IS SPACETIME CAUSALLY WELL-FOUNDED?

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ABSTRACT

In this essay, the causal structure of spacetime is critically examined with a focus on its foundational aspects. The author argues that traditional formulations of the causal order in spacetime often rely on assumptions that may lead to infinite causal regress. To address this, a well-founded causal relation is proposed, in which causation is understood not as a temporal or sequential mechanism but as a grounding relation among events. The analysis suggests that spacetime emerges from a more fundamental layer of reality characterized by ontological priority rather than mere temporal succession. The conclusion posits that the causal structure of spacetime must be well-founded to avoid logical inconsistencies and that this foundation lies beyond spacetime itself, potentially in a network of basal events. This ontological perspective has implications for the interpretation of physical theories and the nature of reality.

1 Introduction

The question of whether spacetime is causally well-founded lies at the intersection of metaphysics, mathematical logic, and theoretical physics. This paper undertakes a rigorous investigation into the foundational causal structure of spacetime, addressing the possibility of a "first cause" and the implications of infinite causal regress. Drawing upon formal definitions from set theory and modal logic, the analysis challenges the assumption that spacetime possesses a globally well-founded causal order.

In Section 2, we lay the groundwork by introducing the concept of a *well-founded relation*, a mathematical structure that ensures the absence of infinite descending chains. We then motivate its relevance to the causal ordering of events in spacetime, highlighting the connection between foundational metaphysical notions and physical interpretation.

Section 3 builds a modal-logical framework to analyze causal precedence using possible world semantics. Definitions of modal necessity and possibility are formalized, and a structured argument is presented to demonstrate the inconsistency of attributing necessary causal power to an infinite sequence of effects. This culminates in a contradiction that undermines the existence of a globally minimal cause in the causal hierarchy of spacetime. The discussion transitions to the physical domain of general relativity. By examining the causal properties of smooth, globally hyperbolic Lorentzian manifolds, we test the metaphysical conclusions against standard models of relativistic spacetime. The argument demonstrates that, under reasonable physical assumptions, no point within the interior of such a manifold can function as a first cause, owing to the existence of past-directed timelike geodesics in any neighborhood. It also extends this argument to boundary points of spacetime manifolds. Even when the first cause is posited to lie on the boundary, the analysis reveals that such a configuration implies geodesic incompleteness and signals the breakdown of classical general relativity. Thus, if a first cause exists, it must lie outside the domain of standard spacetime physics.

Finally, in Section 4, we consider the role of quantum cosmology in addressing singularity formation. By examining the Wheeler–DeWitt equation within a minisuperspace framework and incorporating operator-ordering ambiguities, we explore scenarios under which the wavefunction of the universe remains finite as the scale factor approaches zero. This provides a mechanism for singularity avoidance and supports the hypothesis that the universe may emerge from a quantum regime without a classical initial singularity.

Taken together, the results of this inquiry suggest that the causal structure of spacetime cannot be globally well-founded without encountering logical or physical pathologies. Instead, the emergence of spacetime from a deeper, non-temporal foundation—potentially described by a quantum theory of gravity—is a necessary precondition for a coherent understanding of causality and the universe's origin.

2 Definitions and Philosophical Motivations

Philosophy has long served as a foundation for many physical theories. Just as Mach's principle inspired Einstein in formulating the theory of relativity [1], it is worthwhile to examine key philosophical concepts that may inform our exploration of whether spacetime is causally *well-founded*. In what follows, we first discuss relevant philosophical notions before presenting an argument grounded in modal logic. This approach provides a preliminary perspective on the issue which we then apply to the physical framework of general relativity.

2.1 Definitions and Preliminaries

We aim to examine the relations in which every nonempty subclass of a given class contains an R minimum element. Relations satisfying this property are termed well-founded. Since quantification of class symbols is not allowed within our framework, the definition must instead be formulated in terms of subsets, supplemented by auxiliary conditions that ensure that the property extends to arbitrary subclasses[2].

