A Proof of Riemann Hypothesis for Large |t|

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Abstract

In this article, it is proved that for large |t|, all the non-trivial zeros of the Riemann zeta function must lie on the critical line, as per Riemann hypothesis.

Keywords: Riemann zeta function, Riemann hypothesis, Non-trivial zeros, Critical line

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1. Key Equations and Fundamental Formulas

Riemann zeta function

The Riemann zeta function was originally defined as [1],

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}, \Re(s) > 1 \tag{1}$$

where $\Re(s)$ denotes the real part of s.

Analytic continuation

The Dirichlet eta function, also known as the alternating zeta function, is defined as [2],

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \qquad \Re(s) > 0 \tag{2}$$

The Dirichlet eta function is related to $\zeta(s)$ by,

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})} \eta(s), \qquad \Re(s) > 0 \tag{3}$$

where $s \neq 1 + \frac{2\pi k i}{\log(2)}, k = 0, \pm 1, \pm 2 \ldots$

This identity provides an analytic continuation of $\zeta(s)$ to the half-plane $\Re(s) > 0$ excluding $s = 1$.

As an extension to $\mathbb{C}$ as a meromorphic function, Riemann introduced the functional equation,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\,\zeta(1-s), \qquad s \in \mathbb{C} \setminus \{0, 1\} \tag{4}$$

Approximate formulas

Due to Theorem 4.11 in [3], for $0 < \sigma < 1, x > Ct/(2\pi), C > 1$,

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \tag{5}$$

Zeros of the Riemann zeta function

The trivial zeros of the Riemann zeta function occur at the negative even integers; that is, $\zeta(-2n) = 0, n \in \mathbb{N}$ [1] while the non-trivial zeros lie in the critical strip, $0 < \Re(s) < 1$ [4, 5]. It is verified that the Riemann Hypothesis is true until the 10^{13}-th zero [6]. The non-trivial zeros of $\zeta(s)$ are symmetric with respect to both the critical line and the real axis, that is $\zeta(s) = \zeta(1-s) = \zeta(\bar{s}) = \zeta(1-\bar{s}) = 0$. This can be proved using the functional equation (4) and the complex conjugation properties.

Poisson summation formula (PSF)

For a smooth, complex valued function which decays at infinity with all derivatives (Schwartz function) [7], we have,

$$\sum_{n \in \mathbb{Z}} f(n) e^{2\pi i n \beta} = \sum_{k \in \mathbb{Z}} \hat{f}(k - \beta), \qquad \beta \in \mathbb{R} \tag{6}$$

In particular,

$$\sum_{n \in \mathbb{Z}} (-1)^n f(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k - 1/2) \tag{7}$$

where the Fourier coefficients are given by,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx \tag{8}$$

Stationary phase method

For real functions $\theta(x)$ and $g(x)$, and the interval (a, b) along the real axis, if $\theta'(x_0) = 0$ and $\theta''(x_0) \neq 0$, then for large t [8, 9],

$$\int_a^b g(x)e^{it\theta(x)}dx \sim \sqrt{\frac{2\pi}{t|\theta''(x_0)|}}g(x_0)\exp\left[it\theta(x_0) + \frac{i\pi}{4}sgn(\theta''(x_0))\right] \tag{9}$$

where $sgn(\cdot)$ is the sign function.

Oscillatory integral bounds

Due to Lemma 4.3 in [3], for real functions $\theta(x), h(x)$ where $\theta'(x)/h(x)$ is monotonic through an interval $[a, b]$ and $\theta'(x)/h(x) \geq m > 0$, or $\theta'(x)/h(x) \leq -m < 0$, then

$$\left|\int_a^b h(x)e^{i\theta(x)}dx\right| \leq \frac{4}{m} \tag{10}$$

2. Riemann Hypothesis

All the non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = 1/2$.

Proof. Assume that $\zeta(s) = 0$, where $0 < \sigma \leq 1/2, t \gg 1$.

According to equation (2), $\eta(s)$ must be zero too. Let us write $\eta(s)$ as follows.

$$\eta(s) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -\sum_{n=1}^{\infty} \frac{\exp[in\pi]}{n^{\sigma+it}} = -\sum_{n=1}^{\infty} g(n)\exp[i\theta(n)] \tag{11}$$

where

$$\begin{aligned} g(x) &= \frac{1}{x^\sigma} \\ \theta(x) &= \pi x - t\log x \quad \Rightarrow \theta'(x) = \pi - \frac{t}{x}, \quad \theta''(x) = \frac{t}{x^2} \neq 0 \end{aligned} \tag{12}$$

So, there is a stationary point, $x_0 \in [1, \infty]$, which is,

$$\theta'(x) = \pi - \frac{t}{x} = 0 \Rightarrow x_0 = \frac{t}{\pi} \tag{13}$$

Let us isolate split the sum of $\eta(s)$ into two sums as follows,

$$\eta(s) = \sum_{n=1}^{j-1} \frac{(-1)^{n+1}}{n^s} + \sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{14}$$

where $j = \lfloor t/\pi \rfloor$. This ensures that $[j, \infty]$ encloses half of the stationary phase interval. Figure 1 shows the graphs of the real and imaginary parts of $(-1)^{n+1}/n^s$ where $s = \sigma + 10000i$ for different values of σ.

