THE NFW DM PROFILE IS COMPATIBLE WITH THE DECAYING LAW OF VELOCITY WITH -0.25 AS POWER OF RADIUS -V2

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Abstract

The purpose of this study is to demonstrate that the Navarro–Frenk–White (NFW) dark matter (DM) density profile is consistent with the velocity decay law characterized by a radial dependence of $r^{-0.25}$ within the halo regions of the Milky Way (MW) and Andromeda (M31) galaxies. The velocity law, expressed as $v=a\cdot r^{-0.25}$, was derived within the framework of the DMbQG theory.

Since the NFW profile has been extensively validated over the past three decades across thousands of galaxies, it is crucial to examine whether a novel framework such as DMbQG theory produces a total mass function that is effectively indistinguishable from NFW predictions. At the core of the DMbQG framework lies the direct mass formula, $M_{TOTAL}(r) = \frac{a^2\cdot\sqrt{r}}{G}$

which yields the total enclosed mass within a sphere of radius r. As discussed in Chapter 8 of Abarca [1], the total mass depends on the parameter a^2, where **a** is the proportionality constant in the velocity decay law $v = a\cdot r^{-0.25}$.

These two expressions, which share the same constant **a**, provide dual approaches for its determination: (i) directly from the galactic rotation curve in the halo region, and (ii) indirectly from the virial mass and radius derived via the NFW model and using the direct mass to calculate the parameter **a**. In practice, the total galactic virial mass can be estimated by adding the baryonic mass—primarily confined to the bulge and disk—to the virial DM mass obtained through the NFW prescription.

Consequently, if the parameter a derived from the rotation curve coincides with the value of **a** obtained using the direct mass formula with the total mass inferred from the NFW formalism, the central thesis of this work is validated.

The structure of this paper is as follows: In Chapter 2, the parameter **a** is calculated for the MW using rotation curve datasets from two independent authors. Chapter 3 introduces the NFW formalism and its principal expressions. Chapter 4 presents the derivation of a^2 from the direct mass formula using the virial mass obtained through the NFW method. In Chapter 5, the parameter **a** obtained through both approaches is compared for the MW using two independent datasets. Chapter 6 mirrors this analysis for the M31 galaxy, again employing two distinct datasets. Across these four independent comparisons, the empirical results provide strong support for the central thesis of this study.

1. Introduction

Since 2014 up to 2025, I have published several papers studying DM in galactic halos, especially in M31 and Milky Way although also I have published some papers studying other galaxies and clusters. Through those works a new and original theory, called Dark matter by quantum gravitation theory, hereafter DMbQG, has been developed.

The reader must to have at least a general knowledge about this original theory to understand this paper. The paper [1] Abarca (2024), is the best work about the DMbQG theory, so the reader may consult such paper when is cited in this work. In [1] Abarca, there are some tests of validation of the theory using results published for MW, for M31 and for the Local Group. By other side, in the paper [2] Abarca (2024) the theory was extended to cluster of galaxies and its theoretical findings are tested using results published of the Virgo cluster. In few words, the DMbQG theory is well developed and tested into the two previously cited papers.

In addition, in the paper [3] Abarca (2025) also it is demonstrated the equivalence of the direct mass function and the total mass using the NFW DM function mass plus the baryonic mass, but that study is more complex than this work.

The novelty of this paper is that it is focused on the comparison of the parameter **a**, obtained by two different methods, and one of them involves the NFW DM mass formula.

The purpose of this paper is to demonstrate that the NFW dark matter density profile is consistent with the velocity decay law characterized by a radial dependence of $r^{-0.25}$ within the halo regions of the MW and M31 galaxies and consequently the NFW DM mass function plus the baryonic function is equivalent to the direct mass formula, which was derived within the framework of the DMbQG theory.

Given that the NFW profile has been extensively validated over the past three decades across thousands of galaxies, it is essential to verify whether a novel approach such as DMbQG yields a total mass function effectively indistinguishable from the NFW predictions.

The DMbQG theory, developed in [1] Abarca (2024), postulates that dark matter arises from an as-yet unidentified quantum gravitational phenomenon induced by the gravitational field generated by the baryonic component of a galaxy. A distinctive feature of this hypothesis, in contrast to the NFW paradigm, is the assumption that the DM halo is unbounded rather than confined.

At the core of the DMbQG framework lies the *direct mass formula*: $M_{TOTAL}(r) = \frac{a^2 \cdot \sqrt{r}}{G}$

which yields the total enclosed mass within a sphere of radius r, been the radius unbounded. As discussed in Chapter 8 of [1] Abarca, the mass depends on the parameter a, where **a** is the proportionality constant in the velocity decay law $v = a \cdot r^{-0.25}$.

