THE NFW DM PROFILE IS COMPATIBLE WITH THE DECAY LAW OF VELOCITY WITH -0.25 AS POWER OF RADIUS –V3

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Abstract

The purpose of this study is to demonstrate that the Navarro–Frenk–White (NFW) dark matter (DM) density profile is consistent with the velocity decay law characterized by a radial dependence of $r^{-0.25}$ within the halo regions of the Milky Way (MW) and Andromeda (M31) galaxies.

The velocity law, expressed as $v=a\cdot r^{-0.25}$, was derived from a novel framework known as the **DM by Quantum Gravitation** theory (hereafter referred to as the **DMbQG** theory). This theory proposes that dark matter (DM) arises from the propagation of the gravitational field through an as-yet-unknown quantum gravitational phenomenon.

Since the NFW profile has been extensively validated over the past three decades across thousands of galaxies, it is crucial to examine whether a novel framework such as DMbQG theory produces a total mass function that is effectively indistinguishable from NFW predictions. At the core of the DMbQG framework lies the direct mass formula, $M_{TOTAL}(r) = \frac{a^2 \cdot \sqrt{r}}{G}$

which yields the total enclosed mass within a sphere of radius r. As discussed in Chapter 8 of Abarca [1], the total mass depends on the parameter a^2, where **a** is the proportionality constant in the velocity decay law $v = a\cdot r^{-0.25}$.

These two expressions, which share the same constant **a**, provide dual approaches for its determination: (i) directly from the galactic rotation curve in the halo region, and (ii) indirectly from the virial mass and radius derived via the NFW model and using the direct mass to calculate the parameter **a**. In practice, the total galactic virial mass can be estimated by adding the baryonic mass—primarily confined to the bulge and disk—to the virial DM mass obtained through the NFW prescription.

Consequently, if the parameter **a** derived from the rotation curve coincides with the value of **a** obtained using the direct mass formula with the total mass inferred from the NFW formalism, the central thesis of this work is validated.

The structure of this paper is as follows: In Chapter 2, the parameter **a** is calculated for the MW using rotation curve datasets from two independent authors. Chapter 3 introduces the NFW formalism and its principal expressions. Chapter 4 presents the derivation of a^2 from the direct mass formula using the virial mass obtained through the NFW method. In Chapter 5, the parameter **a** obtained through both approaches is compared for the MW using two independent datasets. Chapter 6 mirrors this analysis for the M31 galaxy, again employing two distinct datasets. Chapter 7 study the compatibility of NFW method with the decay law velocity in two galaxies outside the Local Group. Across these four galaxies, inside and outside the Local Group, the empirical results provide strong support for the central thesis of this paper.

1. Introduction

Since 2014 up to 2025, I have published several papers studying DM in galactic halos, especially in M31 and Milky Way although also I have published some papers studying other galaxies and clusters. Through those works a new and original theory, called Dark matter by quantum gravitation theory, hereafter DMbQG, has been developed.

The reader must have at least a general knowledge about this original theory to understand this paper. The paper [1] Abarca (2024), is the best work about the DMbQG theory, so the reader may consult such paper when is cited in this work. In [1] Abarca, there are some tests of validation of the theory using results published for MW, for M31 and for the Local Group. By other side, in the paper [2] Abarca (2024) the theory was extended to cluster of galaxies and its theoretical findings are tested using results published of the Virgo cluster. In few words, the DMbQG theory is well developed and tested into the two previously cited papers.

In addition, in the paper [3] Abarca (2025) also it is demonstrated the equivalence of the direct mass function and the total mass using the NFW DM function mass plus the baryonic mass, but that study is more complex than the present work.

The novelty of this paper is that it is focused on the comparison of the parameter **a**, obtained by two different methods, and one of them involves the NFW DM mass formula.

The purpose of this paper is to demonstrate that the NFW dark matter density profile is consistent with the velocity decay law characterized by a radial dependence of $r^{-0.25}$ within the halo regions of the MW and M31 galaxies and consequently the NFW DM mass function plus the baryonic function is equivalent to the direct mass formula, which was derived within the framework of the DMbQG theory.

Given that the NFW profile has been extensively validated over the past three decades across thousands of galaxies, it is essential to verify whether a novel approach such as DMbQG yields a total mass function effectively indistinguishable from the NFW predictions.

The DMbQG theory, developed in [1] Abarca (2024), postulates that dark matter arises from an as-yet unidentified quantum gravitational phenomenon induced by the gravitational field generated by the baryonic component of a galaxy. A distinctive feature of this hypothesis, in contrast to the NFW paradigm, is the assumption that the DM halo is unbounded rather than confined.

At the core of the DMbQG framework lies the *direct mass formula*: $M_{TOTAL}(r) = \frac{a^2 \cdot \sqrt{r}}{G}$

which yields the total enclosed mass within a sphere of radius r, been the radius unbounded. As discussed in Chapter 8 of [1] Abarca, the mass depends on the parameter a, where **a** is the proportionality constant in the velocity decay law $v=a \cdot r^{-0.25}$.

These two relations, sharing the same constant **a**, enable dual pathways to its determination: on the one hand, from the galactic rotation curve in the halo region; on the other, from the virial mass and radius conventionally derived through the NFW model and using the direct mass to calculate the parameter **a**. In practice, the total galactic mass may be estimated by adding the baryonic mass—primarily confined to the bulge and disk—to the virial DM mass obtained from the NFW prescription.

Finally, it is important to note that in Abarca (2025) [3], a comparison is made between the direct mass function and the NFW total mass function across a wide region of the galactic halo for both the MW and M31 galaxies. Specifically, Table 16 presents the comparison for the MW, while Table 23 shows the results for M31 and both tables are plotted to see graphically the data sets. In both cases, the relative difference remains quite small, staying below 15% throughout the halo region up to the virial radius.

2. Proportionality constant associated with the velocity decay law in the DMbQG theory

In the framework of DMbQG theory, into the halo region, the velocity decays according to $v = a \cdot r^{-0.25}$, being the parameter **a** constant, whose value may be calculated by a couple of data: $a = v \cdot r^{0.25}$.

It is simple to check that the dynamical mass $M_{DYN} = V^2 \cdot Radius / G$ is mathematically equivalent to direct mass, $M_{DIRECT} = M_{TOTAL}(r) = \frac{a^2 \cdot \sqrt{r}}{G}$ if it is assumed the velocity law $v = a \cdot r^{-0.25}$. The *direct mass formula* represents the total mass enclosed by the sphere with a specific radius, according the DMbQG theory.

The formula to calculate the parameter **a** comes from [1] Abarca (2024), see epigraph 8.7 where is explained that according the DMbQG theory, into the halo, the velocity decays according the formula $V = a \cdot r^{-0.25}$.

As the rotation curve data have non negligible errors, it is calculated the parameter **a** for the whole dataset and afterwards it is made its average value. In the epigraph 10.1 of the paper [1] it is explained why the halo begins at 30 kpc for the MW Galaxy.

As the total mass depend on the parameter $\mathbf{a^2}$, this value is highlighted even more than parameter **a**, because with the direct mass formula it is known the distribution of total mass into the whole halo region.

2.1 Calculation of parameter a using the [4] Huang et al. (2016) Milky Way data

Using the procedure explained before, it is got in the table 1 the parameter **a**, which is a 3 % lower than the one got by Sofue. See table 2.