Let

$$f(x) = \frac{e^{-x/N}}{x^s}\mu(x) = \frac{1}{x^\sigma}\mu(x)e^{-it\log x} = g(x)\mu(x)e^{-it\log x} \tag{15}$$

where $e^{-x/N}, N \gg |t|$ is a damping factor to ensure rapidly decaying Fourier transform and hence convergence and $\mu(x)$ is a smooth bump function supported on $[j, \infty]$ (see Figure 2). It satisfies the following conditions,

i) $\mu(x) = 1$ for $x \in [j, \infty)$.

ii) $\mu(x)$ decays near-linearly from 1 at $x = j$ to 0 at $x = j - 1$.

iii) $\mu(x)$ is infinitely differentiable (C^∞).

iv) $\lim_{x \to j-1} \mu^{(m)}(x) = 0, m = 1, 2 \cdots$.

So, $f(x)$ lies in Schwartz space where PSF can be applied leading to,

$$\begin{aligned}\sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s} &\approx -\sum_{n\in\mathbb{Z}}(-1)^n f(n) = -\sum_{k\in\mathbb{Z}}\hat{f}(k-1/2)\\ &= -\int_{-\infty}^{\infty} e^{-x/N}g(x)\mu(x)\exp\left[i\pi(1-2k)x - it\log x\right]dx\end{aligned} \tag{16}$$

The phase is,

$$\theta(x) = \pi(1-2k)x - t\log x \quad \Rightarrow \theta'(x) = \pi(1-2k) - \frac{t}{x}, \quad \theta''(x) = \frac{t}{x^2} \neq 0 \tag{17}$$

The stationary points can be found as,

$$\theta'(x) = \pi(1-2k) - \frac{t}{x} = 0 \Rightarrow x_0 = \frac{t}{\pi(1-2k)} \tag{18}$$

So, the main contribution arises from the $k = 0$ term where $\hat{f}(-1/2)$ is the only coefficient that includes a stationary point which occurs at $x_0 = t/\pi \in [j-1, \infty)$.

The stationary phase method given by equation (9) can be generalized for our case as follows.

Let us expand the phase around the stationary point using Taylor series as,

$$\theta(x) \approx \theta(x_0) + \theta'(x_0)(x - x_0) + \frac{\theta''(x_0)}{2}(x - x_0)^2 \approx \theta(x_0) + \frac{\theta''(x_0)}{2}(x - x_0)^2 \tag{19}$$

Then, $\hat{f}\left(-\frac{1}{2}\right)$ can be written as,

$$\begin{aligned} \hat{f}\left(-\frac{1}{2}\right) &\approx \int_{j-1}^{\infty} e^{-x/N} g(x)\mu(x) \exp\left\{i\left[\theta(x_0) + \frac{\theta''(x_0)}{2}(x - x_0)^2\right]\right\} dx \\ &\approx e^{-x_0/N} g(x_0) e^{i\theta(x_0)} \int_{x_0}^{\infty} \exp\left[i\frac{\theta''(x_0)}{2}(x - x_0)^2\right] dx \\ &\approx \frac{1}{2} e^{-x_0/N} g(x_0) e^{i\theta(x_0)} \int_{-\infty}^{\infty} \exp\left[i\frac{\theta''(x_0)}{2}(x - x_0)^2\right] dx \end{aligned} \tag{20}$$

where $e^{-x/N} g(x)\mu(x) \approx e^{-x_0/N} g(x_0)\mu(x_0) = e^{-x_0/N} g(x_0)$ in the neighborhood of the stationary point and the $1/2$ factor introduced in the last step is due to the extension of the Gaussian integral to be from $-\infty$ including the left half of the stationary phase interval.

By evaluating the Gaussian integral [10],

$$\hat{f}\left(-\frac{1}{2}\right) \sim \frac{1}{2}\sqrt{\frac{2\pi}{|\theta''(x_0)|}} e^{-x_0/N} g(x_0) \exp\left[i\theta(x_0) + \frac{i\pi}{4} sgn(\theta''(x_0))\right] \tag{21}$$

Substituting by $g(x_0) \approx (t/\pi)^{-\sigma}, \theta(x_0) \approx t - t\log(t/\pi), \theta''(x_0) \approx \pi^2/t$, we get,

$$\hat{f}\left(-\frac{1}{2}\right) \sim \frac{\sqrt{2}}{2}\left(\frac{t}{\pi}\right)^{1/2-s} \exp\left[i\left(t + \pi/4\right)\right] e^{-t/(N\pi)} \tag{22}$$

Due to rapid oscillation, lack of stationary points and the small size of the integrands near the end points, the contributions of the other Fourier coefficients are negligible. This can be verified as follows.

