The General Solution of Sextic Equations

in Terms of Fractional Sequences

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Abstract:

This paper reports a general solution for the sextic equations, which is an explicit power series of two parameters and fit for equations with real and/or complex coefficients.

The general sextic equation can be simplified by the Tschirnhausen transformations and expressed with four items in a type, called normal type. And it can further be simplified with only two non-constant coefficients into a form, called standard form. This fact means that the resolution of the sextic is a problem of two degree of freedoms.

There are totally 10 types and each type contains 6 forms. Among the total 60 forms, each correspondents to a power series, the coefficients in most of series are fractional sequences, some integer sequences.

If the series converges, the solution is found. Otherwise, successive Tschirnhausen transformations can be employed to obtain a series of new forms until the condition of convergence is satisfied. And then a reverse procedure is needed to find an original root. The experiment results show that it is always possible to satisfy the convergence condition and find the roots of transformed equations after several iterations.

The convergence of power series in all the 60 forms are different. The most favorite type and form are recommended.

Similar method can be used to the resolution of higher degree of polynomial equations.

Keywords:

polynomial equation, sextic equation, Tschirnhausen transformation, normal type, standard form, power series, fractional sequence, integer sequence, convergence

1. Introduction

The effort for the resolution of higher order polynomial equations has a long history. Abel (1826) and Galois (1832) had shown that it is impossible to solve the general polynomial equations of degree higher than the fourth in radicals.[1] Other solutions have been sought, like Kampe de Feriet functions [2], hypergeometric functions [3], elliptic modular functions, and inverse Lagrangian formula etc.[4] Some of them are not explicit or restricted by the convergence. [5-8]

Recently, Longfellow gave a general solution of the sextic with real coefficients via Tschirnhausen transformations and inverse regularized beta functions. [4] Wildberger and Rubine discussed the possibility of the resoluton of higher degree polynomial equation with a hyper-Catalan series. [9-10] For the quintic equation, we gave a general solution with power series of integer and fractional sequences. [11-12]

This paper reports a general solution for the sextic equations, which is an explicit power series of two parameters with fractional or integer sequences and fit for equations with real and/or complex coefficients.

If the series converges, the solution is found. Otherwise, successive Tschirnhausen transformations can be employed to obtain a series of new equations until the condition of convergence is satisfied. The convergence of power series derived from 60 forms are different. The most favorite type and form are recommended.

Similar method can be used to the resolution of higher degree of polynomial equations.

In this paper, the normal types and the standard forms of the sextic are firstly discussed in section 2 and 3 and then the power series of favorite forms are listed in section 4. The resolution of general sextic equation is shown in section 5. Lastly, some results and discussions are summed up in section 6.

2. Simplification and classification of sextic equations

Suppose the general sextic polynomial equation is

$$a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 = 0 \qquad (1)$$

where: $a_k, k = 0,1,2,...,6$ are real or complex numbers and $a_0a_6 \neq 0$.

To simplify (1), Tschirnhausen transformation can be used. [13-14] And two successive transformations of quadric and quartic can be applied to cancel three items of degree 5, 4 and 3, to obtain a equation with four items,

$$b_0 + b_1 y^1 + b_2 y^2 + b_6 y^6 = 0 \qquad (2)$$

where: $b_0 b_6 \neq 0$.

We name (2) as type 6210 to indicate the degrees of remained four items.

In fact, the first quadric Tschirnhausen transformation can be used to cancel the items of degree 5 and 4. And the second quartic to cancel the degree 3 item by solving a third degree equation to get (2), type 6210. But at the same time, we can choose to cancel the degree 2 item by solving a fourth degree equation to get type 6310, or to cancel the degree 1 item by solving a fifth degree equation to get type 6320, provided that the fifth degree equation can be solved similarly.

Theoretically, there are 10 types for the sextic in the four items:

6210
6310, 6320
6410, 6420, 6430
6510, 6520, 6530, 6540

As mentioned above, among them only three types 6210, 6310, and 6320 can be obtained directly from the general equation (1) .
If let $y = \frac{1}{t}$ the reciprocal transform in (2), it becomes type 6540, this property is called duality。

The correspondence of duality in 10 types are

6210	6310	6320	6410	6420	6430	6510	6520	6530	6540
6540	6530	6430	6520	6420	6320	6510	6410	6310	6210

where: type 6420 and 6510 are self-duality.

Notice that type 6420 can be solved with radicals by a third degree equation with square variable substitution. We list it here just for the wholeness. The type 6410, 6510 and 6520 are special cases, which are not accessible by Tschirhausen transformations from the general equation (1).

