On the Classical Explicit Formula for Bernoulli Numbers

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Abstract

This paper presents the proof of the classical explicit formula for Bernoulli numbers, expressed as a sum involving Stirling numbers of the second kind. The approach follows a combinatorial and polynomial comparison method similar to that used by Maurice d'Ocagne. Starting from the explicit formula of Stirling numbers and using known relations with falling factorials, we derive the closed-form expression systematically.

Introduction

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are a sequence of rational numbers defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

The first few Bernoulli numbers are:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots$$

These numbers play important roles in number theory, analysis, and combinatorics, particularly in expressing sums of powers of consecutive integers in closed form.

The first person who seems to have obtained an explicit expression for the Bernoulli numbers is Worpitzky [3]. Several explicit formulas for Bernoulli numbers appeared subsequently.

The approach followed in this paper is the same as that in reference [2] by d'Ocagne, where a classical treatment of Bernoulli numbers is presented. We also make use of techniques introduced by Grünert [1], connecting summation methods with Bernoulli numbers.

1 Grunert's Operational Formula

Define recursively the operator ϑ^n acting on a sufficiently differentiable function $Y = Y(x)$ by

$$\vartheta^n Y = x \frac{d}{dx} \left(\vartheta^{n-1} Y \right),$$

with the initial condition $\vartheta^0 Y = Y$. Explicit calculations for low values of n yield

$$\begin{aligned}
\vartheta Y &= xY', \\
\vartheta^2 Y &= xY' + x^2 Y'', \\
\vartheta^3 Y &= xY' + 3x^2 Y'' + x^3 Y^{(3)}, \\
\vartheta^4 Y &= xY' + 7x^2 Y'' + 6x^3 Y^{(3)} + x^4 Y^{(4)}.
\end{aligned}$$

By induction, it follows that

$$\vartheta^n Y = \sum_{k=0}^{n} S(n,k)\, x^k\, Y^{(k)}, \tag{1}$$

where $S(n,k)$ are the Stirling numbers of the second kind. This identity is known as *Grunert's operational formula.*

1.1 Exponential Function and Explicit Formula for $S(n,k)$

Let us now consider the exponential function $Y(x) = e^x$. Since all derivatives satisfy $Y^{(k)} = e^x$, Grunert's formula (1) becomes

$$\vartheta^n e^x = e^x \sum_{k=0}^{n} S(n,k) x^k,$$

or equivalently,

$$e^{-x}\, \vartheta^n e^x = \sum_{k=0}^{n} S(n,k) x^k.$$

Expanding e^x and e^{-x} into their power series, we obtain

$$\vartheta^n e^x = \sum_{i=0}^{\infty} \frac{\vartheta^n x^i}{i!}, \qquad e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}.$$

Since

$$\vartheta^n x^i = i^n x^i,$$

it follows that

$$\vartheta^n e^x = \sum_{i=0}^{\infty} \frac{i^n x^i}{i!}.$$

Therefore,

$$e^{-x}\, \vartheta^n e^x = \left(\sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!} \right) \left(\sum_{i=0}^{\infty} \frac{i^n x^i}{i!} \right).$$

Multiplying these two series via the Cauchy product gives

$$\sum_{k=0}^{\infty} \left(\frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \right) x^k.$$

Comparing this with the right-hand side

$$\sum_{k=0}^{n} S(n,k)x^k,$$

and noting that the latter is a polynomial of degree n, we obtain the explicit formula

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n, \qquad k = 0, 1, \ldots, n. \tag{2}$$

2 From Stirling Numbers to Bernoulli Numbers

2.1 Binomial Inversion and Falling Factorials

From the explicit formula for Stirling numbers of the second kind obtained above, we may apply the *Pascal's inversion formula* to deduce the identity

$$k^n = \sum_{i=0}^{k} S(n,i)(k)_i, \qquad k \in \{0, 1, \ldots, n\}, \tag{3}$$

where $(x)_i = x(x-1)\cdots(x-i+1)$ denotes the falling factorial.

2.2 Summing Over k

Summing both sides of (3) over k from 0 to n yields

$$\begin{aligned}\sum_{k=0}^{n} k^n &= \sum_{k=0}^{n}\sum_{i=0}^{k} S(n,i)(k)_i \\ &= \sum_{i=0}^{n} S(n,i)\sum_{k=i}^{n}(k)_i. \end{aligned} \tag{4}$$

The inner sum is a standard combinatorial identity:

$$\sum_{k=i}^{n}(k)_i = \frac{(n+1)_{i+1} - (i)_{i+1}}{i+1}.$$

Since $(i)_{i+1} = 0$, this simplifies to

$$\sum_{k=i}^{n}(k)_i = \frac{(n+1)_{i+1}}{i+1}.$$

Thus, (4) becomes

$$\sum_{k=0}^{n} k^n = \sum_{i=0}^{n} S(n,i)\frac{(n+1)_{i+1}}{i+1}. \tag{5}$$

2.3 Polynomial Expansion

The falling factorial can be expanded as a polynomial in $(n+1)$:

$$(n+1)_{i+1} = \sum_{j=0}^{i} c_j (n+1)^{j+1},$$

for suitable coefficients c_j. Substituting this into (5), we find

$$\begin{aligned}\sum_{k=0}^{n} k^n &= \sum_{i=0}^{n} S(n,i)\,\frac{1}{i+1}\sum_{j=0}^{i} c_j (n+1)^{j+1} \\ &= \sum_{j=0}^{n}\left(\sum_{i=j}^{n} S(n,i)\,\frac{c_j}{i+1}\right)(n+1)^{j+1}.\end{aligned}$$

2.4 Comparison with Faulhaber's Formula

On the other hand, Faulhaber's classical formula for sums of powers states:

$$\sum_{k=0}^{n} k^n = \sum_{j=0}^{n}\left(\frac{\binom{n+1}{j+1}}{n+1} B_{n-j}\right)(n+1)^{j+1}.$$

Comparing coefficients of like powers of $(n+1)$ in the two expansions, we obtain

$$\frac{\binom{n+1}{1}}{n+1} B_n = \sum_{i=0}^{n} S(n,i)\frac{c_0}{i+1}.$$

2.5 Explicit Formula for Bernoulli Numbers

We known that $c_0 = (-1)^i i!$, and hence we arrive at

$$B_n = \sum_{i=0}^{n} S(n,i)\frac{(-1)^i i!}{i+1}. \tag{6}$$

Finally, substituting the explicit formula

$$i!S(n,i) = \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} j^n,$$

into (6), we obtain the celebrated closed form:

$$B_n = \sum_{i=0}^{n} \frac{1}{i+1}\sum_{j=0}^{i}\binom{i}{j}(-1)^j j^n.$$

This completes the derivation.

3 Conclusion

This paper demonstrated how the operator $\vartheta = x\frac{d}{dx}$ and Grunert's operational formula provide a powerful tool to unify the treatment of Stirling numbers, Bernoulli numbers, and sums of powers. The explicit connections obtained highlight the rich interplay between combinatorics, analysis, and number theory.

References

[1] Johann A. Grünert, *Über die Summation der Reihen ...*, Journal für die reine und angewandte Mathematik, vol. 25, 1843, pp. 240–279.

[2] Maurice d'Ocagne, *Sur Les Nombres de Bernoulli*, Bulletin de la Société Mathématique de France, 1889, pp. 107–109.

[3] J. Worpitzky, *Studien über die Bernoullischen und Eulerschen Zahlen*, Journal für die reine und angewandte Mathematik, vol. 94, 1883, pp. 203–232.