For $k \neq 0$,

$$\hat{f}\left(k - \frac{1}{2}\right) = \int_{j-1}^{\infty} \frac{e^{-x/N}\mu(x)}{x^{\sigma}} e^{i\theta(x)} dx = \int_{j-1}^{\infty} \frac{e^{-x/N}\mu(x)}{ix^{\sigma}\theta'(x)} \frac{d}{dx}\left[e^{i\theta(x)}\right] dx \tag{23}$$

Integrating by parts,

$$\hat{f}\left(k-\frac{1}{2}\right) = \left[\frac{e^{-x/N}\mu(x)}{ix^{\sigma}\theta'(x)}e^{i\theta(x)}\right]_{j-1}^{\infty} + \int_{j-1}^{\infty}\left\{\frac{-\mu'(x)e^{-x/N}}{ix^{\sigma}\theta'(x)} + \mu(x)e^{-x/N}\left[\frac{\sigma}{ix^{\sigma+1}\theta'(x)} + \frac{\theta''(x)}{ix^{\sigma}\left[\theta'(x)\right]^2} + \frac{1}{iNx^{\sigma}\theta'(x)}\right]\right\}e^{i\theta(x)}dx$$
$$= \int_{j-1}^{\infty}\left\{\frac{-\mu'(x)e^{-x/N}}{ix^{\sigma}\theta'(x)} + \mu(x)e^{-x/N}\left[\frac{\sigma}{ix^{\sigma+1}\theta'(x)} + \frac{\theta''(x)}{ix^{\sigma}\left[\theta'(x)\right]^2} + \frac{1}{iNx^{\sigma}\theta'(x)}\right]\right\}e^{i\theta(x)}dx \tag{24}$$

since $\lim_{x\to j-1}\mu(x) = \lim_{x\to\infty} 1/x^{\sigma} = 0$

Let us split the above integral as follows,

$$\hat{f}\left(k-\frac{1}{2}\right) = \int_{j-1}^{j}\left\{\frac{-\mu'(x)e^{-x/N}}{ix^{\sigma}\theta'(x)} + \mu(x)e^{-x/N}\left[\frac{\sigma}{ix^{\sigma+1}\theta'(x)} + \frac{\theta''(x)}{ix^{\sigma}\left[\theta'(x)\right]^2} + \frac{1}{iNx^{\sigma}\theta'(x)}\right]\right\}e^{i\theta(x)}dx + \int_{j}^{\infty}\left\{\frac{-\mu'(x)e^{-x/N}}{ix^{\sigma}\theta'(x)} + \mu(x)e^{-x/N}\left[\frac{\sigma}{ix^{\sigma+1}\theta'(x)} + \frac{\theta''(x)}{ix^{\sigma}\left[\theta'(x)\right]^2} + \frac{1}{iNx^{\sigma}\theta'(x)}\right]\right\}e^{i\theta(x)}dx \tag{25}$$

Since $\mu(x) = 1, \mu'(x) = 0$ for $x > j$,

$$\hat{f}\left(k-\frac{1}{2}\right) = \int_{j-1}^{j}\frac{-\mu'(x)e^{-x/N}}{ix^{\sigma}\theta'(x)}e^{i\theta(x)}dx + \int_{j-1}^{j}\frac{\sigma\mu(x)e^{-x/N}}{ix^{\sigma+1}\theta'(x)}e^{i\theta(x)}dx + \int_{j-1}^{j}\frac{\theta''(x)\mu(x)e^{-x/N}}{ix^{\sigma}\left[\theta'(x)\right]^2}e^{i\theta(x)}dx + \int_{j-1}^{j}\frac{\mu(x)e^{-x/N}}{iNx^{\sigma}\theta'(x)}e^{i\theta(x)}dx + \int_{j}^{\infty}\frac{\sigma e^{-x/N}}{ix^{\sigma+1}\theta'(x)}e^{i\theta(x)}dx + \int_{j}^{\infty}\frac{\theta''(x)e^{-x/N}}{ix^{\sigma}\left[\theta'(x)\right]^2}e^{i\theta(x)}dx + \int_{j}^{\infty}\frac{e^{-x/N}}{iNx^{\sigma}\theta'(x)}e^{i\theta(x)}dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \tag{26}$$