3. The degree of freedoms of the sextic and 6 forms of each types

The coefficients of remained four items in (2) are not independent. For example, we can choose to divide out b_0 and let variable change to get

$$1 + z^1 + c_2 z^2 + c_6 z^6 = 0 \qquad (3)$$

where: $z = \frac{b_1}{b_0}\ y$, etc.

There are left only two non-constant coefficients, or parameters.

Generally, a four-item equation of the 10 types can have an expression,

$$b_0 + b_m y^m + b_n y^n + b_6 y^6 = 0 \qquad (4)$$

where: $b_6 b_0 \neq 0, 1 \leq m < n \leq 5.$

It is always possible to express (4) with only two parameters, which is actually the DOFs. We would like to express this fact as

Preposition 1: The general sextic polynomial equation has two independent non-constant parameters, or two degree of freedoms.

We will see that this result is of fundamental meaning for the resolution of the sextic with power series.

If the four-item equation (4) is expressed in two non-constant parameters, there are totally 6 cases, called form. We can define an indicator of forms with four digits in ascending order of variable, each digit correspondent to an item, where digit 1 for the position of the non-constant parameter, 0 for constant. For example, the indicator of the form in (3) is "0011". Again we use an 8-tuple numbers in parentheses to name and express the combination of types and forms, with the first four digits the type, the last four digits the form. Thus the indictor of the combination in (3) is (6210 0011).

Suppose the two parameters be $p\ \text{and}\ q$, the 6 forms of (4) are

<1> 1100 $p + q\ z^m + z^n + z^6 = 0$ (5)
<2> 1010 $p + z^m + q\ z^n + z^6 = 0$
<3> 1001 $p + z^m + z^n + q\ z = 0$
<4> 0110 $1 + p\ z^m + q\ z^n + z^6 = 0$
<5> 0101 $1 + p\ z^m + z^n + q\ z^6 = 0$
<6> 0011 $1 + z^m + p\ z^n + q\ z^6 = 0$

Obviously, there are totally 60 forms in (5). And there are two groups, {<1>, <2>, <3>} and {<4>, <5>, <6>}. It is clear that one can get <4>, <5> and <6> from the general equation (1) by Tshcirnhausen transformations, but not <1>, <2> and <3>. The group {<1>, <2>, <3>} can be changed into group {<4>, <5>, <6>}, but not vice versa.

4. Fractional and integer sequences accompanying different forms of sextic equations

Now according to preposition 1, the resolution of the sextic is a problem of two DOFs and among the 60 forms, each contains two parameters. Thus we have reason to suppose that the root of the sextic (1) or exactly (5) can be expressed in a power series of two parameters p, q as

$$z(p, q) = \sum_{i+j \geq 0} \alpha_{i,j}\ p^i q^j \qquad (6)$$

If (6) converges, it is a root of (5) and the root of the original (1) can be found by inverse Tschirnhausen transformations.

The definition of power series in (6) is straight forward for different types and forms. One only need to substitute (6) into one of forms in (5) to find $\{\alpha_{i,j}\}$ by comparing the coefficients of powers of p and q. The experimentation shows that $\{\alpha_{i,j}\}$ are fractional sequences in most cases, some are integer sequences. And to authors knowledge, the sequences are first reported, their formula are almost unknown.

For example， for (6210 0011) in (3), the power series are

$$z(p, q) = \ -1 - p - q - 2p^2 - 8pq - 6q^2 - 5p^3 - 45pq^2 - 91q^2p - 51q^3 \ldots, \ldots$$

where: $p = c_2, q = c_6$

Here care must be taken, if the positive and negative signs of the four items in (5) are considered, each form has 8 cases, their series are somehow different. For example, we change one of signs in (3),

$$1 - z^1 + pz^2 + qz^6 = 0$$

the power series becomes

$$z(p, q) = \ 1 + p + q + 2p^2 + 8pq + 6q^2 + 5p^3 + 45pq^2 + 91q^2p + 51q^3 \ldots, \ldots$$

From the viewpoint of resolution, different power series has different convergence radius. If the items of a sequence are all positive integer, its convergence radius must be small.

To distinguish 8 cases, we use the ninth digit to indicate. The power series of 8 cases of (6210 0110) are

<1> (6210 0110 1) (7)

$$1 + pz^1 + qz^2 + z^6 = 0$$

$$z(p,q) = \ t - \frac{p}{6t^4} - \frac{q}{6t^3} - \frac{p^2}{24t^9} - \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \dots, \dots$$

where: $t = \pm I, \pm\frac{\sqrt{2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{2+2I\sqrt{3}}}{2}$; I, imaginary unit.

<2> (6210 0110 2)

$$-1 + pz^1 + qz^2 + z^6 = 0$$

$$z(p,q) = \ t - \frac{p}{6t^4} - \frac{q}{6t^3} - \frac{p^2}{24t^9} - \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \dots, \dots$$

where: $t = \pm 1, \pm\frac{\sqrt{-2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{-2+2I\sqrt{3}}}{2}$.