These two relations, sharing the same constant **a**, enable dual pathways to its determination: on the one hand, from the galactic rotation curve in the halo region; on the other, from the virial mass and radius conventionally derived through the NFW model and using the direct mass to calculate the parameter **a**. In practice, the total galactic mass may be estimated by adding the baryonic mass—primarily confined to the bulge and disk—to the virial DM mass obtained from the NFW prescription.

2. Proportionality constant associated to the velocity decaying law in DMbQG theory

In the framework of DMbQG theory, into the halo region, the velocity decays according to $v = a \cdot r^{-0.25}$, being the parameter **a** constant, whose value may be calculated by a couple of data: $a = v \cdot r^{0.25}$**.**

It is right to check that the dynamical mass $M_{DYN} = V^2 \cdot \text{Radius} / G$ is mathematically equivalent to direct mass, $M_{DIRECT} = M_{TOTAL}(r) = \frac{a^2 \cdot \sqrt{r}}{G}$ if it is assumed the velocity law $v = a \cdot r^{-0.25}$. The *direct mass formula* represents the total mass enclosed by the sphere with a specific radius, according the DMbQG theory.

The formula to calculate parameter **a** comes from [1] Abarca (2024), see epigraph 8.7 where is explained that according the DMbQG theory, into the halo, the velocity decays according the formula $V = a \cdot r^{-0.25}$.

As the rotation curve data have non negligible errors, it is calculated the parameter **a** for the whole dataset and afterwards it is made its average value. In the epigraph 10.1 of the paper [1] it is explained why the halo begins at 30 kpc for the MW Galaxy.

As the total mass depend on the parameter $\mathbf{a}^2$, this value is highlighted even more than parameter **a**, because with the direct mass formula it is known the distribution of total mass into the whole halo region.

2.1 Calculus of parameter a using the [4] Huang et al. (2016) MW data

Using the procedure explained before, it is got in the table 1 the parameter a, which is a 3 % lower than the one got by Sofue. See table 2.

Table 1 parameter a using the Huang (2016) data for the MW				
Velocity	Radius	Radius	Velocity	Parameter a
km/s	kpc	m	m/s	$m^{1.25}$ /s
211,2	31,29	9,65516E+20	211200	3,7229E+10
217,93	33,73	1,04081E+21	217930	3,9143E+10
219,33	36,19	1,11671E+21	219330	4,0094E+10
213,31	38,73	1,19509E+21	213310	3,9661E+10
200,05	41,25	1,27285E+21	200050	3,7786E+10
190,15	43,93	1,35555E+21	190150	3,6486E+10
198,95	46,43	1,43269E+21	198950	3,8706E+10
192,91	48,71	1,50304E+21	192910	3,7984E+10
198,9	51,56	1,59099E+21	198900	3,9724E+10
185,88	57,03	1,75977E+21	185880	3,8071E+10
173,89	62,55	1,93011E+21	173890	3,6448E+10
196,36	69,47	2,14364E+21	196360	4,2251E+10
175,05	79,27	2,44603E+21	175050	3,8929E+10
			Total sum	5,0251E+11
			Average **a**	3,8655E+10
			Average $\mathbf{a}^2$	1,4942E+21

Notice that the maximum of relative difference between the average a^2 and the biggest or the lowest value is about 15 %.

The data of rotation curve, See [4] Huang, page 2633, table 3, have an average error of velocities about 10%. The author does not give error of radius, so supposing that the radius error is zero then $d(a^2)/a^2 = 2 \cdot dv/v$ i.e. the relative error of a^2 is the twice that the one of the velocities. Therefore $d(a^2)/a^2$ is about 20%.

2.2 Calculus of parameter a using the [5] Sofue (2020) MW data

The rotation curve data (radius and velocity) have been taken from table 6, page 25 of [5] Sofue (2020).

