

# Iterative Expansion of Generalized Mapping Theory and Description of Taylor Series Expansion

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## Abstract

Based on the quadruple framework (object set A, operation set F, result set B, and generation relation  $\vdash$ ) proposed in the Generalized Mapping Theory <sup>[1]</sup>, this paper makes a key extension to address the limitations of its recursive mechanism, constructing an enhanced generalized mapping framework for dynamic mathematical process modeling. Inspired by the ideas of "explicit operation set" in Chapter 2 and "multi-stage process modeling" in Chapter 4 of the preprint, this study innovatively introduces a state transfer mechanism to solve the core problem of original function information loss in Taylor series expansion. By modeling the Taylor expansion as a recursive sequence of operations, the new framework achieves three breakthroughs: first, it explicitly separates the derivative calculation rules from the state transfer path, extending the "behavior mathematization" concept of the preprint to the field of pure mathematics; second, it establishes a state-driven recursive paradigm that completely retains the function itself and expansion point parameters. Using Taylor expansion as a paradigmatic case, this study verifies the universal descriptive ability of the enhanced generalized mapping for continuous mathematical processes, providing a mathematical tool with both operation traceability and structural expressiveness for dynamic systems such as numerical solutions of differential equations and function approximation, further realizing the theoretical vision of "establishing a complete path of behavior-result generation relations" proposed in the preprint.

**Keywords:**Generalized Mapping Theory,Taylor Series,Iterative Expansion

## Introduction

Generalized function theories (such as Schwartz's distribution theory <sup>[1]</sup> and Colombeau algebra <sup>[2]</sup>) demonstrate excellent performance in describing static distribution problems, but they have fundamental limitations when dealing with generative relationships between dynamic behaviors and results. Taking the Taylor expansion of functions as an example: traditional methods only treat the expansion as a static expression, failing to explicitly model the operational process of increasing derivative orders and the mechanism for continuous access to the original function. This deficiency leads to the loss of key dynamic information (such as derivative calculation paths and truncation decision logic), essentially perpetuating the static paradigm of Schwartz's theory that "emphasizes results over processes" <sup>[2]</sup>.

In recent years, the Generalized Mapping Theory (GMT) proposed by Ling [1] has achieved explicit mathematization of behavioral processes for the first time through a quadruple framework (object set, operation set, result set, and generation relation). However, its recursive mechanism in pure mathematical processes (such as series expansion) is still imperfect, particularly lacking the ability to continuously access original data <sup>[3]</sup>.

Based on the framework of Generalized Mapping Theory (GMT), this paper proposes a formal modeling method for dynamic mathematical processes. Inspired by the quadruple architecture (object set A, operation set F, result set B, and generation relation  $\vdash$ ) proposed therein, we

reconstructed the recursive mechanism of GMT, introducing recursive iteration of the operation set  $F$  and an iteratively expanded form with invariant input set  $A$ .

Two key breakthroughs have been achieved:

1) Explicitization of dynamic processes: Each expansion term of the Taylor formula serves as an independent mapping output, with the operation set explicitly encoding derivative calculation rules, extending the multi-stage process modeling capability described in Chapter 4 of the preprint.

2) Dynamic update of the operation set: Define the generalized mapping iteration of the operation set  $F_k$ , is  $F_{k-1} \vdash F_k | f$ , which indicates that each iteration of the operation set is also described using generalized mapping, while the update rule  $f$  of the operation set remains invariant, thereby realizing explicit iteration of calculation rules.

This work verifies the universality of GMT in modeling pure mathematical processes, providing a mathematical representation tool with traceable operation paths for dynamic systems such as series expansion and numerical solutions of differential equations. It represents a further application expansion of GMT, and we welcome all colleagues to actively participate in related research.

The structure of the full text is as follows: Section 2 reviews the basic definitions of GMT; Section 3 analyzes the mathematical properties of Taylor expansion; Section 4 proposes an operation-driven recursive generalized mapping scheme; Section 5 takes the Taylor expansion of  $\sin x$  as a case to demonstrate how GMT completely retains the generation path from to ; Section 6 summarizes the theoretical value and looks forward to the future prospects of GMT; Section 7 lists the references.

## 2. Review of Generalized Mapping

There are two forms of expression in Generalized Mapping Theory as follows:

The first form of generalized mapping:  $A \vdash B | F$

The second form of generalized mapping:  $A \vdash B | F/P$

In the above expressions,  $A$  represents the object set, referring to the objects to be operated on, which can also be considered analogous to the original set in traditional set theory mappings.  $B$  denotes the result set, which is the outcome generated after  $A$  undergoes operations.  $F$  is the operation set, which can be either a traditional functional relationship or an operation rule. The second form is a probabilistic extension of the first form, and this paper does not involve operations of the second form.

## 3. Taylor's Formula and the Expansion Process of $\sin x$

### 3.1 Derivation Process of Taylor's Formula

Taylor's formula expands a smooth function into a power series form around a certain point, with its core idea being to approximate the function using a polynomial.

For a function that has  $n$ -th order derivatives at point  $x=a$ , its Taylor polynomial expansion is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where in:

$f^{(k)}(a)$  is  $f(x)$  at point  $x=a$   $k$ -th order derivative

$R_n(x)$  is the remainder term (The Lagrange remainder takes the form of

$$R_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x-a)^{n+1}, \quad \zeta \text{ between } a \text{ and } x$$

When  $a=0$ , It is called the Maclaurin formula (a special case of Taylor's formula)

### 3.2 *sinx* sample

#### 3.2.1 Find the values of the derivatives of all orders at $x=0$

$$f(x)=\sin x, \quad f(0)=0$$

$$f'(x)=\cos x, \quad f'(0)=1$$

$$f^{(2)}(x)=-\sin x, \quad f^{(2)}(0)=0$$

$$f^{(3)}(x)=-\cos x, \quad f^{(3)}(0)=-1$$

$$f^{(4)}(x)=\sin x, \quad f^{(4)}(0)=0$$

The derivatives show periodicity: they cycle repeatedly.