Table 1 parameter a using the Huang (2016) data for the MW				
Velocity	Radius	Radius	Velocity	Parameter **a**
km/s	kpc	m	m/s	$m^{1.25}$ /s
211,2	31,29	9,65516E+20	211200	3,7229E+10
217,93	33,73	1,04081E+21	217930	3,9143E+10
219,33	36,19	1,11671E+21	219330	4,0094E+10
213,31	38,73	1,19509E+21	213310	3,9661E+10
200,05	41,25	1,27285E+21	200050	3,7786E+10
190,15	43,93	1,35555E+21	190150	3,6486E+10
198,95	46,43	1,43269E+21	198950	3,8706E+10
192,91	48,71	1,50304E+21	192910	3,7984E+10
198,9	51,56	1,59099E+21	198900	3,9724E+10
185,88	57,03	1,75977E+21	185880	3,8071E+10
173,89	62,55	1,93011E+21	173890	3,6448E+10
196,36	69,47	2,14364E+21	196360	4,2251E+10
175,05	79,27	2,44603E+21	175050	3,8929E+10
			Total sum	5,0251E+11
			Average **a**	3,8655E+10
			Average $\mathbf{a^2}$	1,4942E+21

Notice that the maximum of relative difference between the average $\mathbf{a^2}$ and the biggest or the lowest value of a^2 in the table 1 is about 19 %.

The data of rotation curve, See [4] Huang, page 2633, table 3, have an average error of the velocities about 10%. The author does not give error of radius, so supposing that the radius error is zero then $d(a^2)/a^2 = 2 \cdot dv/v$ i.e. the relative error of a^2 is the twice that the one of the velocities. Therefore $d(a^2)/a^2$ is about 20%. As the radius error has to be non zero the relative error of a^2 has to be bigger than 20%.

Consequently all these data are compatibles with the parameter **a** as a constant value associated with the galactic halo of MW.

2.2 Calculation of parameter a using the [5] Sofue (2020) Milky Way data

The rotation curve data (radius and velocity) have been taken from table 6, page 25 of [5] Sofue (2020).

Table 2 parameter a using the Sofue (2020) data for the MW				
radius	velocity	radius	velocity	parameter **a**
kpc	km/s	m	m/s	$m^{1.25}/s$
30,448	229,60	9,40E+20	229600	4,01976E+10
33,493	222,50	1,03E+21	222500	3,98939E+10
36,842	215,00	1,14E+21	215000	3,94787E+10
40,527	207,10	1,25E+21	207100	3,89453E+10
44,579	200,30	1,38E+21	200300	3,85746E+10
49,037	194,70	1,51E+21	194700	3,84004E+10
53,941	189,80	1,66E+21	189800	3,83367E+10
59,335	186,20	1,83E+21	186200	3,85164E+10
65,268	184,70	2,01E+21	184700	3,91273E+10
71,795	183,90	2,22E+21	183900	3,98973E+10
78,975	181,40	2,44E+21	181400	4,03040E+10
86,872	175,50	2,68E+21	175500	3,99333E+10
95,560	167,70	2,95E+21	167700	3,90787E+10
			Total sum	5,10684E+11
			Average a	3,92834E+10
			Average a^2	1,543186E+21

Notice that the maximum of the relative difference regarding the average a^2 is about 5 %.

The data of rotation curve, See [5] Sofue, page 15, have an average error of velocities about 6%. The author does not give error of radius, so by the reason explained in the previous epigraph the relative error of a^2 is bigger than 12%.

The average a^2 got using the Sofue data is a 3 % bigger than the one got by Huang. See table 1. Therefore if it is considered the experimental errors both values of parameter a^2 may be considered as equivalents.

3. The NFW profile for the Dark Matter

In this chapter it will be explained the most usual DM profile, that was developed by Navarro, Frenk and White, hereafter NFW DM profile. This DM profile has been used widely by the scientific community to study the DM in thousands of galaxies during last 30 years. Previously to study this DM profile, will be explained two concepts usually connected with him: M_{200} and R_{200}, also known as virial mass and radius.

3.1 M_{200} and R_{200} or the virial data in the NFW method

In the framework of NFW method, R_{200} is the radius of a sphere whose mean density of DM is 200 times bigger than the critic density of Universe

$$\rho_C = \frac{3H^2}{8\pi G} = 9.2055 \cdot 10^{-27}\, kgm^{-3} \text{ and } 200\rho_C = 1.8411 \cdot 10^{-24}\, kgm^{-3} \quad (3.1)$$

(In this work it will be considered H = 70 km/s/Mpc)

and M_{200} is the DM mass enclosed by the sphere with the radius R_{200}.

Considering the spherical volume formula, it is simple to get the following relations between both concepts.

$$R_{200}^3 = \frac{G \cdot M_{200}}{100 \cdot H^2} \qquad (3.2)$$

$$\text{or } M_{200} = \frac{100 H^2 R_{200}^3}{G} \qquad (3.3)$$

$$\text{or } \frac{M_{200}}{R_{200}^3} = \frac{100 H^2}{G} \qquad (3.4)$$

3.2 The DM mass formula into the NFW method

The NFW profile for the DM density is defined by two parameters: the scale radius and the characteristic density.

It is supposed a spherical symmetry and its formula it is $\rho(r) = \frac{\rho_0}{x \cdot (1+x)^2}$ (3.5)

being ρ_0 a characteristic density, x= r/R_0 a dimensionless magnitude related with radius by R_0, which is called the scale radius.

By integration it is obtained the formula for the Dark matter enclosed by a sphere with radius r.

$$M_{DM}(< r) = K_{NFW} \cdot f(x) \qquad (3.6)$$

being $K_{NFW} = 4\pi\rho_0 R_0^3$, called the NFW characteristic mass (3.7)

and $f(x) = Ln(1 + x) - x/(1 + x)$ (3.8)

where x = r / R_0 , being Ln the natural logarithm.

Two important concepts for NFW profiles are M_{200} and R_{200} both referred to DM only.

So $M_{200\text{-}DM} = M_{DM} (<R_{200}) = K_{NFW} \cdot f(c)$ (3.9)

where c = R_{200} / R_0 (3.10)

is called the concentration parameter and $R_{200} = R_0 \cdot c$

3.3 Calculation of concentration parameter c

As $\frac{M_{200}}{R_{200}^3} = \frac{100 H^2}{G}$ see formula (3.4) then $\frac{M_{200}}{c^3 R_0^3 \cdot} = \frac{100 H^2}{G}$ and $\frac{4 \cdot \pi \rho_0 \cdot R_0^3 \cdot f(c)}{c^3 R_0^3 \cdot} = \frac{100 H^2}{G}$ so

$\frac{c^3}{f(c)} = \frac{4\pi G \rho_0}{100 \cdot H^2}$ That it is the equation for the concentration parameter (3.11)

This equation is quite easy to solve numerically, and it is clear that c depend on the characteristic density only.

With this parameter c, it is easily calculated $M_{200\text{-}DM}$ and R_{200}.

To illustrate this method, in the table 3 are shown the NFW parameters published by [5] Sofue (2020).

Table 3. The NFW parameters for M.W. according Sofue (2020)	
Characteristic density ρ_0	Scale radius R_0
$\mathbf{0,787 \pm 0.037 GeVcm^{-3} = (1.403 \pm 0.066) \cdot 10^{-21}\, kg \cdot m^{-3}}$	$\mathbf{10.94 \pm 1.05\, kpc}$

Using the characteristic density it is simple to get the equation

$c^3 / f(c) = 2286.125$ that gives the value c= 16.348, and f(c) = 1.91

So $R_{200} = R_0 \cdot c = 178.85$ kpc

Using (3.7) $K_{NFW} = 3.4 \cdot 10^{11} M_\Theta$ then using (3.9) $M_{200-DM} = 6.498 \cdot 10^{11} M_\Theta$

It is easy to check that the density of the sphere with the previous data M_{200-DM} and R_{200} is virtually equal to $200\rho_C$.