Table 2 parameter a using the Sofue (2020) data for the MW				
radius	velocity	radius	velocity	parameter a
kpc	km/s	m	m/s	$m^{1.25}$/s
30,448	229,60	9,40E+20	229600	4,01976E+10
33,493	222,50	1,03E+21	222500	3,98939E+10
36,842	215,00	1,14E+21	215000	3,94787E+10
40,527	207,10	1,25E+21	207100	3,89453E+10

44,579	200,30	1,38E+21	200300	3,85746E+10
49,037	194,70	1,51E+21	194700	3,84004E+10
53,941	189,80	1,66E+21	189800	3,83367E+10
59,335	186,20	1,83E+21	186200	3,85164E+10
65,268	184,70	2,01E+21	184700	3,91273E+10
71,795	183,90	2,22E+21	183900	3,98973E+10
78,975	181,40	2,44E+21	181400	4,03040E+10
86,872	175,50	2,68E+21	175500	3,99333E+10
95,560	167,70	2,95E+21	167700	3,90787E+10
			Total sum	5,10684E+11
			Average a	3,92834E+10
			Average a^2	1,543186E+21

Notice that the maximum of relative difference between the average a^2 and the biggest or the lowest value is about 9.7 %.

The data of rotation curve, See [5] Sofue, page 15, have an average error of velocities about 6%. The author does not give error of radius, so by the reason explained in the previous epigraph the relative error of a^2 is about 12%.

The average a^2 got using the Sofue data is a 3 % bigger than the one got by Huang. See table 1. Therefore if it is considered the experimental errors both values of parameter a^2 may be considered as equivalents.

3. The NFW profile for the Dark Matter mass density

In this chapter will be explained the most usual DM profile, that was developed by Navarro, Frenk and White, hereafter NFW DM profile. This DM profile has been used widely by the scientific community to study the DM in thousands of galaxies during last 30 years. Previously to study this DM profile, will be explained two concepts usually connected with him: M_{200} and R_{200}, also known as virial mass and radius.

3.1 M_{200} and R_{200} or the virial data in the NFW method

In the framework of NFW method, R_{200} is the radius of a sphere whose mean density of DM is 200 times bigger than the critic density of Universe

$$\rho_C = \frac{3H^2}{8\pi G} = 9.2055 \cdot 10^{-27}\, kgm^{-3} \text{ and } 200\rho_C = 1.8411 \cdot 10^{-24}\, kgm^{-3} \quad (3.1)$$

(In this work it will be considered H = 70 km/s/Mpc)

and M_{200} is the DM mass enclosed by the sphere with the radius R_{200}.

Considering the spherical volume formula, it is right to get the following relations between both concepts.

$$\boldsymbol{R_{200}^3} = \frac{G \cdot M_{200}}{100 \cdot H^2} \quad (3.2)$$

$$\text{or } \boldsymbol{M_{200}} = \frac{100 H^2 R_{200}^3}{G} \quad (3.3)$$

$$\text{or } \frac{M_{200}}{R_{200}^3} = \frac{100 H^2}{G} \quad (3.4)$$

3.2 The DM mass formula into the NFW method

The NFW profile for density is defined by two parameters, the scale radius and the characteristic density.

It is supposed a spherical symmetry and its formula is $\rho(r) = \frac{\rho_0}{x\cdot(1+x)^2}$ (3.5)

being ρ_0 a characteristic density, x= r/R_0 a dimensionless magnitude related with radius by R_0, which is called the scale radius.

By integration it is right to get the Dark matter enclosed by a sphere with radius r.

$M_{DM}(< r) = K_{NFW} \cdot f(x)$ (3.6)

being $K_{NFW} = 4\pi\rho_0 R_0^3$, called the NFW characteristic mass (3.7)

and $f(x) = Ln(1 + x) - x/(1 + x)$ (3.8)

where x = r / R_0 , being Ln the natural logarithm.

Two important concepts for NFW profiles are M_{200} and R_{200} both referred to DM only.

So $M_{200\text{-}DM} = M_{DM} (<R_{200}) = K_{NFW} \cdot f(c)$ (3.9)

where c = R_{200} / R_0 (3.10)

is called the concentration parameter and $R_{200} = R_0 \cdot c$

3.3 Calculus of concentration parameter

As $\frac{M_{200}}{R_{200}^3} = \frac{100H^2}{G}$ see formula (3.4) then $\frac{M_{200}}{c^3 R_0^3 \cdot} = \frac{100H^2}{G}$ and $\frac{4\cdot\pi\rho_0\cdot R_0^3\cdot f(c)}{c^3 R_0^3\cdot} = \frac{100H^2}{G}$ so

$\frac{c^3}{f(c)} = \frac{4\pi G\rho_0}{100\cdot H^2}$ that it is the equation for the concentration parameter (3.11)

This equation is quite easy to solve numerically, and it is clear that c depend on the characteristic density only.

With this parameter c, it is rightly calculated $M_{200\text{-}DM}$ and R_{200}.

To illustrate this method, in the table 3 are shown the NFW parameters published by [5] Sofue (2020) for MW.