For I_1,

$$|I_1| = \left|\int_{j-1}^{j}\frac{\mu'(x)e^{-x/N}}{x^{\sigma}\theta'(x)}e^{i\theta(x)}dx\right| = \left|\int_{j-1}^{j}h_1(x)e^{i\theta(x)}dx\right| \tag{27}$$

By approximating $\mu(x) \approx x - \lfloor t/\pi \rfloor + 1 + \delta$ for $x \in [j-1, j]$ where $0 < \delta \ll 1$ to avoid singularity at $j-1$ when dividing by $\mu(x)$, then $\mu'(x) \approx 1$ and we have,

$$\frac{\theta'(x)}{h_1(x)} = \frac{\theta'(x)}{\mu'(x)e^{-x/N}/[x^{\sigma}\theta'(x)]} \approx [\theta'(x)]^2 x^{\sigma} e^{x/N} \tag{28}$$

which is increasing and we have,

$$\frac{\theta'(x)}{h_1(x)} \geq \left[\pi(1-2k) - \frac{t}{\lfloor t/\pi \rfloor - 1}\right]^2 (\lfloor t/\pi \rfloor - 1)^\sigma e^{[\lfloor t/\pi \rfloor - 1]/N} \approx (-2\pi k)^2 \left(\frac{t}{\pi}\right)^\sigma e^{[\lfloor t/\pi \rfloor - 1]/N} > 0 \tag{29}$$

By applying equation (10),

$$|I_1| \lesssim \frac{4}{4\pi^2 k^2 (t/\pi)^\sigma e^{[\lfloor t/\pi \rfloor - 1]/N}} = \frac{t^{-\sigma}}{\pi^{2-\sigma} k^2 e^{[\lfloor t/\pi \rfloor - 1]/N}} \tag{30}$$

Similarly,

$$|I_2| = \left| \int_{j-1}^{j} \frac{\sigma \mu(x) e^{-x/N}}{x^{\sigma+1} \theta'(x)} e^{i\theta(x)} dx \right| = \left| \int_{j-1}^{j} h_2(x) e^{i\theta(x)} dx \right| \tag{31}$$

Using $\theta'(x) \approx (-2\pi k)$ for $x \in [j-1, j]$, let us check the monotonicity of $\theta'(x)/h_2(x)$

$$\frac{\theta'(x)}{h_2(x)} = \frac{x^{\sigma+1} [\theta'(x)]^2}{\sigma \mu(x) e^{-x/N}} \approx \frac{(-2\pi k)^2}{\sigma} \left[\frac{x^{\sigma+1} e^{x/N}}{x - \lfloor t/\pi \rfloor + 1 + \delta} \right] = \frac{4\pi^2 k^2}{\sigma} r_2(x) \tag{32}$$

where

$$r_2(x) \approx \frac{x^{\sigma+1} e^{x/N}}{x - \lfloor t/\pi \rfloor + 1 + \delta} \tag{33}$$

$$\begin{aligned} r_2'(x) &\approx e^{x/N} \frac{\{x^{\sigma+1}/N + (\sigma+1)x^\sigma\} [x - \lfloor t/\pi \rfloor + 1 + \delta] - x^{\sigma+1}}{[x - \lfloor t/\pi \rfloor + 1 + \delta]^2} \\ &\approx e^{x/N} x^\sigma \frac{\sigma x - (\sigma+1) [\lfloor t/\pi \rfloor - 1 - \delta]}{[x - \lfloor t/\pi \rfloor + 1 + \delta]^2} < 0 \text{ for } x \in [j-1, j] \end{aligned} \tag{34}$$

Therefore, $\theta'(x)/h_2(x)$ is decreasing, and we have,

$$\frac{\theta'(x)}{h_2(x)} \gtrsim e^{t/(N\pi)} \frac{4\pi^2 k^2}{\sigma(1+\delta)} \left(\frac{t}{\pi}\right)^{\sigma+1} > 0 \tag{35}$$

$$\Rightarrow |I_2| \lesssim \frac{4\sigma(1+\delta) e^{-t/(N\pi)}}{4\pi^2 k^2 (t/\pi)^{\sigma+1}} \approx O\left(\frac{t^{-\sigma-1}}{k^2}\right) e^{-t/(N\pi)} \tag{36}$$