4. Calculation of parameter a² using the direct mass formula

In the chapter 8, of paper [1] Abarca, M. 2024 was demonstrated that the direct mass formula

$$M_{TOTAL}(<r) = \frac{a^2 \cdot \sqrt{r}}{G} \qquad (4.1)$$

is the most appropriate method to estimate the total mass (baryonic plus dark matter) enclosed within a sphere of a given radius extending into the galactic halo. The halo corresponds to the region where the density of baryonic matter becomes negligible compared to the dark matter density. For example, the halo of the Milky Way extends beyond 30 kpc, while that of M31 extends beyond 40 kpc (see Abarca, M., 2024, Chapters 5 and 10).

For example $M_{TOTAL}(< R_{200}) = \dfrac{a^2 \cdot \sqrt{R_{200}}}{G}$ gives the total mass enclosed by the sphere with radius R_{200}. Notice that M_{TOTAL} (< R_{200}) is bigger than M_{200}. Namely M_{TOTAL} (<R_{200}) = M_{200} + M_{BA}. . Where M_{BA} represents the baryonic mass of the galaxy. As it is well known, virtually all of the baryonic mass is contained in the bulge and the galactic disc. Hereafter M_{TOTAL} (<R_{200}) will be renamed as $M_{200-TOTAL}$

4.1. The parameter a² formula depending on the virial radius and the virial mass.

Since the direct mass formula has only one parameter, knowing the total mass associated with a specific radius is enough to calculate the parameter a^2.

So from this formula $M_{TOTAL}(< R_{200}) = \dfrac{a^2 \cdot \sqrt{R_{200}}}{G}$ it is got

$$a^2 = \frac{G \cdot M_{200-TOTAL}}{\sqrt{R_{200}}} \qquad (4.2)$$

In the following epigraphs the parameter a^2 will be calculated using the data provided by three authors.

4.1.1 Calculation of parameter a² using the Karukes virial data

For example, in the table 4, are shown the virial data for MW published by [6] E. Karukes,(2020) into the table 6, page 25 of his paper.

The first column shows the DM enclosed into the sphere with radius R_{200} and the second one shows the total mass (DM plus baryonic).

Using the value $M_{200-TOTAL}$ and R_{200} into the formula (4.2) it is got $a^2 = 1.54 \cdot 10^{21}\ m^{5/2}/s^2$, that it is virtually equal to the value got for this parameter in the chapter 2, using the rotation curve data method with the data provided by Huang or Sofue.

By other side, using M_{200-DM} and R_{200} it is right to check that the average density is virtually $200\rho_C$.

Table 4 The virial data for M.W. according Karukes (2020)			Calculus of parameter a^2 by the virial data method
$M_{200\text{-}DM}$	$M_{200\text{-}TOTAL}$	R_{200}	
$M_\odot$	$M_\odot$	kpc	$m^{5/2}/s^2$
$\mathbf{8.26^{+1.2}_{-0.8} \cdot 10^{11}}$	$\mathbf{8.95^{+1}_{-0.8} \cdot 10^{11}}$	$\mathbf{193^{+9}_{-6}}$	$1.54 \cdot 10^{21}$
$M_{BA} = M_{200\text{-}TOTAL} - M_{200\text{-}DM} = 7 \cdot 10^{10}\ M_\odot$			

In Table 5, the parameter $\mathbf{a^2}$ is calculated using a different set of data pairs. The only value to be excluded is that corresponding to 28 kpc, as it deviates significantly from the remaining data points. This discrepancy can be attributed to the fact that the galactic halo is considered to begin at approximately 30 kpc. At radii smaller than 30 kpc, the contribution of baryonic matter is non-negligible, and therefore the direct mass estimation formula is not applicable.

The direct mass formula is so simple because its domain is the halo region, where the baryonic matter density is negligible compared to the dark matter density.

As it is shown in the table 5, into the halo region the parameter a^2 is virtually constant.

Table 5 Total Mass and its radius according [6] Karukes		
Radius	M_{TOTAL}	Parameter a^2
(kpc)	(x 10^{11}) $M_\odot$	x10^{21} $m^{2.5}/s^2$
28.33	$3.04^{+0.10\,(0.19)}_{-0.08\,(0.17)}$	1.3653
45.79	$4.27^{+0.22\,(0.43)}_{-0.19\,(0.37)}$	1.5085
74.0	$5.68^{+0.40\,(0.83)}_{-0.37\,(0.65)}$	1.5784
119.57	$7.26^{+0.66\,(1.40)}_{-0.58\,(1.03)}$	1.5872
193.24	$8.95^{+0.98\,(2.07)}_{-0.84\,(1.48)}$	1.54

4.1.2 Calculation of parameter a^2 using the Sofue virial data

In the table 6 are collected the NFW parameters of MW from the paper [5] Sofue, table 3, page 12. The baryonic mass may be calculated from the figure 3 (for the mass of bulge) and the table 3 for the mass of disc. Notice how different is the baryonic mass regarding the value calculated by Karukes. See table 4.

Table 6. NFW parameters for M.W. according Sofue (2020) Baryonic mass of MW $M_{BA} = 1.8 \cdot 10^{11}\ M_\odot$	
Characteristic density $\mathbf{D_o}$	Scale radius $\boldsymbol{R_0}$
$\mathbf{1.40 \cdot 10^{-21}\ kg \cdot m^{-3}}$	10.94 kpc

With the same data of table 6, in the epigraph 3.3 was calculated the virial mass $M_{200\text{-DM}} = 6.5 \cdot 10^{11}\ M_\Theta$ and R_{200} = 178.85 kpc. So adding the baryonic mass it is got $M_{200\text{-TOTAL}} = M_{200\text{-DM}} + M_{BA} = 8.3 \cdot 10^{11}\ M_\Theta$ and by the formula (4.2) it is calculated easily the parameter $a^2 = 1.484 \cdot 10^{21}\ m^{2.5}/s^2$. In the table 7 are summarized the results.

Table 7. Parameter a^2 using the Sofue virial data		
$M_{200\text{-TOTAL}}$	R_{200}	Parameter a^2
$8.3 \cdot 10^{11}\ M_\Theta$	178.85 kpc	$1.484 \cdot 10^{21}\ m^{2.5}/s^2$

4.1.3 Calculation of parameter a² using the Huang virial data

In the table 8 are the NFW parameters provided by this author and the baryonic mass of the MW. These data have been got from table 4 at the page 2636 from [4] Huang.

Table 8. The NFW parameters for M.W. according Huang (2016)	
Characteristic density $\mathbf{D_0}$	Scale radius $\boldsymbol{R_0}$
$\mathbf{8.196 \cdot 10^{-22}\ kg \cdot m^{-3}}$	14.39 kpc
Baryonic mass of MW $M_{BA} = 8.1 \cdot 10^{10}\ M_\Theta$	

The formula for the concentration factor is $\frac{c^3}{f(c)} = \frac{4\pi G \rho_0}{100 \cdot H^2} = 1335.50$ whose solution is c = 13.2046 and f(c) =1.7240

Therefore $R_{200} = 13.2046 \cdot 14.39 = 190.014$ kpc and $M_{200} = K_{NFW} \cdot f(c) = 7.81 \cdot 10^{11}\ M_\Theta$ where K_{NFW} is the characteristic mass of NFW profile, $K_{NFW} = 4.531 \cdot 10^{11}\ M_\Theta$.