Table 3. The NFW parameters for M.W. according Sofue (2020)	
Characteristic density $\boldsymbol{\rho_0}$	Scale radius $\boldsymbol{R_0}$
$\mathbf{0,787 \pm 0.037 GeV cm^{-3} = 1.403 \cdot 10^{-21}\, kg \cdot m^{-3}}$	$\mathbf{10.94 \pm 1.05\, kpc}$

Using the characteristic density it is right to get the equation

$c^3 / f(c) = 2286.125$ that gives the value c= 16.348, and f(c) = 1.91

So $R_{200} = R_0 \cdot c = 178.85$ kpc

Using (3.7) $K_{NFW} = 3.4\cdot 10^{11}\, M_\odot$ then using (3.9) $M_{200\text{-}DM} = 6.498\cdot 10^{11}\, M_\odot$

It is right to check that the density of the sphere with the previous data M_{200} and R_{200} is virtually equal to $200\rho_C$

4. Calculus of parameter a² using the direct mass formula

In chapter 8, of paper [1] Abarca, M. 2024 was demonstrated that the direct mass formula

$$M_{TOTAL}(<r) = \frac{a^2 \cdot \sqrt{r}}{G} \qquad (4.1)$$

is the most suitable formula to calculate the total mass (baryonic plus DM) enclosed by a sphere with a specific radius that ranges into the galactic halo.

The halo is the region where the density of baryonic matter is negligible versus the D.M. density. e.g. the halo for Milky Way have a radius bigger than 30 kpc, or the halo for M31 have a radius bigger than 40 kpc. See [1] Abarca, M.(2024) chapters 5 and 10.

For example $M_{TOTAL}(< R_{200}) = \frac{a^2 \cdot \sqrt{R_{200}}}{G}$ gives the total mass enclosed by the sphere with radius R_{200}. Notice that M_{TOTAL} (< R_{200}) is bigger than M_{200}. Namely M_{TOTAL} (<R_{200}) = M_{200} + M_{BA}. Where M_{BA} represents the baryonic mass of the galaxy, which is virtually enclosed into the bulge and the galactic disc. Hereafter M_{TOTAL} (<R_{200}) will be renamed as $M_{200\text{-}TOTAL}$

4.1. Parameter a² formula depending on virial radius and virial mass.

Due to the fact that the direct mass formula has one parameter only, is enough to know the total mass associated to a specific radius to be able to calculate the parameter a^2. So from this formula

$M_{TOTAL}(< R_{200}) = \frac{a^2 \cdot \sqrt{R_{200}}}{G}$ it is got

$$a^2 = \frac{G \cdot M_{200-TOTAL}}{\sqrt{R_{200}}} \qquad (4.2)$$

4.1.1 Calculus of parameter a² using the Karukes virial data

For example, in the table 4, are shown the virial data for MW published by [6] E. Karukes,(2020) into the table 6, page 25 of his paper.

The first column shows the DM enclosed into the sphere with radius R_{200} and the second one shows the total mass (DM plus baryonic).

Using the value $M_{200\text{-}TOTAL}$ and R_{200} into the formula (4.2) it is got $a^2 = 1.54 \cdot 10^{21}$ $m^{5/2}/s^2$, that it is virtually equal to the value got for this parameter in the chapter 2, using the data provided by Huang or Sofue and using a different method.

By other side, using $M_{200\text{-}DM}$ and R_{200} it is right to check that the average density is virtually $200\rho_C$.

Table 4 The virial data for M.W. according Karukes (2020)			Calculus of parameter a^2 by the virial data method
$M_{200\text{-}DM}$	$M_{200\text{-}TOTAL}$	R_{200}	
$M_\odot$	$M_\odot$	kpc	$m^{5/2}/s^2$
$\mathbf{8.26^{+1.2}_{-0.8} \cdot 10^{11}}$	$\mathbf{8.95^{+1}_{-0.8} \cdot 10^{11}}$	$\mathbf{193^{+9}_{-6}}$	$1.54 \cdot 10^{21}$

$M_{BA} = M_{200\text{-}TOTAL} - M_{200\text{-}DM} = 7{\cdot}10^{10}\ M_{\Theta}$

In addition in the table 5 it is calculated the parameter a^2 using another different data pairs. The only one to reject is the one associated to 28 kpc as it is too different to the others. This fact it is explained because the halo begins at 30 kpc. For lower radius than 30 kpc the baryonic density is not negligible and the direct mass formula does not work.

The direct mass is a formula so simple because its dominion is the halo region where the density of baryonic matter is negligible regarding the DM density.

