}{d}\right)\\
=\;&\sum_{d|x}\mu(d)\log\left(\frac{x}{d}\right)\sum_{m\leq x/d}F\left(\frac{x}{md}\right)
\end{aligned}$$

On the other hand, according to our definition of G, we have:

$$G\left(\frac{x}{d}\right)=\log\left(\frac{x}{d}\right)\sum_{m\leq x/d}F\left(\frac{x}{md}\right)$$

Thus, in terms of G, the formula established above reads:

$$F(x)\log x+\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)=\sum_{d|x}\mu(d)G\left(\frac{x}{d}\right)$$

But this is exactly the formula in the statement, as desired. □

Next, we have the following key formula, due to Selberg:

THEOREM 16.16. *We have the following formula,*

$$\psi(x)\log x+\sum_{n\leq x}\Lambda(n)\psi\left(\frac{x}{n}\right)=2x\log x+O(x)$$

valid for any $x>0$.

PROOF. This is something quite tricky, the idea being as follows:

(1) Let us first apply the formula in Proposition 16.15 to the function $F_1(x)=\psi(x)$. The function G_1 constructed there is subject to the following estimate:

$$\begin{aligned}G_1(x)&=\log x\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\\&=x\log^2x-x\log x+O(\log^2x)\end{aligned}$$

As for the formula obtained by applying Proposition 16.15, this is as follows:

$$\psi(x)\log x+\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\Lambda(n)=\sum_{d\leq x}\mu(d)G_1\left(\frac{x}{d}\right)$$

(2) Now let us first apply as well the formula in Proposition 16.15 to the function $F_2(x)=x-\gamma-1$. The function G_2 there is subject to the following estimate:

$$\begin{aligned}G_2(x)&=\log x\sum_{n\leq x}\left(\frac{x}{n}-\gamma-1\right)\\&=x\log^2x-x\log x+O(\log x)\end{aligned}$$

As for the formula obtained by applying Proposition 16.15, this is as follows:

$$(x-\gamma-1)\log x+\sum_{n\leq x}\left(\frac{x}{n}-\gamma-1\right)\Lambda(n)=\sum_{d\leq x}\mu(d)G_2\left(\frac{x}{d}\right)$$

(3) Let us compare now what we have in (1,2). As a first observation, we have:

$$G_1(x)-G_2(x)=O(\log^2x)$$

Next, let us substract the formulae coming from Proposition 16.15. By using the above estimate for the difference G_1-G_2, we obtain the following estimate:

$$\begin{aligned}&\psi(x)\log x+\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\Lambda(n)-(x-\gamma-1)\log x-\sum_{n\leq x}\left(\frac{x}{n}-\gamma-1\right)\Lambda(n)\\&=\sum_{d\leq x}\mu(d)G_1\left(\frac{x}{d}\right)-\sum_{d\leq x}\mu(d)G_2\left(\frac{x}{d}\right)\\&=\sum_{d\leq x}\mu(d)\left[G_1\left(\frac{x}{d}\right)-G_2\left(\frac{x}{d}\right)\right]\\&=O(x)\end{aligned}$$

(4) And with this, we are almost done. Indeed, rearranging the terms gives:

$$\begin{aligned}
& \psi(x)\log x+\sum_{n\leq x}\Lambda(n)\psi\left(\frac{x}{n}\right)\\
= \ & (x-\gamma-1)\log x+\sum_{n\leq x}\left(\frac{x}{n}-\gamma-1\right)\Lambda(n)+O(x)\\
= \ & 2x\log x+O(x)
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Getting now to the Prime Number theorem, this can be now proved as follows:

THEOREM 16.17. *The Selberg formula established above, namely*

$$\psi(x)\log x+\sum_{n\leq x}\Lambda(n)\psi\left(\frac{x}{n}\right)=2x\log x+O(x)$$

can be used in order to prove the Prime Number theorem.

PROOF. This is something quite tricky, that we will not explain here in detail, the idea being as follows. Consider the following function:

$$\sigma(x)=e^{-x}\psi(e^x)-1$$

In terms of this function, the Selberg formula gives an estimate as follows:

$$|\sigma(x)|x^2\leq 2\int_0^x\int_0^y|\sigma(t)|dtdy+O(x)$$

Consider now the following quantity, with the function σ being as above:

$$C=\limsup_{x\to\infty}|\sigma(x)|$$

By reasoning by contradiction, the Selberg formula ultimately gives:

$$C=0$$

But this latter estimate proves the Prime Number theorem, as desired. So, this was for the idea, and in practice, exercise of course for you, to read more about this. □

So long for the Selberg proof of the Prime Number theorem. We should mention that there is as well a third known proof, due to Newman, which is more recent than Selberg's, also not using zeta, and a bit shorter. As usual, exercise for you, to read about this.

