Tensor Extension of Generalized Mapping and Conversion Between Its Two Forms

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Abstract
Traditional generalized function theories (such as Schwartz's distribution theory and Colombeau algebra) have limitations in describing certain phenomena [1]. To address this issue, the Generalized Mapping theory is proposed, which realizes the mathematical modeling of dynamic behavior-result correlations through a four-element framework consisting of an object set A, an operation set F, a result set B, and a generative relation $\vdash$. However, when the Generalized Mapping theory was initially proposed, its two forms were formulated and defined using sets. Although set-based definitions can describe the basic logic of the theory, they are insufficient in practical applications, especially in computer simulations. Since vector, matrix, and even tensor computations are frequently required (despite sets being occasionally treatable as vectors), it is necessary to theoretically extend the Generalized Mapping theory to tensors. Additionally, there are scenarios where conversion between its two forms is needed. Therefore, this paper presents the tensor extension of the Generalized Mapping theory and discusses the theoretical conditions required for converting between its two forms.

Keywords: Generalized Mapping theory; generalized functions; tensor computation

Introduction

Traditional generalized function theories, such as Schwartz's distribution theory and Colombeau algebra, have provided essential tools for addressing problems like singular integrals and differential equations. However, they exhibit limitations in describing the generative relationship between dynamic behaviors and outcomes, making it difficult to accurately model the dynamic correlation phenomena between behaviors and results [1]. To tackle this issue, the Generalized Mapping theory has been proposed, which achieves systematic mathematical modeling of dynamic behavior-result correlations through a four-element framework comprising an object set A, an operation set F, a result set B, and a generative relation$\vdash$.

The Generalized Mapping theory was initially defined in two core forms using set theory: deterministic mapping (without considering probabilistic effects) and probabilistic mapping (incorporating probabilistic regulation), which clearly illustrate its fundamental logic. Nevertheless, in practical applications—particularly in computer simulations—set-based definitions cannot directly support efficient computations at the vector, matrix, or tensor levels. This has restricted the theory's in-depth application in fields such as multimodal signal processing and time-varying system modeling [2].

In light of this, extending the Generalized Mapping theory to the tensor domain (upgrading set-based formulations to tensor forms) and establishing a conversion mechanism between its two mapping forms have become critical for the theory's practical implementation. Building on the matrix extension of Generalized Mapping, this paper further advances theoretical extension at the tensor level and systematically explores the conversion conditions between deterministic and probabilistic mappings, aiming to provide more robust mathematical support for the theory's application in modeling complex dynamic systems.

2. Review of Generalized Mapping

The Generalized Mapping theory exists in two forms:

The first form of Generalized Mapping: $A \vdash B|F$

The second form of Generalized Mapping: $A \vdash B|F/P$

In the above expressions, A denotes the object set, referring to the objects to be operated on, which can be analogized to the original set in traditional set theory mappings. B represents the result set, i.e., the outcomes generated after operating on A. F is the operation set, which can be a traditional functional relationship or an operation rule.

The second form of Generalized Mapping introduces P as the probability set to account for probabilistic influences. For example, in some multimodal artificial intelligence systems, the final output may depend on Module 1 with a probability of 0.3 and Module 2 with a probability of 0.7 [3].

However, in practical applications, it has been found that in various computer simulation calculations using Generalized Mapping, programs actually extend A, B, and P into tensors. F, sometimes as an operation rule, also needs to be designed as a tensor. Therefore, it is necessary to theoretically carry out tensor extension for the Generalized Mapping theory. Since matrices are a special form of tensors [4], it is first necessary to extend the Generalized Mapping theory using matrices.

3. Matrix Extension of Generalized Mapping

Since the Generalized Mapping theory has two forms, the matrix extension of Generalized Mapping is discussed in two forms as follows. First, we discuss the first form.

The matrix extension of the first form of Generalized Mapping ($A \vdash B|F$) is divided into two cases. In the first case, F cannot be regarded as a matrix. In such scenarios, only A or B can be extended into a matrix. Examples of this exist, such as treating A as a square matrix and B as the determinant of the square matrix A. Here, F serves merely as a specification or definition.

In the second case, F can be regarded as a matrix. In this situation, F, as a matrix-based representation of a rule, can undergo matrix multiplication with the matrix A. That is, the first form of Generalized Mapping degenerates into a matrix multiplication form:

$$B=F \cdot A$$

Next, we consider the matrix extension of the second form of Generalized Mapping: ($A \vdash B|F/P$). Similar to the first form, the second form also has two cases. The first case, where F cannot be represented as a matrix but acts as a rule, is not discussed here. We focus on the second case where F must be represented as a matrix. In this scenario, P becomes a vector, expressed as:

$$B=p1(F_1 \cdot A)+p2(F_2 \cdot A)+\text{---}+pn(F_n \cdot A) \quad \text{(where n is a natural number)}$$