As it is shown in the table 5, into the halo region the parameter a^2 is virtually constant.

Table 5 Total Mass and its radius according [6] Karukes		
Radius	M_{TOTAL}	Parameter a^2
(kpc)	$(\times 10^{11})\ M_{\Theta}$	$\times 10^{21}\ m^{2.5}/s^2$
28.33	$3.04^{+0.10\,(0.19)}_{-0.08\,(0.17)}$	1.3653
45.79	$4.27^{+0.22\,(0.43)}_{-0.19\,(0.37)}$	1.5085
74.0	$5.68^{+0.40\,(0.83)}_{-0.37\,(0.65)}$	1.5784
119.57	$7.26^{+0.66\,(1.40)}_{-0.58\,(1.03)}$	1.5872
193.24	$8.95^{+0.98\,(2.07)}_{-0.84\,(1.48)}$	1.54

4.1.2 Calculus of parameter a^2 using the Sofue virial data

In the table 6 are collected the NFW parameters of MW from the paper [5]Sofue, table 3, page 12. The baryonic mass may be calculated from the figure 3 (for the mass of bulge) and the table 3 for the mass of disc. Notice how different is the baryonic mass regarding the value calculated by Karukes. See table 4.

Table 6. NFW parameters for M.W. according Sofue (2020)	
Characteristic density $\mathbf{D_0}$	Scale radius $\boldsymbol{R_0}$
$\mathbf{1.40 \cdot 10^{-21}\ kg \cdot m^{-3}}$	10.94 kpc
Baryonic mass of MW $M_{BA} = 1.8{\cdot}10^{11}\ M_{\Theta}$	

In the epigraph 3.3 was calculated the virial mass $M_{200\text{-}DM} = 6.5{\cdot}10^{11}\ M_{\Theta}$ and $R_{200} = 178.85$ kpc . So adding the baryonic mass it is got $M_{200\text{-}TOTAL} = M_{200\text{-}DM} + M_{BA} = 8.3{\cdot}10^{11}\ M_{\Theta}$ and by the formula (4.2) it is calculated rightly the parameter $a^2 = 1.484{\cdot}10^{21}\ m^{2.5}/s^2$. In the table 7 are summarized the results.

Table 7. Parameter a^2 using the Sofue virial data		
$M_{200\text{-}TOTAL}$	R_{200}	Parameter a^2
$8.3{\cdot}10^{11}\ M_{\Theta}$	178.85 kpc	$1.484{\cdot}10^{21}\ m^{2.5}/s^2$

4.1.3 Calculus of parameter a^2 using the Huang virial data

In the table 8 are the NFW parameters provided by this author and the baryonic mass of the MW. These data have been got from table 4 at the page 2636 from [4] Huang.

Table 8. The NFW parameters for M.W. according Huang (2016)	
Characteristic density $\mathbf{D_0}$	Scale radius $\boldsymbol{R_0}$
$\mathbf{8.196 \cdot 10^{-22}\ kg \cdot m^{-3}}$	14.39 kpc
Baryonic mass of MW $M_{BA} = 8.1 \cdot 10^{10}\ M_\Theta$	

The formula for the concentration factor is $\frac{c^3}{f(c)} = \frac{4\pi G \rho_0}{100 \cdot H^2} = 1335.50$ whose solution is c = 13.2046 and f(c) =1.7240

Therefore $R_{200} = 13.2046 * 14.39 = 190.014$ kpc and $M_{200} = K_{NFW} \cdot f(c) = 7.81 \cdot 10^{11}\ M_\Theta$ where K_{NFW} is the characteristic mass of NFW profile, $K_{NFW} = 4.531 \cdot 10^{11}\ M_\Theta$.

As M_{200} is the DM enclosed by the sphere with radius R_{200}, to calculate the total mass it is added the baryonic mass, so $M_{200\text{-TOTAL}} = (7.81+0.81) \cdot 10^{11}\ M_\Theta = 8.62 \cdot 10^{11}\ M_\Theta$

Finally it is possible to calculate the parameter a^2 using $M_{200\text{-TOTAL}}$ and R_{200} getting by the formula (4.2)

$a^2 = 1.4949 \cdot 10^{21}\ m^{2.5}/s^2$ that it is incredibly close to the one got in the table 1, $a^2 = 1.4942 \cdot 10^{21}\ m^{2.5}/s^2$.

In the table 9 are summarized the results.

Table 9 Parameter a^2 using the virial data provided by Huang et al.			
M_{200}	$M_{200\text{-TOTAL}}$	R_{200}	Parameter a^2
$7.81 \cdot 10^{11}\ M_\Theta$	$8.62 \cdot 10^{11}\ M_\Theta$	190.014 kpc	$1.4949 \cdot 10^{21}\ m^{2.5}/s^2$.