For I_3,

$$\begin{aligned} |I_3| &= \left| \int_{j-1}^{j} \frac{\theta''(x) \mu(x) e^{-x/N}}{x^\sigma [\theta'(x)]^2} e^{i\theta(x)} dx \right| \leq \int_{j-1}^{j} \frac{t e^{-t/(N\pi)}}{x^{\sigma+2} (-2\pi k)^2} dx = \frac{t e^{-t/(N\pi)}}{4(\sigma+1)\pi^2 k^2} \left[\frac{-1}{x^{\sigma+1}} \right]_{j-1}^{j} \\ &\approx \frac{t e^{-t/(N\pi)}}{4(\sigma+1)\pi^2 k^2} \left[\frac{1}{(t/\pi - 1)^{\sigma+1}} - \frac{1}{(t/\pi)^{\sigma+1}} \right] \approx \frac{t e^{-t/(N\pi)}}{4(\sigma+1)\pi^2 k^2} \left(\frac{\pi}{t}\right)^{\sigma+1} \left[(\sigma+1)\frac{\pi}{t} + O\left(\frac{1}{t^2}\right) \right] \\ &\approx O\left(\frac{t^{-\sigma-1}}{k^2}\right) e^{-t/(N\pi)} \end{aligned} \tag{37}$$

For I_4,

$$|I_4| = \left| \int_{j-1}^{j} \frac{\mu(x)e^{-x/N}}{Nx^\sigma \theta'(x)} e^{i\theta(x)} dx \right| = \left| \int_{j-1}^{j} h_4(x) e^{i\theta(x)} dx \right| \tag{38}$$

Let us check the monotonicity of $\theta'(x)/h_4(x)$

$$\frac{\theta'(x)}{h_4(x)} = \frac{x^\sigma N[\theta'(x)]^2}{\mu(x)e^{-x/N}} \approx (-2\pi k)^2 N \left[\frac{x^\sigma e^{x/N}}{x - \lfloor t/\pi \rfloor + 1 + \delta} \right] = 4\pi^2 k^2 N r_4(x) \tag{39}$$

where

$$r_4(x) \approx \frac{x^\sigma e^{x/N}}{x - \lfloor t/\pi \rfloor + 1 + \delta} \tag{40}$$

$$\begin{aligned} r_4'(x) &\approx e^{x/N} \frac{\{x^\sigma/N + \sigma x^{\sigma-1}\} [x - \lfloor t/\pi \rfloor + 1 + \delta] - x^\sigma}{[x - \lfloor t/\pi \rfloor + 1 + \delta]^2} \\ &\approx e^{x/N} x^{\sigma-1} \frac{(\sigma - 1)x - \sigma [\lfloor t/\pi \rfloor - 1 - \delta]}{[x - \lfloor t/\pi \rfloor + 1 + \delta]^2} < 0 \text{ for } x \in [j-1, j] \end{aligned} \tag{41}$$

Therefore, $\theta'(x)/h_4(x)$ is decreasing, and we have,

$$\frac{\theta'(x)}{h_4(x)} \gtrsim e^{t/(N\pi)} \frac{4\pi^2 k^2 N}{(1+\delta)} \left(\frac{t}{\pi}\right)^\sigma > 0 \tag{42}$$

$$\Rightarrow |I_4| \lesssim \frac{4(1+\delta)e^{-t/(N\pi)}}{4\pi^2 k^2 N (t/\pi)^\sigma} \approx O\left(\frac{t^{-\sigma}}{k^2}\right) \frac{e^{-t/(N\pi)}}{N} \tag{43}$$

For I_5,

$$|I_5| = \left| \int_{j}^{\infty} \frac{\sigma e^{-x/N}}{x^{\sigma+1} \theta'(x)} e^{i\theta(x)} dx \right| = \left| \int_{j}^{\infty} h_5(x) e^{i\theta(x)} dx \right| \tag{44}$$

Let us check the monotonicity of $\theta'(x)/h_5(x)$

$$\frac{\theta'(x)}{h_5(x)} = \frac{[\theta'(x)]^2 x^{\sigma+1} e^{x/N}}{\sigma} \tag{45}$$

which is increasing, and we have,

$$\frac{\theta'(x)}{h_5(x)} \gtrsim \frac{4\pi^2 k^2}{\sigma} \left(\frac{t}{\pi}\right)^{\sigma+1} e^{t/(N\pi)} > 0 \tag{46}$$

$$\Rightarrow |I_5| \lesssim \frac{4\sigma e^{-t/(N\pi)}}{4\pi^2 k^2 (t/\pi)^{\sigma+1}} \approx O\left(\frac{t^{-\sigma-1}}{k^2}\right) e^{-t/(N\pi)} \tag{47}$$

For I_6,

$$|I_6| = \left| \int_j^\infty \frac{\theta''(x) e^{-x/N}}{x^\sigma \left[\theta'(x)\right]^2} e^{i\theta(x)} dx \right| \tag{48}$$

