A curious identity involving the Appell hypergeometric series

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Abstract

We present an identity that relates the Appell F1 function and the constant Pi

I Introduction: The Apell series F1

Appell series are a natural two-variable extensión of hypergeometric series. They are treated with detail in Erdélyi et al. [4], the classical reference for special functions.

The Appell series F1 is defined as

$$F1(a; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{m! \, n! \, (c)_{m+n}} x^m y^n \quad , \qquad |x|, |y| < 1$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \dots (a+n-1) \ \ if \ n = 1,2,3,\dots; \ \ (a)_0 = 1$$

$$\Gamma(x) = \int_0^{\infty} e^{-t} \, t^{x-1} \, dt \ , \qquad x > 0$$

Transformations formulas for F1:

$$F1(a; b, b'; c; x, y) = (1-x)^{-b}(1-y)^{-b'} F1\left(c - a; b, b'; c; \frac{x}{x-1}, \frac{y}{y-1}\right)$$

$$F1(a; b, b'; c; x, y) = (1-x)^{-a} F1\left(a; c - b - b', b'; c; \frac{x}{x-1}, \frac{y-x}{1-x}\right)$$

$$F1(a; b, b'; c; x, y) = (1-y)^{-a} F1\left(a; b, c - b - b'; c; \frac{x-y}{1-y}, \frac{y}{y-1}\right)$$

$$F1(a; b, b'; c; x, y) = (1-x)^{c-a-b}(1-y)^{-b'} F1\left(c - a; c - b - b', b'; c; x, \frac{x-y}{1-y}\right)$$

$$F1(a; b, b'; c; x, y) = (1-x)^{-b}(1-y)^{c-a-b'} F1\left(c - a; b, c - b - b'; c; \frac{y-x}{1-x}, y\right)$$

Integral for F1

$$F1(a; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'} du$$

$$\Re(a) > 0, \Re(c-a) > 0$$

For details see Ref. [4], Ref. [5, chapter 9, pp. 1018-1021], Ref. [6, chapter 16, pp. 412-415].

II A curious identity involving F1

Entry 1. For $m, n \in \{0,1,2,3,\dots\}, i = \sqrt{-1}$ we have

$$\frac{\pi}{2} = \sum_{k=0}^{m} \frac{n+1}{(n+k+1)(n+k+2)} F1\left(n+2; 1,1, k+n+3; \frac{1-i}{2}, \frac{1+i}{2}\right)$$

$$+ \sum_{k=0}^{n} \frac{m+1}{(m+k+1)(m+k+2)} F1\left(k+1; 1,1, k+m+3; \frac{1-i}{2}, \frac{1+i}{2}\right)$$

where π is the famous constant Pi:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)$$

Equivalent expressions are given in entries 2 and 3.

Entry 2. For $m, n \in \{0,1,2,3,\dots\}, i = \sqrt{-1}$ we have

$$\frac{\pi}{4} = \sum_{k=0}^{m} \frac{n+1}{(n+k+1)(n+k+2)} F1(k+1; 1,1, k+n+3; i, -i)$$

$$+ \sum_{k=0}^{n} \frac{m+1}{(m+k+1)(m+k+2)} F1(m+2; 1,1, k+m+3; i, -i)$$

Entry 3. For $m, n \in \{0,1,2,3,\dots\}$ we have

$$\frac{\pi}{4} =$$

$$\sum_{k=0}^{m} \frac{n+1}{(n+k+1)(n+k+2)} 3F2\left(\left\{1, \frac{1+k}{2}, 1+\frac{k}{2}\right\}, \left\{\frac{n+k+3}{2}, 2+\frac{n+k}{2}\right\}, -1\right)$$

$$+ \sum_{k=0}^{n} \frac{m+1}{(m+k+1)(m+k+2)} 3F2\left(\left\{1, 1+\frac{m}{2}, \frac{3+m}{2}\right\}, \left\{\frac{m+k+3}{2}, 2+\frac{m+k}{2}\right\}, -1\right)$$

Remark: $3F2$ is the generalized hypergeometric function.

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III Future Research

Entry 4. For $m, n \in \{0,1,2,3,\dots\}$ we have

$$\frac{\pi}{4} = \sum_{r=0}^{\infty} (-1)^r \sum_{k=0}^{m} \frac{\binom{n+k}{k}}{\binom{2r+k+n+1}{2r+k}(2r+k+n+2)}$$
$$+ \sum_{r=0}^{\infty} 2^{-r-1} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \sum_{k=0}^{n} \frac{\binom{m+k}{k}}{\binom{2s+k+m+1}{k}(2s+k+m+2)}$$

IV References

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