In the paper [4] the authors give a different virial data because they have considered an over density = 95 instead 200. In the table 10 are shown the virial data provided by the authors. See the table 4 at the page 2636.The procedure to calculate the parameter a^2 is the same.

Table 10 Parameter a^2 using the virial data provided by Huang et al.			
R_{VIR}	M_{VIR}	$M_{VIR\text{-TOTAL}}$	Parameter a^2
255.69 kpc	$9 \cdot 10^{11}$	$9.81 \cdot 10^{11}$	$1.467 \cdot 10^{21}\ m^{2.5}/s^2$.

This value of parameter a^2 is only 1.8 % lower than the value one got by the Huang rotation curve data, see the table 1.

5. Parameter a^2 got by Rotation curve versus the one got by virial data in MW halo

This chapter is the core of this paper. In the previous chapters 2 and 4, the parameter a^2 has been calculated by two different methods. Namely, in the chapter 2, the parameter a^2 is got by the method rightly related with the velocity decaying law whereas in the chapter 4, the parameter a^2 is got using the virial data calculated by the NFW and adding the baryonic mass to the virial DM mass.

In this chapter it will be compared both values considering its procedural errors and concluding that the NFW method is compatible with the decaying law of velocity stated in the title of this paper because the relative difference of parameter a^2 got between the two different methods is clearly lower than the specific error associated to each method.

Now it will be summarized the analysis of error associated to each method:

Regarding the first method, in the epigraph 2.1 was shown that the relative error of parameter a^2 produced by the velocities errors is about 20% and in the epigraph 2.2 the same error is about 12% as the velocities error is about 6%.

Regarding the second method, in the epigraph 4.1.1 in the table 4 is shown that the relative error of virial total mass is about 11% and the one of the virial radius is about 4.7%. With those values, when it is calculated the relative error to a^2 through the formula (4.2) the result is about 13%. The formula used to calculate this result is $d(a^2)/a^2 = dm/m + (1/2)\cdot dr/(r)$

In the table 11 are summarized the different results of parameter a^2 got in the previous chapters 2 and 4. In the third column is shown the relative difference of parameter a^2 calculated by the two different methods. Namely the relative difference using the Huang data is 1.8 % and regarding the Sofue data the same difference is about 3.8 %

In conclusion the relative difference of parameter a^2 using the two different methods is much smaller than the characteristic errors associated to each method. Therefore it is possible to state that the NFW DM profile is compatible with the velocity decaying law $v = a\cdot r^{-0.25}$ into the halo of Milky Way.

Table 11 Parameter a^2 by rotation curve (R.C.) versus a^2 by the virial data - a^2 units $m^{2.5}/s^2$		
Parameter a^2 by R.C.	Parameter a^2 by virial data	Relative difference %
Huang et al. data		
$1.4942\cdot 10^{21}$ – see table 1	$1.4949\cdot 10^{21}$ –see table 9	Virtually zero
$1.4942\cdot 10^{21}$ – see table 1	$1.467\cdot 10^{21}$ – see table 10	1.8 %
Sofue data		
$1.5432\cdot 10^{21}$ – see table 2	$1.484\cdot 10^{21}$ –see table 7	3.8 %
Karukes et al. data		
R.C. not published	$1.54\cdot 10^{21}$ –see table 4	

6. Parameter a^2 got by Rotation curve versus the one got by virial data in M31 halo

To illustrate that the NFW DM profile is equivalent with the velocity decaying law v= $a \cdot r^{-0.25}$ for the M31 galaxy as well, it will be used the papers [7] Sofue (2015) and [8] Zhang et al. (2024) where are published two rotation curves of M31.

6.1 The M31 Sofue data (2016)

6.1.1 Calculus of parameter a^2 using the rotation curve data

In the chapter 2 of paper [1] it is introduced the rotation curve data published by Sofue and in the chapter 8 it is calculated the parameter **a** using such rotation curve. Namely the parameter $\mathbf{a^2} = 2.235 \cdot 10^{21}\ m^{5/2}/s^2$ is the one got by the rotation curve data of M31.

6.1.2 Virial data and calculus of parameter a^2 by the direct mass formula

In the table 2 of paper [7] it is introduced the NFW parameters shown in the table 12.