As M_{200} is the DM enclosed by the sphere with radius R_{200}, to calculate the total mass it is added the baryonic mass, so $M_{200\text{-TOTAL}} = (7.81+0.81) \cdot 10^{11}\ M_\Theta = 8.62 \cdot 10^{11}\ M_\Theta$

Finally it is possible to calculate the parameter a^2 using $M_{200\text{-TOTAL}}$ and R_{200} getting by the formula (4.2)

$a^2 = 1.4949 \cdot 10^{21}\ m^{2.5}/s^2$ that it is incredibly close to the one got in the table 1, $a^2 = 1.4942 \cdot 10^{21}\ m^{2.5}/s^2$.

In the table 9 are summarized the results.

Table 9 Parameter a^2 using the virial data provided by Huang et al.			
M_{200}	$M_{200\text{-TOTAL}}$	R_{200}	Parameter a^2
$7.81 \cdot 10^{11}\ M_\Theta$	$8.62 \cdot 10^{11}\ M_\Theta$	190.014 kpc	$1.4949 \cdot 10^{21}\ m^{2.5}/s^2$.

In the paper [4] the authors give a different virial data because they have considered an over density = 95 instead 200. In the table 10 are shown the virial data provided by the authors. See the table 4 at the page 2636.The procedure to calculate the parameter a^2 is the same.

Table 10 Parameter a^2 using the virial data provided by Huang et al.			
R_{VIR}	M_{VIR}	$M_{VIR\text{-TOTAL}}$	Parameter a^2
255.69 kpc	$9 \cdot 10^{11}$	$9.81 \cdot 10^{11}$	$1.467 \cdot 10^{21}\ m^{2.5}/s^2$.

This value of the parameter a^2 is only 1.8 % lower than the value one got by the Huang rotation curve data, see the table 1.

5. Comparison of parameter a^2 from rotation curve and virial data in the MW halo

This chapter represents the core of this paper. In Chapters 2 and 4, the parameter $\mathbf{a^2}$ has been calculated using two different methods. Specifically, in Chapter 2, $\mathbf{a^2}$ was obtained through the method directly related to the velocity decay law in the halo region, whereas in Chapter 4, $\mathbf{a^2}$ was derived using the virial data calculated from the NFW profile and by adding the baryonic mass to the virial dark matter mass.

In this chapter, both values will be compared, taking into account their procedural errors. It will be shown that the NFW method is consistent with the velocity decay law stated in the title of this paper, since the relative difference in $\mathbf{a^2}$ obtained by the two methods is clearly smaller than the specific error associated with each method.

A summary of the error analysis for each method is as follows:

- **First method:** In Section 2.1, it was shown that the relative error of $\mathbf{a^2}$ due to velocity uncertainties is about 20%. In Section 2.2, this error was reduced to about 12%, given that the velocity error is approximately 6%.
- **Second method:** In Section 4.1.1, Table 4 shows that the relative error of the virial total mass is about 11%, while that of the virial radius is about 4.7%. Using these values, and applying formula (4.2), the relative error of $\mathbf{a^2}$ is about 13%.
  The formula used for the relative error is: $d(a^2)a^2= dm/m + 1/2 \cdot dm/m$

Table 11 summarizes the different values of $\mathbf{a^2}$ obtained in Chapters 2 and 4. The third column shows the relative differences between the two methods: using Huang's data, the relative difference is 1.8%, while with Sofue's data, it is about 3.8%.

Table 11 Parameter a^2 by rotation curve (R.C.) versus a^2 by the virial data - a^2 units $m^{2.5}/s^2$		
Parameter a^2 by rotation curve	Parameter a^2 by virial data	Relative difference %
Huang's et al. data		
$1.4942 \cdot 10^{21}$ – see table 1	$1.4949 \cdot 10^{21}$ –see table 9	Virtually zero
$1.4942 \cdot 10^{21}$ – see table 1	$1.467 \cdot 10^{21}$ – see table 10	1.8 %
Sofue's data		
$1.5432 \cdot 10^{21}$ – see table 2	$1.484 \cdot 10^{21}$ –see table 7	3.8 %
Karukes's et al. data		
Rotation curve not published	$1.54 \cdot 10^{21}$ –see table 4	

In conclusion, the relative difference in $\mathbf{a}^2$ obtained by the two methods is much smaller than the characteristic errors associated with each method. Therefore, it can be stated that the NFW dark matter profile is compatible with the velocity decay law $v = a \cdot r^{-0.25}$ within the halo of the Milky Way.

6. Comparison of parameter a² from rotation curve and virial data in the M31 halo

To illustrate that the NFW DM profile is equivalent with the velocity decay law $v = a \cdot r^{-0.25}$ into the halo region, for the M31 galaxy as well, it will be used the papers [7] Sofue (2015) and [8] Zhang et al. (2024) where are published two rotation curves of M31.

6.1 The M31 Sofue data (2015)

6.1.1 Calculation of parameter a² using the rotation curve data

In the chapter 2 of paper [1] it is introduced the rotation curve data published by [7] Sofue, Y. (2015) and in the chapter 8 of paper [1] it is calculated the parameter **a** using such rotation curve. Namely $\mathbf{a}^2 = 2.235 \cdot 10^{21}\ m^{5/2}/s^2$ is the value obtained by the rotation curve data of M31.

Unfortunately, the author does not provide the numerical data for the rotation curve, and the associated errors cannot be reliably inferred from the graph alone. As a result, it is not possible to conduct an error analysis for this method when calculating the parameter a^2.

6.1.2 Virial data and calculation of parameter a² by the direct mass formula

In the table 2 of paper [7] it is introduced the NFW parameters shown in the table 12.

Table 12. The NFW parameters for M31 according Sofue (2015)	
Characteristic density $\mathbf{D_o}$	Scale radius $\boldsymbol{R_0}$
$(\mathbf{1.51 \pm 0.15}) \cdot \mathbf{10^{-22}\ kg \cdot m^{-3}}$	34.6 ±2.1 kpc
Baryonic mass of M31 $M_{BA} = 1.6 \cdot 10^{11}\ M_\Theta$	

The formula for the concentration factor is $\dfrac{c^3}{f(c)} = \dfrac{4\pi G \rho_0}{100 \cdot H^2} = 246.048$ whose solution is c = 6.579 and f(c) =1.1573

Therefore $R_{200} = 6.579 \cdot 34.6 = 227.63$ kpc and $M_{200} = K_{NFW} \cdot f(c) = 1.342 \cdot 10^{12}\ M_\Theta$ where K_{NFW} is the characteristic mass of the NFW DM profile, $K_{NFW} = 1.16 \cdot 10^{12}\ M_\Theta$

As M_{200} is the DM enclosed by the sphere with radius R_{200}, to calculate the total mass it is added the baryonic mass, so $M_{200\text{-TOTAL}} = (1.34+0.16) \cdot 10^{12} M_\Theta = 1.50 \cdot 10^{12}\ M_\Theta$

Finally it is possible to calculate the parameter a^2 using $M_{200\text{-TOTAL}}$ and R_{200} getting the value $a^2 = 2.377 \cdot 10^{21}\ m^{2.5}/s^2$ that it is only 6 % bigger than the value one got by the Sofue rotation curve data, whose value is $a^2 = 2.235 \cdot 10^{21}\ m^{5/2}/s^2$.

In this case, the error analysis is more complex because the total mass depends on the parameter c, which is obtained by solving a transcendental equation involving c^3. The characteristic density D_0 carries an uncertainty of approximately 10%. Consequently, the total mass must have an error significantly larger than 10%, which in turn propagates to the parameter a^2. Despite this, the relative difference in a^2 between the two methods is only 6%. Therefore, it can be concluded that both methods are consistent, i.e., the NFW dark matter profile is compatible with a velocity decay law in the halo region of the form $v=a\cdot r^{-0.25}$, mirroring what was found in the chapter 5 for the MW galaxy.