16d. Riemann hypothesis

Time to end this book, and with no book on number theory avoiding the Riemann hypothesis, we will talk of course about this, the Riemann hypothesis.

So, let us go back to the classical proof of the Prime Number Theorem, by Hadamard and de la Vallée Poussin, explained before in this chapter. This proof was crucially based on the following fact, which technically was our Theorem 16.7 above:

FACT 16.18. *The Riemann zeta function, analytically continuating*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

has no zeroes on the lines $Re(s) = 0$ *and* $Re(s) = 1$.

The point now is that this can be regarded as a particular case of the Riemann hypothesis, stating that the zeroes of zeta in the strip $0 \leq Re(s) \leq 1$ must satisfy:

$$Re(s) = \frac{1}{2}$$

So, this is the famous Riemann hypothesis, and with the above Fact 16.18 standing as a first piece of motivation for it, and with the proof of the Prime Number theorem using it, discussed earlier in this chapter, explaining the relation with the prime numbers.

In practice now, many things can be said about the Riemann hypothesis, and it is beyond our scope here, at the end of this book, which was meant to be an introduction to arithmetic in general, and to the zeta function in particular, to get into this.

* * *

However, one thing that we can do, now that we know what the main problem is, is to review the material from chapter 15, or perhaps from this whole book, with this idea in mind, explaining in the simplest possible words what the main problem is.

At the beginning of everything, assuming a bit of calculus and complex numbers known, as per the level of chapter 9 in this book, we have the following fact:

THEOREM 16.19. *We can talk about the Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at any $s \in \mathbb{C}$ *with* $Re(z) > 1$.

PROOF. We have indeed the following computation, with $s = r + it$ and $r > 1$:

$$\begin{aligned}
|\zeta(s)| &\leq \sum_{n=1}^{\infty} \frac{1}{|n^s|} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^r} \\
&< 1 + \int_1^{\infty} \frac{1}{x^r}\, dx \\
&= 1 + \frac{1}{r-1}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Next, we have the following result, which was something more tricky:

THEOREM 16.20. *We have the following formula,*

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

which can stand as definition for ζ, in the strip $0 < Re(s) < 1$.

PROOF. We can define the Dirichlet function η as being the signed version of ζ, in the following way, and with the convergence for $Re(s) > 0$ being elementary:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

Now observe that ζ and η are connected at $Re(s) > 1$, in the following way:

$$\begin{aligned}
\zeta(s) + \eta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\
&= 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^s} \\
&= 2^{1-s} \zeta(s)
\end{aligned}$$

But this gives the following formula, which is the one in the statement:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\, \eta(s)$$

Thus, basically done, and in order to finish, as to be fully complete, we can invoke a bit of complex analysis, and we are led to the conclusion in the statement. □

And with this, good news, skipping many other things that can be said about zeta, that we learned the hard way in chapter 15, and skipping as well Fact 16.8, that we learned in this chapter, we can formulate the main problem in arithmetic, as follows:

CONJECTURE 16.21 (Riemann hypothesis). *The zeroes of zeta in the critical strip*

$$0 < Re(s) < 1$$

can only appear on the critical line, $Re(s) = 1/2$.

As a first comment, feel of course free to believe that this is correct, or not.

To be more precise, barring any bizarre phenomena from logic, this conjecture being wrong would more or less amount in having a certain complex number z in the critical strip, saying to us "here I am, the mighty $z \in \mathbb{C}$, all mathematics being about me".

And can this be plausible, or not? Not very clear, but speaking "monsters" in mathematics, there are actually quite a few of them, especially in representation theory.

In practice now, many things can be said, about the Riemann hypothesis, for all tastes, and for more on this, there are many good books available. Enjoy.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter. But of course, if looking for some difficult number theory exercises, there are many of them. So, have a look at the various books referenced below, read some of them as to get to research level, on a topic that you like, and then start solving problems.

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