Table 12. The NFW parameters for M31 according Sofue (2015)	
Characteristic density $\mathbf{D_0}$	Scale radius $\boldsymbol{R_0}$
$\mathbf{1.51 \cdot 10^{-22}\ kg \cdot m^{-3}}$	34.6 kpc
Baryonic mass of M31 $M_{BA} = 1.6 \cdot 10^{11}\ M_\Theta$	

The formula for the concentration factor is $\frac{c^3}{f(c)} = \frac{4\pi G \rho_0}{100 \cdot H^2}$ = 246.048 whose solution is c = 6.579 and f(c) =1.1573

Therefore R_{200} = 6.579·34.6 = 227.63 kpc and $M_{200} = K_{NFW} \cdot f(c) = 1.342 \cdot 10^{12}\ M_\Theta$ where K_{NFW} is the characteristic mass of NFW profile, $K_{NFW} = 1.16 \cdot 10^{12}\ M_\Theta$

As M_{200} is the DM enclosed by the sphere with radius R_{200}, to calculate the total mass it is added the baryonic mass, so $M_{200\text{-TOTAL}} = (1.34+0.16) \cdot 10^{12} M_\Theta = 1.50 \cdot 10^{12}\ M_\Theta$

Finally it is possible to calculate the parameter a^2 using $M_{200\text{-TOTAL}}$ and R_{200} getting $a^2 = 2.377 \cdot 10^{21}\ m^{2.5}/s^2$ that it is 6 % bigger than the value one got by the Sofue rotation curve data, whose value is $a^2 = 2.235 \cdot 10^{21}\ m^{5/2}/s^2$.

In the following epigraph, the relative difference of parameter a^2 between the two methods is about 10 %, and it will be made an error analysis of parameter a^2 got by the two different methods.

As in this case, with the Sofue data, the relative difference is only 6% is omitted the error analysis because it is sure that the specific error associated to the parameter a^2 calculus is greater than 10%. So rightly is stated that both methods are equivalent i.e. the NFW DM profile is compatible with a velocity decaying law into the halo region equal to $v = a \cdot r^{-0.25}$.

6.2 Zhang et al. data (2024)

The data provided by these authors have error measures bigger regarding the Sofue data and the relative difference associated to the parameter a^2 also is bigger. Therefore it will be made an error analysis to the both methods. The results back the thesis of this work as well, although the extension of this epigraph is a bit longer because of error analysis.

6.2.1 Calculus of parameter a² using the rotation curve data

In the page 9 of paper [8] Zhang, it is tabulated the R.C. data. In the chapter 5 of paper [1] Abarca it is estimated that for radius bigger than 40 kpc the baryonic density is negligible regarding the DM density, so it is right to calculate the parameter **a** since 40 kpc. However the data for 46 and 52 kpc have a velocity too low, so both values have been rejected.

In the table 13 are tabulated the other values. The average a^2 is a 19% lower than the one calculated by the Sofue rotation curve data.

Table 13 Calculus of a^2 using the R.C. of M31. Data Zhang et al.				
Radius	Velocity	Radius	Velocity	Param. a^2
kpc	km/s	m	m/s	$m^{2.5}/s^2$
54,85	196,28	1,6925E+21	196280,00	1,5850E+21
67,26	202,02	2,0754E+21	202020,00	1,8593E+21
98,74	192,59	3,0468E+21	192590,00	2,0473E+21
123,56	168,53	3,8127E+21	168530,00	1,7538E+21
			Total sum	7,2453E+21
			Average a^2	1,8113E+21

6.2.1.1 Calculus of parameter a² error using the author data

Through the formula of parameter $a^2 = v^2 \cdot r^{\,0.5}$ it is right to calculate its relative error $d(a^2)/a^2 = 2dv/v + (1/2)\cdot dr/r$ where dv/v is the relative error of the velocity and dr/r is the relative error of radius.

In the table 14 are shown the rotation curve data and its errors provided by Zhang.

Table 14. Rotation curve clipped. [8] Zhang et al.			
Radius	Rad. error	Velocity	V. error
Kpc	kpc	km/s	km/s
52.08	1.63	182.05	38.15
54.85	1.67	196.28	38.81
67.26	14.95	202.02	40.67
98.74	17.19	192.59	42.23
123.56	11.6	168.53	41.34

In the table 15 are shown the rotation curve data and its error using the international system of units, also it is shown the relative error of parameter a^2 associated to each data and its average value is 43 %.

If it is considered this relative error then the value of a^2 got by Zhang (1.8E+21) clearly may be equivalent with the one got by Sofue (2.2E+21).