#### 3.2.2. Substituting into the Maclaurin formula

Retaining only the non-zero terms, we obtain:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

when  $n \rightarrow \infty$   $R_{2n+1}(x)$  go to 0, Therefore, the infinite series expansion is:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (-\infty < x < +\infty)$$

This expansion has important applications in fields such as calculating approximate values of and solving differential equations.

## 4. Recursive Expansion of Generalized Mapping Based on Operation Set

By observing the iterative general term of Taylor's formula, we find that each iterative general term can be modeled into the following form using generalized mapping:

$$A \vdash B|F$$

where A is the input set, containing  $f(x)$  and  $x=a$

B is the output set, and the output of the Nth term is:  $\frac{f^{(n)}(a)}{n!} (x-a)^n$ , Although B is iterated, it is

derived from A through operations of F, while A remains unchanged each time. Therefore, what is truly iterated each time is the operation set F.

Through observation, we find that the update of operation set F can also be described using generalized mapping, which can be expressed in the following form:

$$F_{k-1} \vdash F_k|f$$

$F_{k-1}$  is the previous operation set, which corresponds to the input set A in this special generalized mapping. Here, the current operation set corresponds to the set B in this special generalized mapping, while the operation rule f in this generalized mapping remains unchanged.

Up to this point, we have constructed a generalized mapping iteration method that can be used to describe Taylor expansion:

$$A \vdash B_k|F_k \quad (4-1)$$

$$F_{k-1} \vdash F_k|f \quad (4-2)$$

The iteration of these two generalized mappings can describe the Taylor expansion. Among them, the generalized mapping iteration method like (4-2) is called the rule-invariant self-iteration method. As the name implies, it means iterating with itself and keeping the operation rules unchanged. The iteration method of (4-1) is called the original set invariant iteration method,

which means that the original set A of each iteration is the same as the initial original set A without any changes. More specific iteration methods of generalized mapping will be further expanded in the future. We also welcome all colleagues to brainstorm and jointly enrich the theoretical system of generalized mapping.

### 5. Describing Taylor Expansion with Generalized Mapping

The two iterative formulas for Taylor expansion, namely (4-1) and (4-2), have been derived above. To describe Taylor expansion using these two iterative formulas, it can be expressed as follows:

$$A \vdash B_k | F_k \quad (4-1)$$

$$F_{k-1} \vdash F_k | f \quad (4-2)$$

Set A contain  $f(x)$  and  $x=a$

$F_1$ (original operation set) Describing Taylor Expansion with Generalized Mapping: It should be such an operation rule: take the first derivative of the function, take the first-degree term of  $(x-a)$ , and take the first factorial of the denominator.

The rule of  $f$  in (4-2) is to add 1, add 1 to each operation in the operation set of the  $(k-1)$ -th term. So far, the description of Taylor expansion by generalized mapping is completed.

As for the description of the Taylor expansion of  $\sin x$  by generalized mapping, there is no need to consider the specific expansion of  $\sin x$ . It is only necessary to substitute it into the iteration of generalized mapping and replace the initial set A.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$$

That is:

$$A \vdash B_k | F_k \quad (4-1)$$

$$F_{k-1} \vdash F_k | f \quad (4-2)$$

It is only necessary to change the elements in set A to  $\sin x$  and  $x=0$ ; everything else remains unchanged. With this, the description of the Taylor expansion of  $\sin x$  by generalized mapping is completed. This is the charm of generalized mapping: once the mapping structure is formed, similar structures can be described simply by updating the contents within the set.

### 6. Summary

Given the static limitations of traditional generalized function theory in describing dynamic mathematical processes—specifically, its overemphasis on final results while neglecting operational paths (such as the recursive differentiation mechanism in Taylor expansions)—this paper proposes a dual-iteration enhancement framework based on Generalized Mapping Theory (GMT). The core innovation lies in reconstructing the GMT quadruple architecture:

Original set invariant iteration mechanism: The input set retains the original function and expansion point parameters completely, ensuring that the function's intrinsic information is not lost during iteration.

Operation set self-iteration mechanism: Dynamically updates the operation set through fixed rules (such as increasing derivative orders), explicitly encoding the transmission path of calculation rules.

This framework decomposes Taylor expansion into traceable recursive operation sequences, breakthroughly solving the problem of dynamic modeling for purely mathematical processes. Taking the expansion of the sine function at zero as a case study:

The initial input set contains the sine function and zero-point parameters.

The first-step operation set is defined as calculating the first derivative, first power term, and factorial denominator.

Higher-order operation rules are generated through self-iteration of the operation set (e.g., the second-order operation corresponds to calculating the second derivative, second power term, and denominator factorial).

Each output term is generated jointly by the original input set and the current operation set.

This work achieves three theoretical contributions:

Establishing for the first time a complete operational path tracing mechanism for mathematical recursive processes.

Providing a reusable dynamic modeling framework that can be applied to the expansion of any smooth function by simply replacing the input function.

Offering a mathematical tool with both structural expressiveness and operational transparency for dynamic systems such as numerical solutions of differential equations.

The framework verifies the universality of GMT theory in modeling purely mathematical processes and opens up new paths for formal analysis of complex system behaviors. However, there are still many areas in generalized mapping theory that need exploration, and colleagues are warmly invited to actively participate.

## References

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