For $k < 0$, since $[\theta'(x)]^2 > (-2\pi k)^2$ and using Taylor series for $e^{-x/N}$, we have,

$$\begin{aligned} |I_6| &\leq \int_j^\infty \frac{t e^{-x/N}}{x^{\sigma+2} \left(-2\pi k\right)^2} dx = \frac{t}{4\pi^2 k^2} \int_j^\infty \frac{1}{x^{\sigma+2}} \left[1 + O\left(\frac{1}{N}\right)\right] dx \\ &= \frac{t}{\left(4\pi^2 k^2\right)\left(\sigma+1\right)} \left(\frac{t}{\pi}\right)^{-\sigma-1} + O\left(\frac{1}{N}\right) \end{aligned} \tag{49}$$

where interchanging the limit as $N \to \infty$ and integral can be applied since the following conditions of the dominated convergence theorem [11] are satisfied,

$$\begin{aligned} &\frac{e^{-x/N}}{x^{\sigma+2}} \to \frac{1}{x^{\sigma+2}} \text{ for } x \geq j, \\ &\frac{e^{-x/N}}{x^{\sigma+2}} \leq \frac{1}{x^{\sigma+2}}, \quad \int_j^\infty \frac{1}{x^{\sigma+2}} dx < \infty \end{aligned} \tag{50}$$

For $k > 0$, since $[\theta'(x)]^2 > [\pi(1-2k)]^2$ and similarly, we get

$$|I_6| \lesssim \frac{t}{[\pi(1-2k)]^2(\sigma+1)} \left(\frac{t}{\pi}\right)^{-\sigma-1} + O\left(\frac{1}{N}\right) \tag{51}$$

For I_7,

$$|I_7| = \left| \int_j^\infty \frac{e^{-x/N}}{N x^\sigma \theta'(x)} e^{i\theta(x)} dx \right| = \left| \int_j^\infty h_7(x) e^{i\theta(x)} dx \right| \tag{52}$$

Let us check the monotonicity of $\theta'(x)/h_7(x)$

$$\frac{\theta'(x)}{h_7(x)} = [\theta'(x)]^2 x^\sigma N e^{x/N} \tag{53}$$

which is increasing, and we have,

$$\frac{\theta'(x)}{h_7(x)} \gtrsim 4\pi^2 k^2 N \left(\frac{t}{\pi}\right)^\sigma e^{t/(N\pi)} > 0 \tag{54}$$

$$\Rightarrow |I_7| \lesssim \frac{4 e^{-t/(N\pi)}}{4\pi^2 k^2 N (t/\pi)^\sigma} \approx O\left(\frac{t^{-\sigma}}{k^2}\right) \frac{e^{-t/(N\pi)}}{N} \tag{55}$$

Since the following series [12] converge,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \sum_{k=1}^{\infty} \frac{1}{(1-2k)^2} = \frac{\pi^2}{8} \tag{56}$$

then

$$\left| \sum_{k \neq 0} \hat{f}\left(k - \frac{1}{2}\right) \right| \leq \sum_{k \neq 0} \left[|I_1| + |I_2| + |I_3| + |I_4| + |I_5| + |I_6| + |I_7| \right] \approx O\left(t^{-\sigma}\right) \tag{57}$$

Substituting by equations (22) and (57) in equation (16) as $N \to \infty$, we get,

$$\sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s} \sim -\frac{\sqrt{2}}{2} \left(\frac{t}{\pi}\right)^{1/2-s} \exp\left[i\left(t + \pi/4\right)\right] + O(t^{-\sigma}) \tag{58}$$

A numerical verification of this equation for different values of s is provided in Table 1 where $|Error|$ is measured as,

$$|Error| = \left| \frac{\sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s} - \left[-\frac{\sqrt{2}}{2} \left(\frac{t}{\pi}\right)^{1/2-s} \exp\left[i\left(t + \pi/4\right)\right] \right]}{\sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s}} \right| \tag{59}$$

Note that the exact sum can be evaluated as,

$$\sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) - \sum_{n=1}^{j-1} \frac{(-1)^{n+1}}{n^s} \tag{60}$$

However, round-off errors may influence the sum's computation, potentially leading to deviations from the exact value. Note also that $\zeta(0.5 + 10004.6794i) \approx \zeta(0.5 + 100005.5144i) \approx \zeta(0.5 + 1000003.3327i) \approx 0$.