Table 15 R.C. data errors into I.S. units and relative error of a^2				
Radius	Rad. Error	Velocity	Vel. Error	d(a^2)/a^2
m	m	m/s	m/s	%
1,693E+21	5,15312E+19	196280	38810	39,560771
2,075E+21	4,61312E+20	202020	40670	40,374476
3,047E+21	5,30432E+20	192590	42230	43,941868
3,813E+21	3,57941E+20	168530	41340	49,106455
			Total sum	172,98357
			Average	43,245892

6.2.2 Virial data and calculus of parameter a² by the direct mass formula

In the graphic of page 8 of paper [8] Zhang, it is plotted the rotation curves associated to the disc and the bulge mass, so by a simple calculus it is got the baryonic mass of M31, $M_{BA} = 10.5 \cdot 10^{10} M_{\Theta}$. In addition, in the page 11 are given the virial data associated to DM purely, see the table 16 below.

Table 16. Virial data and baryonic mass of M31 according Zhang et al. (2024)		
M_{200}	**R_{200}**	**Baryonic mass**
$M_{vir} = 1.14^{+0.51}_{-0.35} \times 10^{12} M_{\odot}$	$r_{vir} = 220 \pm 25$ kpc.	$M_{BA} = 10.5 \cdot 10^{10}\ M_{\Theta}$

As it is shown in the virial data, table 16, the radius error is about 11% , the up error of mass is 45% and the low error one is 31%.

Using such data it is got $M_{200\text{-}TOTAL} = 1.245 \cdot 10^{12}\ M_{\Theta}$ and as in the previous epigraphs, the parameter a^2 it is calculated rightly, $a^2 = 2.006 \cdot 10^{21}$. This value is only a 10% bigger regarding the value got by the R.C. method (see epigraph 6.2.1).

6.2.2.1 Calculus of parameter a² error using the virial data

It is right to check that the relative error of parameter a^2 got by the virial data is $d(a^2)/a^2 = dm/m + (1/2) \cdot dr/(r)$ and considering the relative error of virial data (see table 16) leads to a relative error of parameter a^2 shown in the table 17.

Table 17 Relative error of parameter a^2		
Relative error M_{VIR}	Relative Error R_{VIR}	Relative Error parameter a^2
Up error 45% Low error 31%	11%	Up 50% Low 36 %

6.2.3 Equivalence of a² got by the two different methods using the Zhang data

In the table 18 are summarized the results got in the two previous epigraphs.

Table 18 Parameter a^2 and its relative errors		
Parameter a^2 by rot. Curve	$a^2 = 1.8 \cdot 10^{21}$	Relative error 43%
Parameter a^2 by virial data	$a^2 = 2.006 \cdot 10^{21}$	Relative low error 36%

If it is considered the relative difference of parameter a^2 between both results (about 10 %) and its own relative errors (about 40 %), clearly may be accepted that both results are equivalents.

One more time the results of parameter a^2 by the two different methods backs the thesis of this work i.e. the NFW DM profile is compatible with the decaying law of velocity $v = a \cdot r^{-0.25}$ into the halo region of M31.

7. Concluding remarks

In this study, the parameter **a** is determined through a double-calculation procedure, employing published datasets for the MW provided by two independent authors, and two distinct datasets for M31 contributed by two additional authors. Across these four independent comparisons, the empirical results strongly support the central thesis of this work.

For the Milky Way, the agreement is particularly remarkable. The relative discrepancy in parameter **a** amounts to only 1.8% when compared with the results of Huang et al. (2016) [4], and 3.8% relative to Sofue (2020) [5], despite the intrinsic uncertainty in the calculation of a^2 being on the order of 20% or greater.

Similarly, the M31 analysis exhibits very good consistency. The relative deviation in a^2 is 6% when benchmarked against the Sofue dataset and 10% with respect to the Zhang dataset. These discrepancies are at least three times smaller than the intrinsic uncertainty associated with the determination of a^2. Further details on the error analysis are provided in Chapters 5 and 6.

As the NFW profile is a reliable method validated in thousands of galaxies, establishing the equivalence between the total mass obtained from the NFW dark matter mass formula plus the baryonic contribution and the direct mass, derived from the double-method test, is crucial for validating the DMbQG theory. This is particularly relevant because the Milky Way (MW) and Andromeda (M31) are the only galaxies for which the rotation curve can be measured across an extended region of their halos.

Once again, the DMbQG theory has successfully passed a new test, complementing the results previously reported in [1] Abarca, M. (2024) and [2] Abarca, M. (2024). The remaining challenge is to ensure broader dissemination of the theory within the scientific community, thereby enabling its evaluation in other galaxies and clusters.

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