Definition 2.1 (Well-Founded Relation)**.** A relation R on a class A is *well-founded* if:

$$R \text{ Fr } A \triangleq (\forall x)[x \subseteq A \land x \neq \emptyset \to (\exists y \in x)[x \cap (R^{-1})``\{y\} = \emptyset]].$$

where:

- R^{-1} denotes the inverse relation
- $(R^{-1})``\{y\}$ represents the R-predecessors of y (i.e., $\{z \mid zRy\}$)

The concept of well-foundedness ensures that no infinite descending chains exist with respect to a given binary relation on a class. Intuitively, a relation R is well-founded on a class A if every non-empty subset of A contains an element that has no R-predecessor within that subset. In other words, there is always a minimal element under R in any such subset. This property precludes the existence of infinite regress in the form of sequences like $\cdots Rx_3Rx_2Rx_1Rx_0$.

A class or set fails to be well-founded—i.e., is *non-well-founded*—when there exists at least one non-empty subset in which every element has an R-predecessor within the subset. In such cases, no minimal element can be found, and one may encounter pathological structures such as infinite descending R-chains or even self-containing sets (e.g., sets x such that $x \in x$), which are disallowed by the Axiom of Regularity in ZFC set theory. However, non-well-founded sets are formally accommodated in alternative set theories such as Aczel's Anti-Foundation Axiom (AFA), where such phenomena are treated consistently.[3].

This definition proves particularly useful in the context of investigating the structure of causal ordering and examining the concept of a so-called *first cause* or *initial event*. One might be inclined to identify such an event as the earliest element in the space-time continuum under a globally defined causal relation. From this perspective, it becomes a matter of logical and mathematical inquiry whether such a first event can coherently exist. If it does, the spacetime structure may be modeled as a well-founded set under the causal relation. Conversely, if no such minimal element exists, the causal structure of spacetime would then exhibit characteristics of a non-well-founded set.

This distinction is not merely of technical interest; it has foundational implications for cosmology, metaphysics, and the study of time. For instance, non-well-founded causal structures are compatible with models that allow causal loops, infinitely descending chains of causes, or other violations of the standard notion of temporal priority. Such models have been discussed in various philosophical and physical contexts [1, 3, 4].

Definition 2.2 (Possible Worlds)**.** Let $\mathcal{L}$ be a logical language, and let $\mathcal{P}$ denote the set of all well-formed propositional formulas over $\mathcal{L}$. A *possible world* is defined as a subset $\omega \subseteq \mathcal{P}$ satisfying the following condition:

$$\forall \varphi \in \mathcal{P},\ \big((\varphi \in \omega) \lor (\neg\varphi \in \omega)\big) \land \big(\varphi \in \omega \Rightarrow \neg\varphi \notin \omega\big).$$

In other words, for every formula φ, a possible world contains either φ or its negation (but not both), ensuring classical consistency and completeness. The set of all such possible worlds is then given by[5, 6]:

$$\mathcal{W} = \big\{\omega \subseteq \mathcal{P} \;\big|\; \forall \varphi \in \mathcal{P},\ \big((\varphi \in \omega) \lor (\neg\varphi \in \omega)\big) \land \big(\varphi \in \omega \Rightarrow \neg\varphi \notin \omega\big)\big\}\,.$$

In the framework of modal logic, a Kripke model is represented by a tuple $M = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ [7, 8], where $\mathcal{W}$ denotes the set of possible worlds, $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is the accessibility relation between worlds, and $\mathcal{V}$ is the valuation function that maps each propositional variable to the set of worlds in which it is true. The accessibility relation $\mathcal{R}$ represents the idea that some worlds are "accessible" from others, reflecting the possibility of transitions between these worlds. The valuation function $\mathcal{V}$, on the other hand, assigns to each propositional variable a set of worlds where that variable holds true, encapsulating the truth conditions for the modal formulas. $M = (\mathcal{W}, \mathcal{R}, \mathcal{V})$ where $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$, and $\mathcal{V} : \mathcal{P} \to 2^{\mathcal{W}}$, where $\mathcal{P}$ is the set of propositional variables. This setup is fundamental in the analysis of modal logic, as it allows for the evaluation of statements about necessity ($\Box$) and possibility ($\Diamond$) by examining the relationships between worlds and the truth values assigned within those worlds.