For the first sum, assume that j is odd, then we have,

$$\sum_{n=1}^{j-1} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{j-1} \frac{1}{n^s} - 2 \sum_{n=1}^{(j-1)/2} \frac{1}{(2n)^s} = \sum_{n=1}^{j-1} \frac{1}{n^s} - 2^{1-s} \sum_{n=1}^{(j-1)/2} \frac{1}{n^s} \tag{61}$$

From equation (5), since $\zeta(s) = 0$,

$$\sum_{n=1}^{j-1} \frac{1}{n^s} = \frac{(j-1)^{1-s}}{1-s} + O\left((j-1)^{-\sigma}\right) \approx O(t^{-\sigma}) \tag{62}$$

and

$$\sum_{n=1}^{(j-1)/2} \frac{1}{n^s} = \sum_{n=1}^{(j+1)/2} \frac{1}{n^s} - \frac{1}{[(j+1)/2]^s} \approx \frac{((j+1)/2)^{1-s}}{1-s} + O\left(((j+1)/2)^{-\sigma}\right) - \frac{1}{[(j+1)/2]^s}$$
$$\approx O(t^{-\sigma}) \tag{63}$$

Similarly, if j is even, then we have,

$$\sum_{n=1}^{j-1} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{j} \frac{(-1)^{n+1}}{n^s} - \frac{(-1)^{j+1}}{j^s} = \sum_{n=1}^{j} \frac{1}{n^s} - 2\sum_{n=1}^{j/2} \frac{1}{(2n)^s} + O(t^{-\sigma})$$
$$= \sum_{n=1}^{j} \frac{1}{n^s} - 2^{1-s} \sum_{n=1}^{j/2} \frac{1}{n^s} + O(t^{-\sigma}) \tag{64}$$

By following similar steps as those for the odd j, it can be shown that it would be $O(t^{-\sigma})$ too.

Substituting by equations (58) and (62) in equation (14),

$$\eta(s) \sim -\frac{\sqrt{2}}{2} \left(\frac{t}{\pi}\right)^{1/2-s} \exp\left[i\left(t + \pi/4\right)\right] + O(t^{-\sigma}) \tag{65}$$

So, for $0 < \sigma \leq 1/2$, $\eta(s)$, and accordingly $\zeta(s)$, can vanish only if $\sigma = 1/2$, otherwise $\eta(s)$ will blow up due to the factor $t^{1/2-s}$ in the first term. Due to the symmetry of the non-trivial zeros of the zeta function about the critical line, $\zeta(s)$ cannot be zero for $1/2 < \sigma < 1$ too. A similar proof can be established for $t \ll -1$. Hence, all the non-trivial zeros of the zeta function for $|t| \gg 1$ must lie on the critical line, $\Re(s) = 1/2$. □

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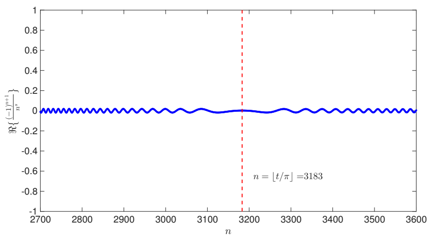
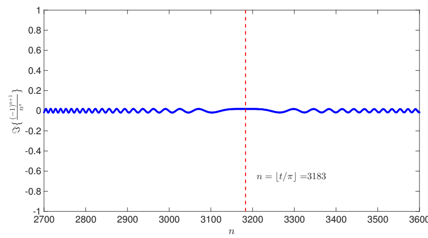
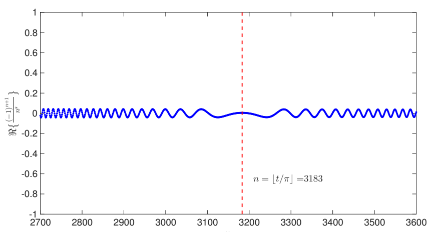


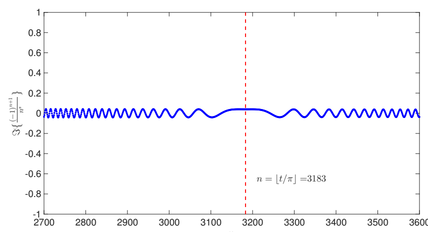
(a) $\Re\{\frac{(-1)^{n+1}}{n^s}\}, s = 0.5 + 10000i.$



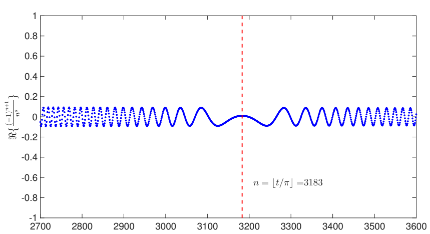
(b) $\Im\{\frac{(-1)^{n+1}}{n^s}\}, s = 0.5 + 10000i.$



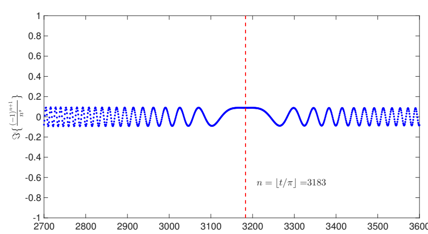
(c) $\Re\{\frac{(-1)^{n+1}}{n^s}\}, s = 0.4 + 10000i.$



(d) $\Im\{\frac{(-1)^{n+1}}{n^s}\}, s = 0.4 + 10000i.$



(e) $\Re\{\frac{(-1)^{n+1}}{n^s}\}, s = 0.3 + 10000i.$



(f) $\Im\{\frac{(-1)^{n+1}}{n^s}\}, s = 0.3 + 10000i.$

Figure 1: Graphs of the real and imaginary parts of $(-1)^{n+1}/n^s$ where $s = \sigma + 10000i$, $k = \lfloor 0.9t/\pi \rfloor$.