6.2 Zhang et al. data (2024)

These authors provide both the rotation curve data and the virial data, each with known uncertainties. The error analysis in this case is somewhat lengthy and is presented in the following sections.

6.2.1 Calculation of parameter a² using the rotation curve data

On page 9 of Zhang [8], the rotation curve data are tabulated. In Chapter 5 of Abarca [1], it is shown that for galactocentric radii exceeding 40 kpc, the baryonic mass density becomes negligible in comparison to the dark matter (DM) density. Consequently, the parameter a^2 is evaluated starting at 40 kpc. Nevertheless, the velocity measurements at 46 kpc and 52 kpc reported by Zhang [8] are anomalously low and have therefore been excluded from the analysis.

In Table 13, the remaining rotation curve data are presented. Their corresponding a^2 values are calculated, along with the average a^2.

Table 13 Calculus of a^2 using the R.C. of M31. Data Zhang et al.				
Radius	Velocity	Radius	Velocity	Param. a^2
kpc	km/s	m	m/s	$m^{2.5}/s^2$
54,85	196,28	1,6925E+21	196280,00	1,5850E+21
67,26	202,02	2,0754E+21	202020,00	1,8593E+21
98,74	192,59	3,0468E+21	192590,00	2,0473E+21
123,56	168,53	3,8127E+21	168530,00	1,7538E+21
			Total sum	7,2453E+21
			Average a^2	1,8113E+21

The average a^2 is a 19% lower than the one calculated by the Sofue rotation curve data, (epigraph 6.1.1).

6.2.1.1 Error analysis of a² based on rotation curve data errors

From the formula of parameter $a^2 = v^2\cdot r^{0.5}$, it is easy to calculate its relative error $d(a^2)/a^2 = 2dv/v + (1/2)\cdot dr/r$ where dv/v is the relative error of the velocity and dr/r is the relative error of radius.

In the table 14 are shown the rotation curve data and its errors provided by Zhang.

Table 14. Rotation curve clipped. [8] Zhang et al.			
Radius	Rad. error	Velocity	V. error
Kpc	kpc	km/s	km/s

52.08	1.63	182.05	38.15
54.85	1.67	196.28	38.81
67.26	14.95	202.02	40.67
98.74	17.19	192.59	42.23
123.56	11.6	168.53	41.34

The table 15 presents the rotation curve data along with their associated errors, expressed in SI units. The table also includes the relative error of parameter a^2 for each data point, with an average value of 43%. Considering this relative error, the value of a^2 obtained by Zhang (1.8×10^{21}) is reasonably consistent with that reported by Sofue (2.2×10^{21}), since their relative difference is only 20%.

Table 15 R.C. data errors into S.I. units and relative error of a^2				
Radius	Rad. Error	Velocity	Vel. Error	$d(a^2)/a^2$
m	m	m/s	m/s	%
1,693E+21	5,15312E+19	196280	38810	39,560771
2,075E+21	4,61312E+20	202020	40670	40,374476
3,047E+21	5,30432E+20	192590	42230	43,941868
3,813E+21	3,57941E+20	168530	41340	49,106455
			Total sum %	172,98357
			Average %	43,24589

6.2.2 Calculation of parameter a² using the direct mass formula

In the graphic of page 8 of paper [8] Zhang, it is plotted the rotation curves associated to the disc and the bulge mass, so by a simple calculus it is got the baryonic mass of M31, $M_{BA} = 10.5 \cdot 10^{10} M_{\Theta}$. In addition, in the page 11 are given the virial data associated to DM purely, see the table 16 below.

Table 16. Virial data and baryonic mass of M31 according Zhang et al. (2024)		
M_{200}	**R_{200}**	**Baryonic mass**
$M_{vir} = 1.14^{+0.51}_{-0.35} \times 10^{12} M_{\odot}$	$r_{vir} = 220 \pm 25$ kpc.	$M_{BA} = 10.5 \cdot 10^{10} M_{\Theta}$

As it is shown in the virial data, table 16, the radius error is about 11%, the up error of mass is 45% and the low error one is 31%.

Using such data it is got $M_{200\text{-}TOTAL} = 1.245 \cdot 10^{12} M_{\Theta}$ and as in the previous epigraphs, the parameter a^2 it is calculated directly, $a^2 = 2.006 \cdot 10^{21}$ $m^{2.5}/s^2$. This value is only a 10% bigger regarding the value got by the rotation curve method $a^2 = 1.8 \cdot 10^{21}$ $m^{2.5}/s^2$ (see epigraph 6.2.1).

6.2.2.1 Error analysis of a² based on the virial data errors

As it was shown in the chapter 5, the relative error of the parameter a^2 got by the formula (4.2) is $d(a^2)/a^2 = dm/m + (1/2) \cdot dr/(r)$ and considering the relative error of the virial data (see table 16) is obtained a relative error of the parameter a^2 shown in the table 17.

Table 17 Relative error of parameter a^2		
Relative error M_{VIR}	Relative Error R_{VIR}	Relative Error of parameter a^2
Up error 45% Low error 31%	11%	Up error 50% Low error 36 %

6.2.3 Equivalence of a² obtained by the two different methods using the Zhang's data

In the table 18 are summarized the results obtained in the two previous epigraphs.

Table 18 The Parameter a^2 got by the two different methods and its relative errors		
Parameter a^2 by Rotation curve	$a^2 = 1.8 \cdot 10^{21}$	Relative error 43%
Parameter a^2 by Virial data	$a^2 = 2.006 \cdot 10^{21}$	Relative low error 36%

If we consider the relative difference in parameter a^2 between the two results—approximately 10%—alongside its own relative uncertainties (around 40%), it is reasonable to conclude that the two results are equivalent.

Once again, the values of a^2 obtained through the two different methods reinforce the central thesis of this work: the NFW dark matter profile is consistent with the velocity decay law $v = a \cdot r^{-0.25}$ in the halo region of M31, mirroring the conclusion obtained in Chapter 5 for the Milky Way galaxy.

7. Galaxies beyond the Local Group

7.1 Selection of the galaxies

The principal difficulty in obtaining adequate rotation curves for these galaxies arises from their large distances. As a consequence, the velocity measurements are affected by larger uncertainties, and the radial extent over which the rotation curves are reliably measured is more limited.

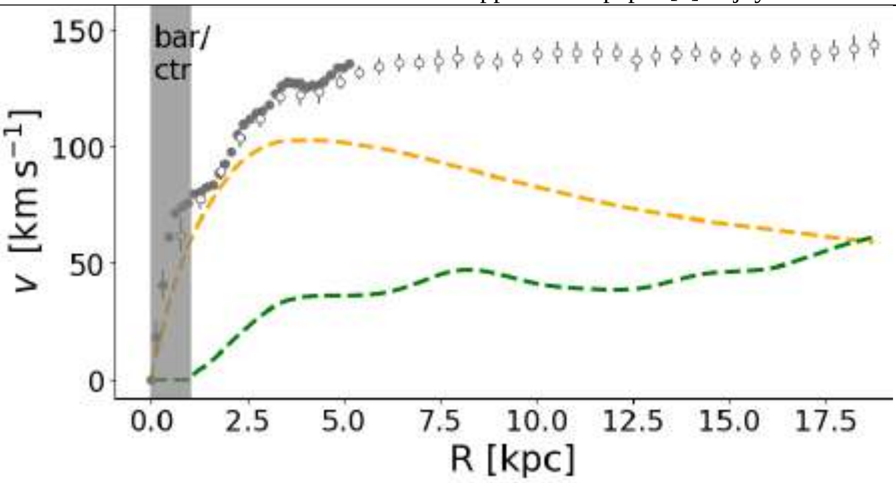
The DMbQG framework requires only a single parameter, a, to define the direct mass formula. Therefore, knowledge of a single point on the rotation curve is, in principle, sufficient to determine the value of a. However, this point must be located at a sufficiently large radius, where most of the baryonic mass is enclosed within the corresponding sphere.

