A Symmetry-Based Proof of Commutativity, Order Dependence, and the Zero Product Property

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Abstract

This paper presents an original proof of the commutative property for addition and multiplication using symmetric term interpretation and the fundamental ambiguity of sequential operations. Unlike traditional approaches, this method reveals how commutativity emerges naturally from the freedom to choose term associations in algebraic expressions. It also examines subtraction and division, showing why they are not commutative, and explains the zero product property in clear mathematical and verbal terms.

1. Introduction

The commutative properties ($a + b = b + a$ and $a \times b = b \times a$) are typically proven through induction or geometric arguments. This work demonstrates them more intuitively by showing how the inherent ambiguity in ungrouped expressions enforces commutativity. The same approach reveals why subtraction and division fail to be commutative, and why multiplying any number by zero always gives zero.

2. Proof of Commutative Addition

Theorem 1: For all natural numbers a and b:

$$a + b = b + a$$

Proof:

1. Begin with the identity:

$$a + a = a + a$$

2. Introduce an arbitrary term m to both sides:

$$a + a + m = a + a + m$$

3. **Key Insight**: The expression $a + a + m$ is ambiguous, we can choose which a is grouped with m:
   - **Option 1**: $a + (a + m)$
   - **Option 2**: $(a + m) + a$

4. Since both options represent the same quantity, we conclude:

$$a + (a + m) = (a + m) + a$$

5. Let $b = a + m$:

$$a + (a + m) = (a + m) + a$$

$$a + b = b + a$$

3. Proof of Commutative Multiplication

Theorem 2: For all natural numbers a and b:

$$a \times b = b \times a$$

Proof:

1. Start with:

$$a \times a = a \times a$$

2. Multiply both sides by m:

$$a \times a \times m = a \times a \times m$$

3. **Key Insight**: The expression $a \times a \times m$ can be read in two ways:
    - **Option 1**: $a \times (a \times m)$
    - **Option 2**: $(a \times m) \times a$
4. Since both represent the same product, we deduce:

$$a \times (a \times m) = (a \times m) \times a$$

5. Let $b = a \times m$:

$$a \times (a \times m) = (a \times m) \times a$$

$$a \times b = b \times a$$

4. Implications: Repeated Operations

The notation:

$$a + a + a = 3a$$

means three instances of a added together.

Because multiplication is repeated addition:

$$3a = 3 \times a = a \times 3$$

5. Subtraction and Why It Is Not Commutative

Start with:

$$a - a = a - a$$

Add –b to both sides:

$$a - a - b = a - a - b$$

This can be read in two ways, but unlike addition, changing the order changes the value.

Example:
Let $-a - b = m$.
Then:

$$a + (-a - b) = a - b - a$$
$$a + m = a - b - a$$

By moving **–b** to the second position on the right-hand side, we do not obtain **–a** – **b** as on the left-hand side. This demonstrates that the order of terms is essential in subtraction.

Conclusion:
For subtraction:

$$b - a \neq a - b$$

Therefore, subtraction is order-dependent, and:

$$a - b = a - b$$

6. Division and Why It Is Not Commutative

Start with:

$$a \div a = a \div a$$

Introduce b:

$$a \div a \div b = a \div a \div b$$

This can be interpreted as:

- $a \div (a \div b)$
- $(a \div b) \div a$

Let $a \div b = m$:

$$a \div (a \div b) = (a \div b) \div a$$

$$a \div m = m \div a$$

But generally:

$$a \div m \neq m \div a$$

Except when $a = m$

Therefore, for division, **order matters**, and we must follow the correct sequence to get the right result.

Special case:

$$a \div a \div b = a \div a \div b$$

$$1 \div b = 1 \div b$$

7. The Zero Product Property

For any number n:

$$n \times 0 = 0 \times n$$

Example with 1000:

$$0\ x\ 1000 = 0\ x\ 1000$$

By definition, 0×10000 means adding zero one thousand times:

$$0 + 0 + 0 + \cdots + 0 = 0$$

Therefore:

$$0 \times 1000 = 1000 \times 0 = 0$$

This works because multiplying by zero means repeated addition of zero, and adding zero never changes the total, it always stays zero.

8. Conclusion

This work has shown:

1. Commutativity of addition and multiplication arises from the freedom to interpret ambiguous expressions.
2. Subtraction and division are order-dependent and not commutative.
3. Multiplication as repeated addition justifies compact notations like $3a, 3 \times a = a \times 3$
4. The zero product property becomes intuitive when explained as repeated addition of zero.

These results highlight that some fundamental truths in mathematics come from the way we choose to interpret and group expressions.