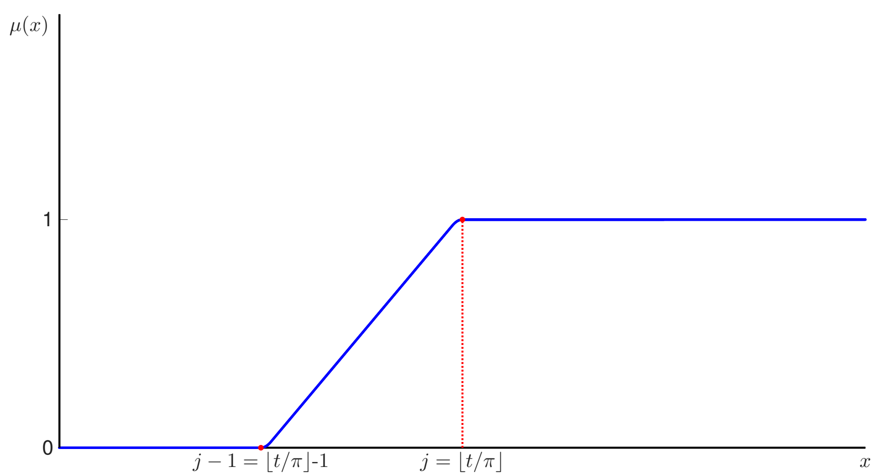


Figure 2: Graph of $\mu(x)$.

Table 1: Comparison between $\sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s}$ and its stationary phase approximation.

σ	t	$\left\lvert \sum_{n=j}^{\infty} \frac{(-1)^{n+1}}{n^s} \right\rvert$	$\frac{\sqrt{2}}{2} \left(\frac{t}{\pi}\right)^{\frac{1}{2}-\sigma}$	$\lvert Error \rvert$
0.1	10000	18.0549	17.8084	0.0155
	10001	18.1571	17.8091	0.0230
	10004.6794	18.2145	17.8117	0.0270
0.2	10000	8.0551	7.9495	0.0153
	10001	8.1006	7.9497	0.0229
	10004.6794	8.1260	7.9506	0.0269
0.3	10000	3.5937	3.5486	0.0151
	10001	3.6140	3.5487	0.0228
	10004.6794	3.6252	3.5489	0.0269
0.4	10000	1.6033	1.5841	0.0150
	10001	1.6124	1.5841	0.0227
	10004.6794	1.6173	1.5841	0.0268
0.5	10000	0.7153	0.7071	0.0149
	10001	0.7194	0.7071	0.0226
	10004.6794	0.7215	0.7071	0.0268
0.1	100000	45.1524	44.7327	0.0118
	100001	44.9807	44.7329	0.0065
	100005.5144	45.0916	44.7337	0.0099
0.2	100000	16.0073	15.8613	0.0118
	100001	15.9464	15.8614	0.0065
	100005.5144	15.9857	15.8616	0.0099
0.3	100000	5.6749	5.6241	0.0117
	100001	5.6533	5.6241	0.0064
	100005.5144	5.6672	5.6242	0.0098
0.4	100000	2.0119	1.9942	0.0117
	100001	2.0042	1.9942	0.0064
	100005.5144	2.0091	1.9942	0.0098
0.5	100000	0.7132	0.7071	0.0117
	100001	0.7105	0.7071	0.0064
	100005.5144	0.7123	0.7071	0.0098
0.1	1000000	112.6757	112.3634	0.0035
	1000001	112.5398	112.3635	0.0018
	1000003.3327	112.688	112.3636	0.0036
0.2	1000000	31.7337	31.6475	0.0035
	1000001	31.6954	31.6476	0.0018
	1000003.3327	31.7372	31.6476	0.0036
0.3	1000000	8.9374	8.9136	0.0035
	1000001	8.9266	8.9136	0.0018
	1000003.3327	8.9384	8.9136	0.0036
0.4	1000000	2.5171	2.5106	0.0035
	1000001	2.5141	2.5106	0.0018
	1000003.3327	2.5174	2.5106	0.0036
0.5	1000000	0.7089	0.7071	0.0035
	1000001	0.7081	0.7071	0.0018
	1000003.3327	0.709	0.7071	0.0036