A possible world ω within a Kripke model is defined as a subset of $\mathcal{W}$, such that for every proposition p, ω satisfies the condition that p holds in ω if and only if the evaluation $\mathcal{V}(p)$ contains ω. Formally, we define $\mathcal{V}$ as follows $\mathcal{V} = \{\omega \subseteq \mathcal{W} \mid v(\omega)(p) = 1\}$ where $v(\omega)(p) = 1$ means that the propositional variable p is true in the possible world ω.

These foundational definitions allow for the development of a rigorous semantic framework for understanding the structure of possible worlds, particularly in modal logic and related fields.

Definition 2.3 (Modally Possible)**.** In modal logic, a proposition φ is said to be modally possible if there exists at least one possible world ω where φ is true. This can be formally expressed as [9]:

$$\exists \omega \left[(v_\omega(\varphi) = 1 \right] \quad \text{iff} \quad \Diamond \varphi$$

where $v_\omega(\varphi) = 1$ indicates that φ is true in world ω, and $\Diamond\varphi$ represents the possibility of φ being true in some world. This equivalence highlights the connection between the semantic notion of possibility and the modal operator $\Diamond$.

Definition 2.4 (Modally Necessary)**.** In modal logic, a proposition ϕ is **modally necessary** if and only if it is true in all possible worlds ω. Formally [9]:

$$\forall \omega \left[v_\omega(\phi) = 1 \right] \quad \text{iff} \quad \Box \phi$$

Definition 2.3 and **Definition 2.4** will serve as the foundational framework for constructing our logical argument, through which we investigate whether spacetime is well-founded under the causal relation (i.e., whether it contains no infinitely descending causal chains). This inquiry forms the philosophical basis of our analysis, guiding our subsequent physical examination of spacetime structure within the context of general relativity and causal set theory.

A well-founded spacetime—one in which every local causal sequence has a minimal element—has significant implications for the ontology of time, quantum gravity, and the cosmological arrow of time [10, 11]. If spacetime is causally well-founded, it supports a temporally discrete and finitistic model of causality, aligning with certain interpretations of causal dynamical triangulation [12] and loop quantum gravity [13]. Conversely, if it permits infinite causal regress, this raises questions about temporal becoming and the nature of singularities [10, 14].

Our approach bridges modal metaphysics and foundational physics, offering a novel perspective on the initial event of the spacetime which we shall call the *the first cause*.

2.2 Metaphysical Rejection of Global Causal Well-Foundedness

A critical examination of *epistemic modality* reveals a fundamental distinction: what holds as metaphysically true may not necessarily align with physical truth, yet what is *metaphysically necessary* must impose constraints on physical necessity. To formalize this, we introduce a causal function $C(x, y)$, interpreted as “x causally precedes y” (irrespective of the specific model of causality), subject to the following axiomatic constraints:

1. **Irreflexivity**: $\forall x \, \neg C(x, x)$ (no event causes itself).
2. **Transitivity**: $C(x, y) \wedge C(y, z) \implies C(x, z)$.
3. **Asymmetry**: $C(x, y) \implies \neg C(y, x)$ (no causal loops).