<3> (6210 0110 3)

$$1 - pz^1 + qz^2 + z^6 = 0$$

$$z(p,q) = \ t + \frac{p}{6t^4} - \frac{q}{6t^3} - \frac{p^2}{24t^9} + \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \dots, \dots$$

where: $t = \pm I, \pm\frac{\sqrt{2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{2+2I\sqrt{3}}}{2}$.

<4> (6210 0110 4)

$$1 + pz^1 - qz^2 + z^6 = 0$$

$$z(p,q) = \ t - \frac{p}{6t^4} + \frac{q}{6t^3} - \frac{p^2}{24t^9} + \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \dots, \dots$$

where: $t = \pm I, \pm\frac{\sqrt{2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{2+2I\sqrt{3}}}{2}$.

<5> (6210 0110 5)

$$1 + pz^1 + qz^2 - z^6 = 0$$

$$z(p,q) = \ t + \frac{p}{6t^4} + \frac{q}{6t^3} - \frac{p^2}{24t^9} - \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \dots, \dots$$

where: $t = \pm 1, \pm\frac{\sqrt{-2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{-2+2I\sqrt{3}}}{2}$.

<6> (6210 0110 6)

$$-1 - pz^1 + qz^2 + z^6 = 0$$

$$z(p,q) = \ t + \frac{p}{6t^4} - \frac{q}{6t^3} - \frac{p^2}{24t^9} + \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \dots, \dots$$

where: $t = \pm 1, \pm\frac{\sqrt{-2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{-2+2I\sqrt{3}}}{2}$.

<7> (6210 0110 7)

$$-1 + pz^1 - qz^2 + z^6 = 0$$

$$z(p,q) = \ t - \frac{p}{6t^4} + \frac{q}{6t^3} - \frac{p^2}{24t^9} + \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \cdots,\ldots$$

where: $t = \pm 1, \pm\frac{\sqrt{-2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{-2+2I\sqrt{3}}}{2}.$

<8> (6210 0110 8)

$$-1 + pz^1 + qz^2 - z^6 = 0$$

$$z(p,q) = \ t + \frac{p}{6t^4} + \frac{q}{6t^3} - \frac{p^2}{24t^9} - \frac{pq}{18t^8} - \frac{q^2}{72t^7} \ \cdots,\ldots$$

where: $t = \pm I, \pm\frac{\sqrt{2-2I\sqrt{3}}}{2}, \pm\frac{\sqrt{2+2I\sqrt{3}}}{2}.$

When let t=1 in <2>,<5>,<6> and <7>, the fractional sequences and power series look relatively simple,

$$1 - \frac{p}{6} - \frac{q}{6} - \frac{p^2}{24} - \frac{pq}{18} - \frac{q^2}{72} \ \cdots,\ldots$$

$$1 + \frac{p}{6} + \frac{q}{6} - \frac{p^2}{24} - \frac{pq}{18} - \frac{q^2}{72} \ \cdots,\ldots$$

$$1 + \frac{p}{6} - \frac{q}{6} - \frac{p^2}{24} + \frac{pq}{18} - \frac{q^2}{72} \ \cdots,\ldots$$

$$1 - \frac{p}{6} + \frac{q}{6} - \frac{p^2}{24} + \frac{pq}{18} - \frac{q^2}{72} \ \cdots,\ldots$$

The experimental results show that when the modulus of p and q are less than 1, all the 8 power series in (6210 0110) converge absolutely.

It is a tedious work to find all the sequences of 480 cases and it is a challenging task to find their formula, which will help to determine the convergence of power series.

5. Solving the sextic with the favorite form and its fractional sequences

As mentioned above, among the 10 types, the type 6210 is convenient, which can be directly accessible from the general equation (1). Other types can be easily transformed to it. And among the 6 forms of each type, we have found that the fourth form 0110 is most favorite, which is a fractional sequence and has good convergence property. Other forms can also be easily transformed to it. It is recommended that (6210 0110) be used to solve the sextic.

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Algorithm 1: Solving the sextic equation using (6210 0110)

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For a given general sextic equation,

If possible, first try to transform it into (6210 0110 n) by scale or other simple transformations, where: n=1,2,3,...,8;

Otherwise

transform it into (6210 0110 n) by Tschirnhausen transformations;

Test the convergence condition of transformed equation;

If it converges, calculate the power series to get a root of transformed equation;

Inverse transformations, if any, to find the original root;

Stop;

Otherwise go back to next Tschirnhausen transformation until the convergence condition is satisfied.