In the work of Vijayakumar et al. [9], three different rotation curves are presented, in which it is possible to distinguish between the contribution of the stellar disc and that associated with the neutral hydrogen (H I) gas. When the gas rotation curve exhibits a declining behavior similar to that of the stellar disc, it can be inferred that the majority of the galactic baryonic mass is enclosed within the outer radius.

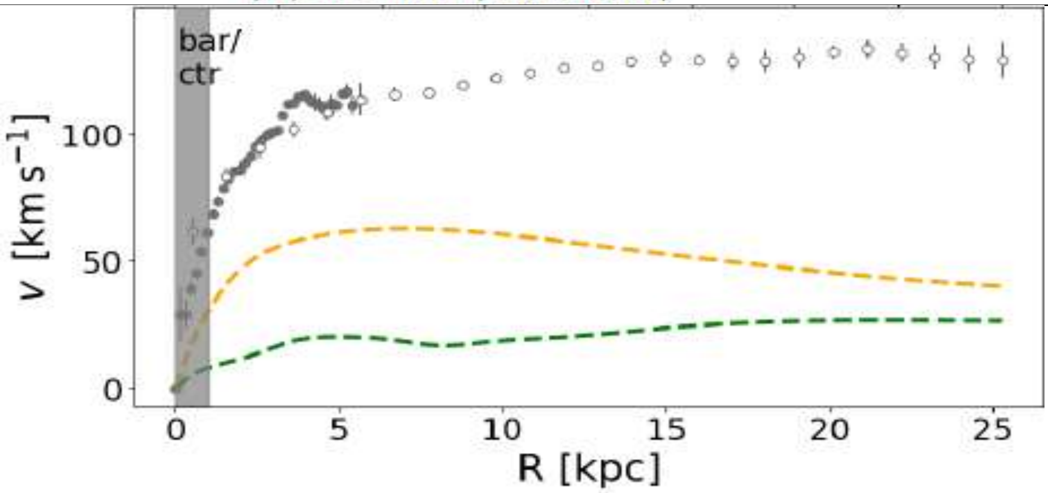
The Table 19 shows three different rotation curves, each graphic have three curves, the total velocity traced by white dots, the velocity associated to the stellar disc traced by the yellow line and the velocity associated to the gas HI, traced by the green line.

It is clear that NGC 3521 is the only galaxy whose gas rotation curve displays a “Keplerian-like” decline at the largest radii. Consequently, it is the only rotation curve from which data can be reliably extracted to determine the parameter a within the framework of the DMbQG theory because it is assumed that the majority of the baryonic mass is enclosed within the outer radius.

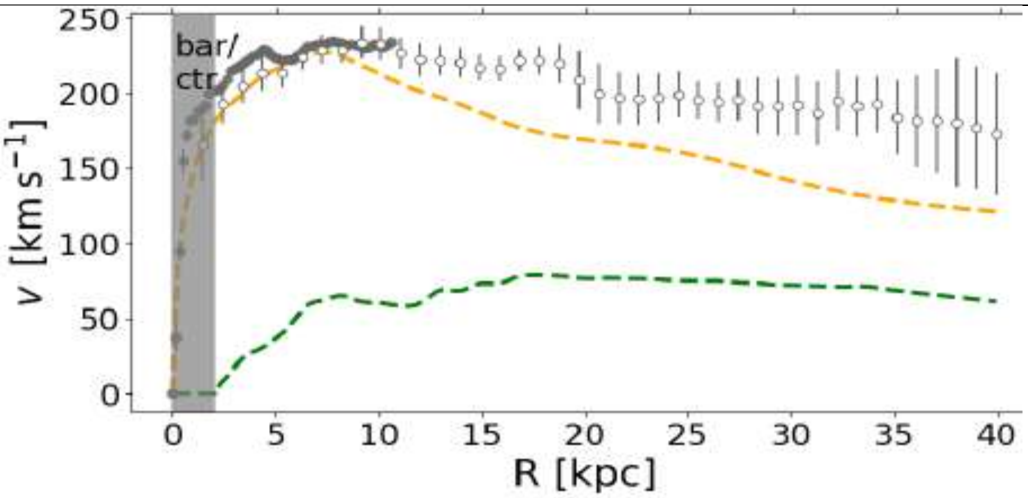
Table 19 Three different rotation curves clipped from paper [9] Vijayakumar V.



(a1) NGC 3621 (original data)



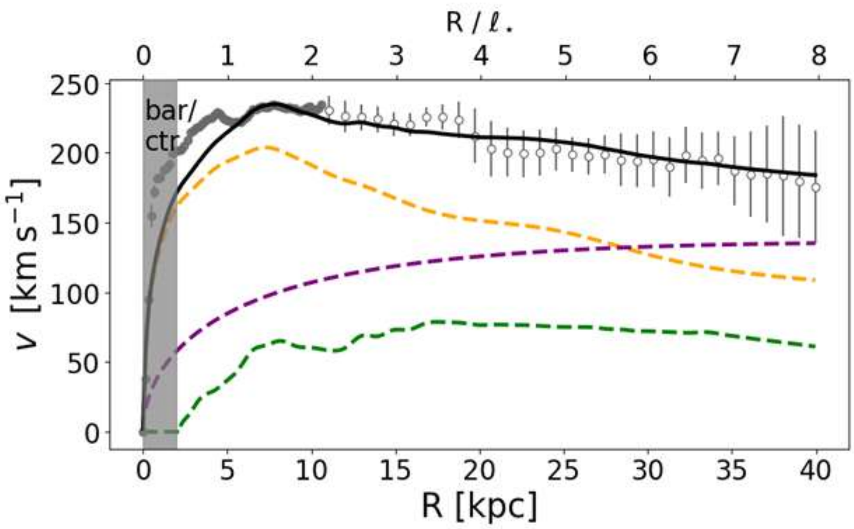
(c1) NGC 3137 (original data)



(b1) NGC 3521 (original data)

7.1 NGC 3521 Galaxy

In this section, the NFW method is compared with the Direct Mass approach. It is ultimately concluded that both methods are equivalent when the range of measurement uncertainties associated with the rotation curve point at 40 kpc is taken into account.

Table 20 Galaxy NGC 3521 data
The data and graphic of NGC 3521 comes from [9] Vijayakumar V.
Distance 13.2 Mpc Stellar mass $M_* = 10^{11}$ M_O
 (b2) NGC 3521 (scaled data and joint rotation curve fit)
The upper graphic show the merged rotation curves and the joint stellar–gas–dark matter fitting results, with the stellar component rescaled according to the maximal disk assumption.
NFW parameters $D_0 = 10^{-2.5}$ $M_\odot/pc^3$ = $2.14\cdot10^{-22}$ kg/m^3 $R_s = 23$ kpc

Baryonic mass calculus

In the table 1 of paper [9] the authors inform about the stellar mass of NGC 3521 $M_* = 10^{11}$ $M_\odot$
As the green line in the graphic represents the rotation curve associated to the gas and at the 40 kpc its slope is similar to the keplerian rotation curve associated to the stars, the yellow line, then it is possible to estimate the total mass of the gas by the dynamical mass formula.
At 40 kpc the estimated velocity of the gas is 61.7 km/s so the $M_{GAS} = 3.5\cdot10^{10}$ $M_\odot$ and consequently the estimated total baryonic mass for NGC 3521 is $M_{BA} = 1.35\cdot10^{11}$ $M_\odot$

Virial data calculus by the NFW method

By the formula $\frac{c^3}{f(c)} = \frac{4\pi G\rho_0}{100\cdot H^2}$ =349 , being f (c) = ln(1+c) – c /(1+c) it is possible to calculate the concentration parameter c.