These properties eliminate pathological causal paths (e.g., closed timelike curves) and are consistent with the time-orientability requirement of general relativistic spacetimes [15]. By excluding non-time-orientable models (e.g., Gödel universes), we preserve the well-foundedness of causal structure—a desideratum for grounding temporal asymmetry [16]. However, this exclusion is provisional; we later revisit causality in time-orientable spacetimes to test the robustness of well-foundedness against quantum gravitational effects [17].

$$1.\quad \forall\omega\,[\,v_\omega(\varphi)=1\,] \;\equiv\; \Box\varphi \qquad \text{(Definition of modal necessity of } \varphi) \tag{1}$$

$$2.\quad \exists\omega\,[\,v_\omega(\varphi)=1\,] \;\equiv\; \Diamond\varphi \qquad \text{(Definition of modal possibility of } \varphi) \tag{2}$$

$$3.\quad \Diamond\psi \implies \exists\delta\,\mathcal{C}(\delta,\psi) \qquad \text{(Premise: if } \psi \text{ is possible, then some } \delta \text{ causes } \psi) \tag{3}$$

$$4.\quad \Box\delta \implies \neg\exists\beta\,\mathcal{C}(\beta,\delta) \qquad \text{(Premise: if } \delta \text{ is necessary, then nothing causes } \delta) \tag{4}$$

$$5.\quad \big(\Box\,\mathcal{C}(\delta,\wedge_{j\in\mathbb{N}}\psi_j)\;\wedge\;\Box\delta\big) \implies \Box\bigwedge_{j\in\mathbb{N}}\psi_j \qquad \text{(From modal distribution / modus ponens)} \tag{5}$$

$$6.\quad \Box\bigwedge_{j\in\mathbb{N}}\psi_j \implies \bot \qquad \text{(Contradiction with (4) and (5))} \tag{6}$$

Premise 1: Definition of Modal Necessity

$$\forall\omega\,[\,v_\omega(\varphi)=1\,] \;\equiv\; \Box\varphi.$$

A formula φ is *necessarily* true (written $\Box\varphi$) exactly when it holds in every possible valuation ω. Equivalently, there is no accessible world in which φ fails.

Premise 2: Definition of Modal Possibility

$$\exists\omega\,[\,v_\omega(\varphi)=1\,] \;\equiv\; \Diamond\varphi.$$

A formula φ is *possibly* true (written $\Diamond\varphi$) exactly when there is at least one valuation ω in which φ holds.

Premise 3: Possibility Entails a Cause

$$\Diamond\psi \implies \exists\delta\,\mathcal{C}(\delta,\psi).$$

If a proposition ψ is merely possible, then there must exist some event or fact δ such that $\mathcal{C}(\delta,\psi)$ holds—i.e. "δ causes ψ." This captures the principle that genuine possibility requires a potential cause.

Premise 4: Necessity Is Uncaused

$$\Box\delta \implies \neg\exists\beta\,\mathcal{C}(\beta,\delta).$$

If an event or proposition δ is necessary (true in all possible worlds), then no prior event β can cause it. In other words, necessary truths stand without causal explanation.

Conclusion: Contradiction from an Infinite Conjunction

$$\Box\Big(\bigwedge_{j\in\mathbb{N}}\psi_j\Big) \implies \bot.$$

From Premises 3 and 4, applying them to each ψ_j in the infinite family and using the distribution of necessity over conjunction, one infers that the entire infinite conjunction $\bigwedge_j \psi_j$ would have to be both necessary and uncaused. This contradicts the uncaused-ness condition (Premise 4), yielding a logical impossibility ($\bot$).

We assume that the causal relation $\mathcal{C}(\delta,\psi)$ is *contingent* in order to avoid a direct contradiction when treating it as necessary.

$$5'.\quad \Diamond\,\mathcal{C}\Big(\delta,\bigwedge_{j\in\mathbb{N}}\psi_j\Big)\;\wedge\;\Box\delta \implies \exists\omega\,\big[\neg\,\mathcal{C}\big(\delta,\bigwedge_j\psi_j\big)\big] =: \Psi^\delta \qquad \text{(from Modus Ponens)} \tag{7}$$

$$6'.\quad \Psi^\delta\;\wedge\;\big[\forall\omega\,v_\omega(\delta)=1\big] \implies \bot \qquad \text{(direct contradiction)} \tag{8}$$

Here,

$\Psi^\delta \;=\; \exists\omega\,\big[\neg\,\mathcal{C}\big(\delta,\wedge_j\psi_j\big)\big]$ asserts that in some possible world δ fails to cause the conjunction.