Here, F_1 to F_n are vectors derived from the decomposition of the operation matrix F, and p1to pn are probability constants, which together form the probability matrix.Thus, the second form of Generalized Mapping can be written as:

$$B=[p1,p2,\text{---},pn]F \cdot A$$

4. Tensor Extension of Generalized Mapping

Before specifically discussing the tensor extension of Generalized Mapping, we consider a concrete example: video processing (extracting frequency-domain features of batch video frames using DFT). In this case, the Fast Fourier Transform (DFT) algorithm is employed, where F can be regarded as the DFT matrix, and A as the set of video frames. This example can be simply expressed in the following mathematical form:

$$F=DFT\ matrix, A=\begin{bmatrix} Video\ frame_1 \\ \vdots \\ Video\ frame_n \end{bmatrix}, B=Frequency\text{-}domain\ features$$

In this example, each video frame is in matrix form, and even some video frames (such as color videos) are inherently tensors, represented as multi-dimensional arrays in programs. At this point, describing A as a set is no longer appropriate. Although describing it as a set is logically sound in mathematical terms, treating A as a tensor for operations has become an unavoidable reality in practical computer simulations. Therefore, tensor extension of Generalized Mapping is imperative. The following uniformly presents the tensor extension for the two forms of Generalized Mapping. Since matrices are a special case of tensors, and vectors, sets, and point sets are all special forms of tensors, describing Generalized Mapping using tensors is natural:

The first form of Generalized Mapping: $A \vdash B|F$

The second form of Generalized Mapping: $A \vdash B|F/P$

The following provides a unified tensor definition, extending A, F, P, and B into tensors: the original set tensor A, operation tensor F, probability tensor P, and result tensor B, respectively.

Original set tensor: $A \in R^{d_1 \times \cdots \times d_p}$

Operation tensor: $F \in R^{k \times m \times n \times r \times \cdots}$ (F may be an operation rule, but even rules such as limit calculation and integral can be regarded as special cases of tensors)

Probability tensor: $P \in R^{c_1 \times \cdots \times c_q}$

Result tensor: $B \in R^{k_1 \times \cdots \times k_p}$

This concludes the tensor extension of Generalized Mapping. Their specific computational scenarios depend on the description of Generalized Mapping obtained from practical situations. Thus, a preliminary theoretical foundation for the Generalized Mapping theory in computer simulation calculations is established. Further theoretical exploration of the tensors in Generalized Mapping is welcome for colleagues to supplement collectively.

5. Conversion Between the Two Forms of Generalized Mapping

In certain scenarios, the two forms of Generalized Mapping can be converted into each other. First, an implicit condition of the second form of Generalized Mapping is that the sum of the probability tensor (vector) P is 1, i.e., the sum of each matrix component is 1. In the case of a vector, this can be expressed as:

$$p1+p2+\cdots+pn=1$$

At this point, the second form of Generalized Mapping cannot be directly converted into the first form because the respective components in the operation tensor F of the second form ($A \vdash B|F/P$) cannot necessarily be merged.

Still using the video processing example for illustration, assume that a batch of video frames undergoes DFT matrix operation, filtering matrix operation, and extraction matrix operation with certain probabilities. Although the sum of the probability vector ((P = [p1, p2, p3])) is 1, the DFT matrix, filtering matrix, and extraction matrix are different operations that cannot be merged. Hence, conversion into the first form is impossible. Only when the DFT matrix, filtering matrix, and extraction matrix are the same type of operation or can be expressed as a unified operation through certain means can the second form of Generalized Mapping be converted into the first form. Essentially, the first form of Generalized Mapping is a special case of the second form, i.e., the case without probabilistic branches.

$$\mathrm{F}=[\textit{DFT matrix},\textit{filtering matrix},\textit{extraction matrix}],\mathrm{A}=\begin{bmatrix}\textit{Video frame}_1\\ \vdots \\ \textit{Video frame}_\mathrm{n}\end{bmatrix},$$

$$\mathrm{B}=\textit{Frequency-domain features},\quad P=[p1,p2,p3]$$

Thus, through the video processing example, the conversion between the two forms of Generalized Mapping and their relationship are clearly explained.

6. Conclusion

The proposal of the Generalized Mapping theory holds significant theoretical and practical importance. It breaks through the limitations of traditional generalized function theories in describing the generative relationship between dynamic behaviors and outcomes. Through the four-element framework of object set, operation set, result set, and generative relation, it provides a new perspective for the mathematical modeling of dynamic behavior-result correlations.

This paper conducts in-depth research on the practical extension of the Generalized Mapping theory and achieves notable results. Based on a review of the two basic forms of Generalized Mapping (deterministic and probabilistic mappings), it systematically completes the matrix and tensor extensions of the theory, upgrading the set-level formulation to matrix and tensor forms that are more suitable for computer simulations. This lays a mathematical foundation for the application of the theory in practical scenarios such as multimodal signal processing and video analysis. Additionally, through specific examples, it clearly elaborates on the conversion conditions between the two mapping forms and reveals the intrinsic relationship where deterministic mapping is a special case of probabilistic mapping.

Currently, the Generalized Mapping theory has begun to be applied in practical fields such as computer simulation calculations. However, further in-depth theoretical support is still needed in areas such as complex dynamic system modeling and high-dimensional tensor operation optimization. In the future, it is expected that more colleagues will join the research to collectively enrich the Generalized Mapping theory system, expand its application boundaries in interdisciplinary fields such as artificial intelligence and time-varying system analysis, and promote the theory's continuous progress from basic modeling to practical implementation.

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