So the equation $c^3 / f(c) = 349$ that solved numerically gives the solution c = 7.626 and consequently $R_{200} = c\cdot R_S$ = 175.4 kpc ; $V_{200} = 10\cdot H\cdot R_{200}$ = 122.8 km/s and $M_{200} = \frac{V_{200}^2 \cdot R_{200}}{G} = 6.1\cdot10^{11} M_\odot$ .Adding the baryonic mass it is obtained the value $M_{200\text{-TOTAL}} = 7.45\cdot10^{11}$ $M_\odot$ enclosed within the virial radius R_{200} = 175.4 kpc

Calculus of the total mass by the parameter a

By the graph it is estimated the velocity of rotation curve for the total mass at 40 kpc equal to 168.9 km/s and by the formula $a^2 = R^{0.5} \cdot V^2$ it is obtained the value $a^2 = 1 \cdot 10^{21}\ m^{5/2}/s^2$ in the framework of DMbQG theory.

Through the direct mass, formula (4.1) it is calculated the total mass associated to the radius $R_{200} = 175.4$ kpc obtaining the value $M_{200\text{-TOTAL}} = 5.54 \cdot 10^{11}\ M_{\odot}$

The relative difference regarding the total mass estimated by the NFW method is 25% .

This relative difference between the virial total masses is compatible with the thesis of this work because this galaxy is quite far away and the measures of rotation curve have high errors. The reader can check how big the error bars are in the graphic, especially the ones associated to radius bigger than 35 kpc.

Therefore one more time this result backs the thesis of this work i.e. the NFW method is compatible with a decaying velocity into the halo region according the law $v = a \cdot v^{-0.25}$, which is the central hypothesis of DMbQG theory.

Relation formula between the NFW parameters and the parameter a^2 in NGC 3521

In the epigraph (7.3) of the paper [3] Abarca,M, it is shown that $a^2 \approx 2\pi G \rho_0 \cdot R_0^{5/2}$ is a very good relation between the parameters of the NFW method and the parameter a^2 for the MW and M31 galaxies.

In this epigraph it will be shown that this relation is quite close for the galaxy NGC 3521

Through the NFW parameters $\rho_{0=} 2.14 \cdot 10^{-22} kg/m^3$ and $R_0 = 23$ kpc it is calculated the expression $2 \cdot \pi \cdot G \cdot \rho_0 \cdot R_0^{2.5} = 1.2 \cdot 10^{21} m^{2.5} \cdot s^{-2}$ which is quite close to the parameter $a^2 = 1 \cdot 10^{21}\ m^{5/2}/s^2$ obtained in the previous paragraph. Namely the relative difference is below 17 %.

7.2 NGC 3621 Galaxy

In the paper of Sorgho et al. [10] there are published two rotation curves where it is discriminated the rotation curves associated to the stellar disc and the gas. It has been selected NGC 3621 because its distance is only 6.6 Mpc whereas the other one is 13.6 Mpc far away. Obviously, the more near the more trustable the measures are.

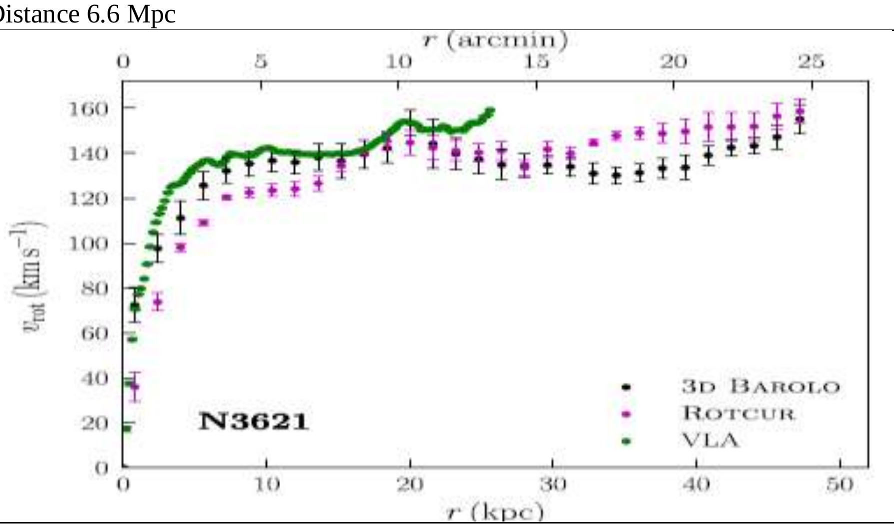
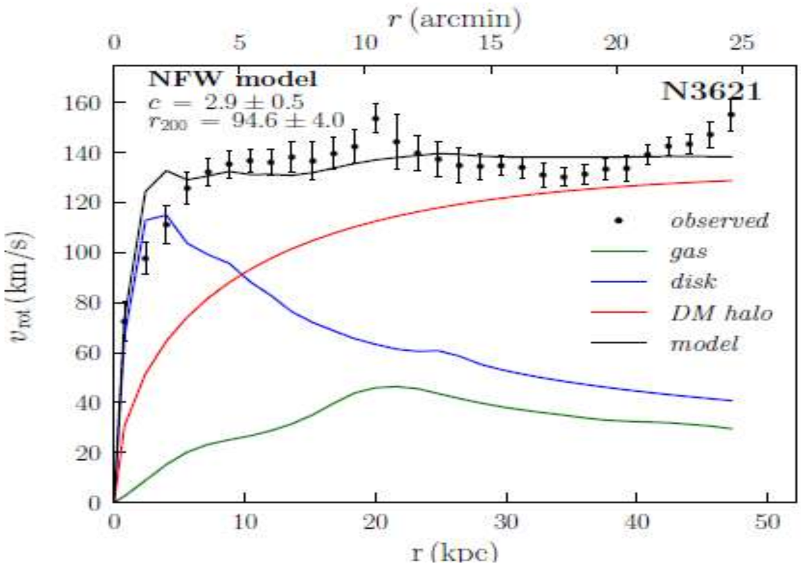
Table 21 Galaxy NGC 3621 data
The data and graphic of this galaxy comes from [10]Sorgho et al.
Distance 6.6 Mpc

Rotation curves of NGC 3621 using three different techniques

Table 21 Galaxy NGC 3621 data

Rotation curves of NGC 3621 associated to the different galactic components



NFW parameters for NGC 3621 galaxy

$\Upsilon_* = 0.5$ $R_{200} = 162.7 \pm 4.0$ kpc c= 5.3±0.5 $\chi^2_{RED} = 2.4$

Baryonic mass calculus

As the green line in the graphic represents the rotation curve associated to the gas and at the 47 kpc its slope is similar to the keplerian rotation curve associated to the stars disk, the blue line, then it is possible to estimate the total mass of the gas by the dynamical mass formula.

At 47 kpc the estimated velocity of the gas is 30.8 km/s so the $M_{GAS} = 1\cdot10^{10}\ M_\odot$

At 47 kpc the estimated velocity of the stars curve is 40 km/s so the $M_{STARS} = M_* = 1.75\cdot10^{10}\ M_\odot$

and consequently the estimated total baryonic mass for NGC 3621 is $M_{BA} = 2.75\cdot10^{10}\ M_\odot$

Virial data calculus by the NFW method

As $R_{200} = 162.7$ kpc and $V_{200} = 10 \cdot H\cdot R_{200} = 113.9$ km/s then $M_{200} = 4.9\cdot10^{11}\ M_\odot$. Adding the baryonic mass it is obtained the value $M_{200\text{-}TOTAL} = 5.17\cdot10^{11}\ M_\odot$ enclosed within the virial radius R_{200} = 162.7 kpc

Calculus of the total mass by the parameter a

As at the maximum radius, 47 kpc, the rotation curves of stars and the gas are keplerian, then may be considered that the majority of baryonic mass is enclosed within the radius 47 kpc.