Contingency We postulate that $\mathcal{C}(\delta, \psi)$ is *not* necessary but contingent. In other words, there may exist accessible worlds in which δ does *not* cause ψ. This allows us to derive a genuine possibility of failure of the causal link.

Step 5'. From the possibility of the causal link $\Diamond\, \mathcal{C}\big(\delta, \bigwedge_j \psi_j\big)$ together with the necessity of the cause $\Box\delta$, we infer by Modus Ponens that there must exist at least one valuation ω in which $\mathcal{C}(\delta, \bigwedge_j \psi_j)$ *does not* hold. We denote this existential claim by Ψ^δ.

Step 6'. The conjunction $\Psi^\delta \wedge [\forall\omega\, v_\omega(\delta) = 1]$ asserts both that δ is true in every world and that in some world δ fails to exercise its purported causal power. These two assertions clash: a necessary cause cannot, in some world, cease to be a cause. Hence we derive a contradiction ($\bot$).

Conclusion. By assuming that the causal relation is necessary, we forced an inconsistency once we recognized that necessity blocks any counter-example world. Allowing the relation to be merely contingent restores coherence but underscores that no *necessary* cause can uniformly bring about an infinite conjunction of effects without collapsing into paradox.

Objection: One might object that in certain possible worlds the relation $\mathcal{C}(\delta, \psi)$ is meaningless, since in those worlds $v(\psi) = 0$ and hence ψ does not obtain. If ψ never occurs, it seems unintelligible to attribute to δ the power to cause ψ.

Defense: The reply is straightforward. If in a world $v(\psi) = 0$, then necessarily $v(\neg\psi) = 1$. Consequently, the causal claim $\mathcal{C}(\delta, \psi)$ remains meaningful, for one may equally consider $\mathcal{C}(\delta, \neg\psi)$. But if causality is taken to be *necessary* (rather than contingent), then δ would have to cause both ψ and $\neg\psi$ in every world, which is patently absurd. This contradiction shows that Step 5's assumption—namely, that the causal relation is a necessary one—cannot be sustained.

This argument furnishes a compelling motivation to reject the existence of a unique minimal event in the causal ordering of the spacetime manifold. Such a hypothesis inexorably gives rise to a logical contradiction within our modal-causal framework and, from the standpoint of epistemic modality[18], is thereby rendered physically impossible. We must consequently re-examine the issue through the formal apparatus of Einstein's General Theory of Relativity, wherein the global causal structure of spacetime is rigorously articulated.

3 Global Causal Structure and the First-Cause Paradox in General Relativity

Let (M, g_{ab}) be a smooth, time-oriented, globally hyperbolic Lorentzian manifold satisfying the strong energy condition. Then there exists no point $p \in M$ such that p is causally preceded by any other event $q \in M$ (i.e., $\nexists q \in M$ with $q \in I^-(p) \setminus \{p\}$).

Definition 3.1 (First Cause)**.** An event $p \in M$ is called a *first cause* if and only if its causal past is empty:

$$I^-(p) = \emptyset \tag{9}$$

where $I^-(p)$ denotes the chronological past of p.

Lemma 1. [15] Let (M, g_{ab}) be a time-orientable spacetime. Then there exists a smooth, nowhere-vanishing, timelike vector field t^a on M.

Theorem 1 (Local Geodesic Convexity[15])**.** Let (M, g_{ab}) be a spacetime manifold, and let $p \in M$ be an arbitrary point. Then:

1. There exists a convex normal neighborhood $U \subset M$ containing p (an open set where any two points are connected by a unique geodesic lying entirely within U).
2. The chronological future $I^+(p)|_U$ consists exactly of points reachable by future-directed timelike geodesics from p within U.
3. This future set $I^+(p)|_U$ is generated by future-directed null geodesics emanating from p and contained in U.

