Inductive Consequences of Goldbach's Conjecture

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Abstract

Assuming the validity of Goldbach's strong conjecture, we derive a family of inductive consequences concerning the representation of natural numbers as sums of primes. As context, we recall the classical theorems of Vinogradov and Chen, which constitute the most celebrated unconditional advances toward Goldbach's conjecture. We then present the inductive statements that would follow immediately if Goldbach's conjecture were true. A proposed elementary proof of Goldbach's conjecture has been provided by the author [1].

1 Introduction

Goldbach's strong conjecture asserts that every even integer greater than or equal to 4 can be written as the sum of two primes. Despite enormous effort, this conjecture remains unresolved.

Important unconditional progress was made in the 20th century by Vinogradov and Chen, who proved remarkable partial results toward Goldbach's conjecture. We recall their theorems for context in Section 2.

In the present work, we assume the truth of Goldbach's conjecture and derive a series of inductive consequences. For a proposed elementary proof of Goldbach's conjecture, see García, Melchor (2025) [1].

2 Context: Vinogradov and Chen

Theorem 2.1 (Vinogradov, 1937)**.** *Every sufficiently large odd integer can be expressed as the sum of three prime numbers.*

Remark 2.2. *Vinogradov's result was obtained through the Hardy–Littlewood circle method. In 2013, Helfgott strengthened the theorem to show that in fact* every odd integer greater than 5 *can be expressed as the sum of three primes.*

Theorem 2.3 (Chen, 1973)**.** *Every sufficiently large even integer can be expressed as the sum of a prime and a number which is either prime or the product of two primes (a semiprime).*

Remark 2.4. *Chen's theorem represents one of the closest unconditional results to Goldbach's strong conjecture, showing that every large even number is "prime + almost-prime."*

3 Goldbach's Conjecture as Hypothesis

For the purposes of this paper, we shall assume the truth of Goldbach's strong conjecture [1]:

Theorem 3.1 (Goldbach's Conjecture, strong form)**.** *Every even integer $2n \geq 4$ can be written as the sum of two primes:*

$$\forall n \geq 2, \exists p, q \in \mathbb{P}, \quad p + q = 2n.$$

On the basis of this assumption, we develop a series of inductive consequences.

4 Induction under Goldbach's Hypothesis

Proposition 4.1 (Odd numbers as sums of three primes)**.** *Every sufficiently large odd integer m can be expressed as the sum of three primes.*

Proof. Let p be a fixed odd prime, for instance $p = 3$. Then $m - p$ is even. For m large enough, $m - p \geq 4$ and by Theorem 3.1, there exist primes p_1, p_2 with $m - p = p_1 + p_2$. Hence

$$m = p + p_1 + p_2.$$

□

Remark 4.2. *This was already established unconditionally by Vinogradov and later Helfgott, but here it arises immediately from Goldbach's conjecture.*

Proposition 4.3 (Even numbers as sums of four primes)**.** *Every sufficiently large even integer n can be expressed as the sum of four primes.*

Proof. Take $p_1 = p_2 = 3$. Then $n - (p_1 + p_2)$ is even and, if n is sufficiently large, greater than 4. By Theorem 3.1, it equals $p_3 + p_4$ with both p_3, p_4 primes. Hence

$$n = p_1 + p_2 + p_3 + p_4.$$

□

Proposition 4.4 (Inductive extension for odd integers)**.** *For every odd $k \geq 3$, there exists N_k such that every odd $n \geq N_k$ can be expressed as the sum of k primes.*

Proof. Proceed by induction on k. The base case $k = 3$ holds by Proposition 4.1. For the inductive step, assume the claim holds for k. Fix two primes $r_1 = r_2 = 3$ and set $n' = n - (r_1 + r_2)$. Then n' is also an odd integer, large enough to be written as $n' = p_1 + \cdots + p_k$ with primes. Therefore

$$n = r_1 + r_2 + p_1 + \cdots + p_k,$$

which is a sum of $k + 2$ primes. □

Proposition 4.5 (Inductive extension for even integers)**.** *For every even $k \geq 2$, there exists M_k such that every even $n \geq M_k$ can be expressed as the sum of k primes.*

Proof. The base case $k = 2$ is exactly Goldbach's Conjecture. Assume the claim holds for k. Fix two primes $r_1 = r_2 = 3$ and define $n' = n - (r_1 + r_2)$, which is again even. By hypothesis $n' = p_1 + \cdots + p_k$, hence

$$n = r_1 + r_2 + p_1 + \cdots + p_k,$$

a sum of $k + 2$ primes. □

5 Conclusion

Assuming Goldbach's strong conjecture, one recovers as immediate consequences the results of Vinogradov and Chen, and in fact much more: any sufficiently large integer n (odd or even) can be expressed as a sum of k primes for every k of the same parity as n.

This establishes a broad inductive framework for additive number theory under Goldbach's hypothesis.

References

[1] Gustavo García, Oscar Melchor. *An Elementary Proof of the Goldbach Conjecture.* viXra preprint, 2025. `https://vixra.org/abs/2508.0034`.