By the above graph it is estimated the velocity of rotation curve at 47 kpc equal to 154 km/s and by the formula $a^2 = R^{0.5}\cdot V^2$ it is obtained the value $a^2 = 9\cdot10^{20}\ m^{5/2}/s^2$ in the framework of DMbQG theory.

Through the direct mass, formula (4.1) it is calculated the total mass associated to the radius $R_{200} = 162.7$ kpc obtaining the value $M_{200\text{-}TOTAL} = 4.8\cdot10^{11}\ M_O$

The relative difference regarding the total mass estimated by the NFW method is about 7%, which is a perfect matching if it is considered the experimental errors.

Therefore one more time this result backs the thesis of this work i.e. the NFW method is compatible with a decaying velocity into the halo region according the law v =a·v $^{-0.25}$, which is the central hypothesis of DMbQG theory.

Relation formula between the NFW parameters and the parameter a in NGC 3621

In the epigraph (7.3) of the paper [3] Abarca,M, it is shown that $a^2 \approx 2\pi G\rho_0 \cdot R_0^{5/2}$ is a very good relation between the parameters of NFW method and the parameter a^2 for the MW and M31 galaxies.

In this epigraph it will be shown that this relation is quite close for the galaxy NGC 3621 as well.

Through the data R_{200} = 162.7 kpc and c= 5.3 it was calculated R_0 = 30.7 kpc and $M_{200} = 4.9 \cdot 10^{11}$ $M_\odot$ and using these data, through the formula of virial mass $M_{200\text{-DM}} = M_{DM}\left(< R_{200}\right) = K_{NFW} \cdot f\left(c\right)$ being $K_{NFW} = 4\pi\rho_0 R_0^3$ and f(c) = 0.99928 it is possible to calculate $\rho_0 = 9.13 \cdot 10^{-23} kg/m^3$

So the expression $2 \cdot \pi \cdot G \cdot \rho_0 \cdot R_0^{2.5} = 1.057 \cdot 10^{21} m^{2.5} \cdot s^{-2}$ and this value is quite close to the parameter $a^2 = 9 \cdot 10^{20}$ $m^{5/2}/s^2$ obtained in the previous paragraph. Namely the relative difference is below 15 %.

In conclusion, it can be stated that for these galaxies far from the Local Group, the virial mass calculated using the NFW method is consistent with a power-law decay, with an exponent of -0.25, for the rotation curve associated with the total mass in the halo region of these galaxies.

According to DMbQG theory, dark matter causes the velocity decay with an exponent of -0.25, which is a result of a quantum gravitational phenomenon. Once again, these galaxies serve as a validation test for this new theory.

8. Concluding remarks

In this study, the parameter **a** is determined through a double-calculation procedure, employing published datasets for the MW provided by two independent authors, and two distinct datasets for M31 contributed by two additional authors. Across these four independent comparisons, the empirical results strongly support the central thesis of this work.

For the Milky Way, the agreement is particularly remarkable. The relative discrepancy in parameter **a** amounts to only 1.8% when compared with the results of Huang et al. (2016) [4], and 3.8% relative to Sofue (2020) [5], despite the intrinsic uncertainty in the calculation of a^2 being on the order of 20% or greater.

Similarly, the M31 analysis exhibits very good consistency. The relative deviation in a^2 is 6% when benchmarked against the Sofue dataset and 10% with respect to the Zhang dataset. These discrepancies are at least three times smaller than the intrinsic uncertainty associated with the determination of a^2. Further details on the error analysis are provided in Chapters 5 and 6.

As the NFW profile is a reliable method validated in thousands of galaxies, establishing the equivalence between the total mass obtained from the NFW dark matter mass formula plus the baryonic contribution and the direct mass, derived from the double-method test, is crucial for validating the DMbQG theory. This is particularly relevant because the Milky Way (MW) and Andromeda (M31) are the only galaxies for which the rotation curve can be measured across an extended region of their halos.

Despite the difficulty of obtaining reliable and accurate rotation curves for galaxies located far beyond the Local Group, Chapter 7 presents two examples that constitute successful tests of the DMbQG theory. Specifically, these examples correspond to the galaxies NGC 3521 and NGC 3621, located at distances of 13.2 Mpc and 6.6 Mpc, respectively.

Once again, the DMbQG theory has successfully passed a new tests, complementing the results previously reported in [1] Abarca, M. (2024) , [2] Abarca, M. (2024) and [3] Abarca,M.(2025) . The remaining challenge is to ensure broader dissemination of the theory within the scientific community, thereby enabling its evaluation in other galaxies and clusters.

BIBLIOGRAPHY

[1]Abarca, M. (2024) A Dark Matter Theory by Quantum Gravitation for Galaxies and Clusters. *Journal of High Energy Physics*, *Gravitation and Cosmology*, **10**, 1749-1784. https://doi.org/10.4236/jhepgc.2024.104100

[2]Abarca, M. (2024) Solving the Conundrum of Dark Matter and Dark Energy in Galaxy Clusters. *Journal of High Energy Physics*, *Gravitation and Cosmology*, **10**, 1785-1805. https://doi.org/10.4236/jhepgc.2024.104101

[3] Abarca,M.(2025) Equivalence between Direct Mass and NFW-Total Mass Formula in MW and M31 Galaxies. Journal of High Energy Physics, Gravitation and Cosmology. Vol.11 No.3, July 2025. https://doi.org/10.4236/jhepgc.2025.113073

[4] Huang,Y. et al. (2016) The Milky Way's rotation curve out to 100 kpc and its constraint on the Galactic mass distribution. MNRAS **463,** 2623–2639 (2016). https://doi:10.1093/mnras/stw2096

[5]Sofue, Y. (2020) Rotation Curve of the Milky Way and the Dark Matter Density. *Galaxies*, **8**, Article 37. https://doi.org/10.3390/galaxies8020037

[6] Karukes, E.V., Benito, M., Iocco, F., Trotta, R. and Geringer-Sameth, A. (2020) A Robust Estimate of the Milky Way Mass from Rotation Curve Data. Journal of Cosmology and Astroparticle Physics , 2020, Article 33. https://doi.org/10.1088/1475-7516/2020/05/033

[7] Sofue, Y. (2015) Dark Halos of M31 and the Milky Way. Publications of the Astronomical Society of Japan, 67, 75. https://doi.org/10.1093/pasj/psv042

[8] Zhang, X., Chen, B., Chen, P., Sun, J. and Tian, Z. (2024) The Rotation Curve and Mass Distribution of M31. Monthly Notices of the Royal Astronomical Society , 528, 2653-2666. https://doi.org/10.1093/mnras/stae025

[9] Vijayakumar V. et al.(2025) Modeling the Mass Distribution and Gravitational Potential of Nearby Disk Galaxies: Implications for the Interstellar Medium Dynamical Equilibrium. The Astrophysical Journal, Volume 989, Number 1**Citation** Vivek Vijayakumar *et al* 2025 *ApJ* **989** 66 **DOI** 10.3847/1538-4357/ade800

[10] Sorgho, A. Carignan,C. et al. (2019) Early observations of the MHONGOOSE galaxies: getting ready for MeerKAT. *Monthly Notices of the Royal Astronomical Society*, Volume 482, Issue 1, January 2019, Pages 1248–1269, https://doi.org/10.1093/mnras/sty2785