Sketch. The existence of convex normal neighborhoods follows from the smoothness of the metric g_{ab} and the inverse function theorem. The characterization of $I^+(p)|_U$ derives from the causal structure of Lorentzian manifolds [15]. □

Assuming p lies in the *interior* of M, Theorem 1 guarantees the existence of a convex normal neighborhood U containing p. Within this neighborhood. For any point $q \in I^-(p)|_U$, there exists a unique past-directed timelike geodesic connecting q to p [19]. If we hypothesize that p is a first cause (i.e., $I^-(p) = \emptyset$ globally), this leads to a contradiction because:

$$\exists q \in U \text{ such that } q \ll p \quad \text{(where } \ll \text{ denotes timelike precedence)}$$

Therefore, no spacetime point in the interior can be a first cause under these conditions since the existence of convex normal neighborhoods (Theorem 1) guarantees:

- Local causal structure where $\forall p \in \text{int}(M), \exists q \in U$ with $q \ll p$
- Global causality violations may still occur, as in spacetimes with topology $S^1 \times \mathbb{R}^3$ where:
  - Closed timelike curves exist (integral curves of $(\partial/\partial t)^a$)
  - $I^+(p) = I^-(p) = M$ for all $p \in M$

This demonstrates that while local causal structure prevents interior first causes ($I^-(p) \neq \emptyset$ in any neighborhood), global pathologies can further undermine any well-defined notion of causal precedence.

Let (M, g_{ab}) be a smooth, time-oriented, Hausdorff Lorentzian manifold, and let $U \subset M$ be a convex normal neighborhood of a point $p \in M$. By definition, for any two events $q, r \in U$ there exists a unique causal (non-spacelike) curve

$$\gamma\colon [t_1, t_2] \longrightarrow M, \qquad \gamma(t_1) = q,\ \gamma(t_2) = r, \quad \gamma([t_1, t_2]) \subset U, \quad t_1 < t_2,$$

which preserves the local causal ordering [20]. Assume, for contradiction, that there exists an *interior* point $p \in M$ satisfying

$$I^-(p) \ = \ \emptyset,$$

i.e. p has no chronological predecessors. We show below that this assumption is incompatible with the standard causal and differentiable structure of (M, g_{ab}).

Proof. **(1) Existence of local cones.** Since M is time-orientable and paracompact, Lemma [1] guarantees a smooth, nonvanishing, future-directed timelike vector field t^a on M [21]. In particular, in any sufficiently small neighborhood of p, there exist both future- and past-directed timelike curves through p. Hence one can always find events $q \neq p$ arbitrarily close to p with $q \in I^-(p)$, contradicting $I^-(p) = \emptyset$.

(2) Convex normal neighborhoods. By Theorem 1, p admits a convex normal neighborhood U. Within U, any two points are connected by a unique geodesic remaining entirely in U. Thus there must exist a past-directed timelike geodesic from some $q \in U$ to p, again contradicting $I^-(p) = \emptyset$ [22].

(3) Breakdown of global hyperbolicity. The only way to evade the preceding local contradictions is to place p on the boundary (or singular edge) of a maximal extension of M. But the Hawking–Penrose singularity theorems then imply past-timelike geodesic incompleteness under the strong energy condition—violating the hypothesis that M is a smooth, causally well-behaved manifold [23].

In every scenario, assuming an interior point with $I^-(p) = \emptyset$ leads to a collapse of the causal structure of (M, g_{ab}). Therefore, no such interior "first–cause" point can exist under the usual hypotheses of general relativity. □

Assume that a point p lies not in the interior of the spacetime manifold M, but rather on its boundary ∂M, such that it satisfies

$$I^-(p) = \emptyset. \tag{10}$$

That is, p has no chronological predecessors within M. While this assumption avoids the contradiction inherent in placing a first-cause event within the manifold's interior (as discussed previously), it leads instead to a more profound geometric and physical pathology—namely, *geodesic incompleteness*.