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The experimental results show that the convergence condition can be satisfied after several transformations, the speed of convergence depends on the concrete equation. And after first two Tschirnhausen transformations, the loop is within the type 6210, the calculation of quartic transformation is relatively simple.

Here are some examples. For accuracy, the calculation of power series are all up to 9 degree, using 10 digits.

Example 1:

$$f(x) = -4 + 3x - 2x^2 + 9x^6$$

this is a type 6210 with four coefficients. It can be transformed to (6210 0110 1)

$$g(y) = 1 + py + qy^2 + y^6$$

where: $p = -0.5674072 - 0.3275927\ I, q = 0.1907857 + 0.3304505\ I,$

$x = (0.7565428 + 0.4367902\ I)\ y.$

The value of p and q satisfies the convergence condition, the calculation of the power series with $t = I$in (7) gives $0.1340426 + 1.0422842\ I$. One of roots of $g(y)$ is $0.1340494 + 1.0422797\ I$. We take the value of the power series to calculate $x = -0.3538506 + 0.8470812\ I$. One of roots of f(x) is $-0.3538435 + 0.8470808\ I$.

If we choose (6210 0110 5) ,

$$g(y) = 1 + py + qy^2 - y^6$$

where: $p = -0.6551853, q = 0.3815714, x = 0.8735805\ y.$

The power series with t =1 in (7) gives $y = 0.9471028$, and $x = 0.8273705$, compared with one of roots of $g(y)$ is 0.9471028, and one of roots of $f(x)$ is $x = 0.8273705$, which are all the same perfectly.

Example 2:

$$f(x) = -1 + 3x - 2x^2 - x^3 + x^4 + x^6$$

this is from [8]. We first delete the item of the fourth degree with a quadric transformation

$$T2:\ y + \frac{1}{3} + (\frac{3}{2} + \frac{1}{6}\sqrt{249})x + x^2 = 0$$

to get

$$g(y) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + y^6$$

where: $c_0 = -5904.598708, c_1 = -1742.850853, c_2 = 480.7092226,$
$c_3 = 156.8163575.$

Then delete the item of the third degree by a quartic transformation

$$T4:\ z + d_0 + d_1 y + d_2 y^2 + d_3 y^3 + y^4 = 0$$

where: $d_0 = 1732.137321, d_1 = 1911.593798, d_2 = -55.06353886,$
$d_3 = 18.00404663.$

to get a type 6210

$$h(z) = a_0 + a_1 z + a_2 z^2 + z^6$$

where: $a_0 = -1.5763195\ \ 10^{23}, a_1 = 2.5216008\ \ 10^{19}, a_2 = 1.2910979\ \ 10^{15}.$

We choose (6210 0110 1) to find

$$k(u) = 1 + pu + qu^2 + u^6$$

where: $p = -1.0182086 - 0.5878630\ \ I, q = -0.2212244 - 0.3831719\ \ I.$

The transformation is

$$T1: z - (6365.091802 + 3674.887466\ \ I)\ \ u = 0$$

The power series with $t = I$ in (7) gives

$$u = 0.0687407 + 1.1265378\ \ I.$$

The nearest root of $k(u)$ is $0.0685330 + 1.1265834\ \ I.$

One of roots of $f(x)$ can be found by inversing transformations. First use u to calculate $z = -3702.358513 + 7423.130766\ \ I$ in T1. Then by using it, the four roots of y can be found in T4. Each of roots of y can be used to find two roots of T2.

Among the 8 roots of x, only one is the target:

$$x = -0.3355671 + 1.5326267\ \ I$$

Compared with the root of the original $f(x)$,

$$x = -0.3356073 + 1.5323945\ \ I$$

The error may be because the p is a little out of the interval of convergence and the number of items of the power series used is limited.

6. Results and discussions

The experimental results show that the two DOFs algorithm fits for the sextic equation with real or complex coefficients. Momentarily the determination of the formula of power series of all the cases and their convergence radii is still underway. The error of calculation may be reduced by adopting more items of power series. Smart selection among the types, forms or cases may help to get more accuracy and be more effective.

The hypothesis of (6) seems not unique, the solution may have other forms. For example, $1 - \mathrm{x} + 0.01\mathrm{x}^2 + 0.02\mathrm{x}^6 = 0$ has a root $x_0 = 1.0353554$, the power series

$$\sum_{n=0}^{10} \sum_{k=0}^{10} \frac{\binom{2n+k}{n+2k}\binom{4n+2k}{k}}{n+4k+1} \left(\frac{0.01^n + 0.02^n}{2}\right)$$

gives 1.0348464, a pretty good approximate value, which be used at least to guess a start point for fast iteration.

The method reported here can be used to higher degree polynomial equations, which will be more complex, and more challenging.

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