According to the Hawking–Penrose singularity theorems [21], a spacetime (M, g_{ab}) satisfying the following conditions:

(i) the strong energy condition (e.g., $R_{ab}v^a v^b \geq 0$ for all timelike vectors v^a),

(ii) the absence of closed timelike curves (causality),

(iii) global hyperbolicity (or at least the existence of a non-compact Cauchy surface),

(iv) the presence of a trapped surface or a suitable focusing condition, must be *geodesically incomplete*. That is, there exists at least one inextendible causal (typically timelike or null) geodesic $\gamma\colon [0, a) \to M$ for which $a < \infty$, yet γ cannot be extended beyond this parameter value. In particular, if $p \in \partial M$ and $I^-(p) = \emptyset$, then any past-directed timelike geodesic terminating at p must be incomplete:

$$\text{Length}(\gamma) < \infty, \quad \text{and} \quad \lim_{\tau \to a^-} \gamma(\tau) = p \in \partial M. \tag{11}$$

Physically, this implies that the manifold M is past-incomplete: spacetime does not continue “behind” p, and no physical observer (following a timelike worldline) can probe any earlier region beyond it. In this sense, spacetime “ends” at p, and p marks a genuine *singularity* or causal boundary of the universe.

It is important to emphasize that p is not part of the differentiable manifold M, and hence does not correspond to a well-defined event in the usual physical sense. As such, while $I^-(p) = \emptyset$ is formally possible on the boundary, it indicates a breakdown of classical general relativity and motivates the search for a more complete theory (e.g., quantum gravity) capable of resolving or regularizing such singularities.

4 Singularity Avoidance via Ordering Operator Constraints p

In classical general relativity, spacetime singularities are often unavoidable under typical energy and causality conditions, as shown by the Hawking–Penrose singularity theorems[21]. However, in the framework of quantum cosmology, particularly within the minisuperspace approximation, it has been proposed that singularities may be avoided due to quantum gravitational effects. One such model is explored in[24].

We consider a cosmological model characterized by a homogeneous, isotropic universe with closed spatial sections (i.e., an approximately spherical cosmological bubble). In the minisuperspace framework, the action reads:

$$S = \frac{1}{16\pi G} \int d^4x \, R \sqrt{-\det(g)} \tag{12}$$

with a Friedmann–Lemaître–Robertson–Walker (FLRW) metric given in natural units by

$$ds^2 = \frac{2G}{3\pi} \left[N^2(t) - a^2(t) d\Omega_3^2 \right] \tag{13}$$

where $N(t)$ is the lapse function and $a(t)$ is the scale factor.

The corresponding Lagrangian is obtained as

$$\mathcal{L} = \frac{1}{2} N(t) a(t) \left[k - \frac{\dot{a}^2(t)}{N^2(t)} \right] \tag{14}$$

with the canonical momentum

$$p_a = -\frac{a\dot{a}}{N}. \tag{15}$$

This yields the Hamiltonian:

$$\mathcal{H} = -\frac{1}{2} \left(\frac{p_a^2}{a} + ka \right). \tag{16}$$

Upon quantization, the momentum operator p_a is promoted to a differential operator. Due to operator ordering ambiguities, the quantized squared momentum becomes:

$$p_a^2 \to -a^{-p} \frac{d}{da} \left(a^p \frac{d}{da} \right) \tag{17}$$

where p represents the operator ordering ambiguity. The Wheeler–DeWitt equation (WDWE) then takes the form:

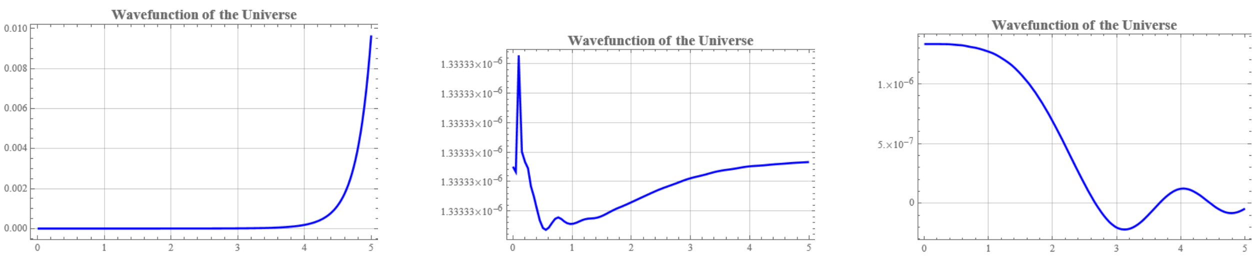
$$\left[-\frac{1}{a^p} \frac{d}{da} \left(a^p \frac{d}{da} \right) - ka^2 \right] \Psi(a) = 0 \tag{18}$$

To assess the presence of singularities, we analyze the behavior of the wavefunction $\Psi(a)$ as $a \to 0$. A "validity function" is defined as:

$$v(a) = \begin{cases} 0, & \text{if } \lim_{a \to 0} |\Psi(a)|^2 < \infty \\ 1, & \text{if } \lim_{a \to 0} |\Psi(a)|^2 \to \infty \end{cases} \tag{19}$$

If $v(a) = 0$, then the wavefunction remains regular at $a = 0$, indicating that the universe may emerge from a non-singular quantum state—supporting the idea of spontaneous creation "from nothing." This behavior depends critically on the choice of operator ordering p. For a range of operator ordering parameters, the wavefunction vanishes or remains finite as $a \to 0$, suggesting the absence of a big bang-type curvature singularity.

Thus, the application of operator ordering constraints in the Wheeler–DeWitt framework provides a viable pathway toward singularity avoidance, and challenges the inevitability of initial singularities predicted by classical general relativity.



(a) (b) (c)

Figure 1: Representative plots showing the real part of the Wavefunction of the Universe, $\mathrm{Re}\,\psi(a)$, as a function of the scale factor a for $p = 4$: (a) With $k = -1$, (b) With $k = 0$, and (c) With $k = 1$.

Conclusion

This paper has systematically investigated the question of whether spacetime is causally well-founded by integrating philosophical, logical, and physical perspectives. Our analysis began with a rigorous examination of well-founded relations in modal logic, where we demonstrated that the assumption of a global first cause leads to logical contradictions under contingent causal frameworks. Specifically, the necessity of an uncaused initial event clashes with the requirement that causal relations remain contingent, as formalized in Definitions 2.3 and 2.4. This philosophical argument was further supported by the metaphysical rejection of global causal well-foundedness, which highlighted the incompatibility of necessary causes with infinite causal chains.

Transitioning to the physical domain, we explored the implications of general relativity for spacetime's causal structure. Theorem 1 and its corollaries established that no interior point in a time-oriented Lorentzian manifold can serve as a first cause, as convex normal neighborhoods guarantee local causal predecessors for every event. This result was extended to boundary points, where the Hawking-Penrose singularity theorems (Section 3) revealed that past-incomplete geodesics and singularities are inevitable under classical energy conditions—further undermining the notion of a well-founded causal order.

In Section 4, we examined quantum cosmological alternatives, showing that operator-ordering constraints in the Wheeler-DeWitt equation may permit singularity-free solutions (Figure 1). While this suggests a potential mechanism for avoiding initial singularities, it does not restore causal well-foundedness; rather, it shifts the discussion to quantum indeterminacy and the emergence of spacetime from a non-singular quantum state.

In summary, our interdisciplinary approach yields three key conclusions: 1. **Logical**: Modal-causal frameworks preclude a metaphysically necessary first cause. 2. **Classical Physical**: General relativity prohibits interior first causes and predicts boundary singularities. 3. **Quantum Physical**: Quantum gravity models may circumvent singularities but do not imply causal well-foundedness.

These findings collectively challenge the hypothesis that spacetime is causally well-founded, while underscoring the need for deeper unification of modal metaphysics with quantum gravitational theories to resolve the paradoxes of cosmic origins.

5 Ethics Declaration

Not applicable.

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