On the Classical Explicit Formula for Bernoulli Numbers

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Abstract

This paper presents the proof of the classical explicit formula for Bernoulli numbers, expressed as a sum involving Stirling numbers of the second kind. The approach follows a combinatorial and polynomial comparison method similar to that used by Maurice d'Ocagne. Starting from the generating function of Bernoulli polynomials and using known relations with falling factorials, we derive the closed-form expression systematically.

Introduction

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are a sequence of rational numbers defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

The first few Bernoulli numbers are:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots$$

These numbers play important roles in number theory, analysis, and combinatorics, particularly in expressing sums of powers of consecutive integers in closed form.

The first person who seems to have obtained an explicit expression for the Bernoulli numbers is Worpitzky [3]. Several explicit formulas for Bernoulli numbers appeared subsequently.

The approach followed in this paper is the same as that in reference [2] by d'Ocagne, where a classical treatment of Bernoulli numbers is presented. We also make use of techniques introduced by Grünert [1], connecting summation methods with Bernoulli numbers.

1 Grunert's Operational Formula

Define recursively the operator ϑ^n acting on a sufficiently differentiable function $Y = Y(x)$ by

$$\vartheta^n Y = x \frac{d}{dx} \left(\vartheta^{n-1} Y \right),$$

with the initial condition $\vartheta^0 Y = Y$. Explicit calculations for low values of n yield

$$\begin{aligned}
\vartheta Y &= xY', \\
\vartheta^2 Y &= xY' + x^2 Y'', \\
\vartheta^3 Y &= xY' + 3x^2 Y'' + x^3 Y^{(3)}, \\
\vartheta^4 Y &= xY' + 7x^2 Y'' + 6x^3 Y^{(3)} + x^4 Y^{(4)}.
\end{aligned}$$

By induction, it follows that

$$\vartheta^n Y = \sum_{k=0}^{n} S(n,k)\, x^k\, Y^{(k)}, \tag{1}$$

where $S(n,k)$ are the Stirling numbers of the second kind. This identity is known as *Grunert's operational formula.*

2 Applications to Exponential and Power Functions

2.1 Exponential Function

Setting $Y = e^x$ and noting that all derivatives satisfy $Y^{(k)} = e^x$, formula (1) simplifies to

$$\vartheta^n e^x = e^x \sum_{k=0}^{n} S(n,k) x^k,$$

which gives

$$e^{-x} \cdot \vartheta^n e^x = \sum_{k=0}^{n} S(n,k) x^k.$$

Expanding e^x and e^{-x} into their power series:

$$\vartheta^n e^x = \sum_{m=0}^{\infty} \frac{\vartheta^n x^m}{m!}, \quad e^{-x} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!},$$

and using that

$$\vartheta^n x^m = m^n x^m,$$

we have:

$$\vartheta^n e^x = \sum_{m=0}^{\infty} \frac{m^n x^m}{m!}.$$

Therefore,

$$e^{-x} \cdot \vartheta^n e^x = \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!} \right) \left(\sum_{m=0}^{\infty} \frac{m^n x^m}{m!} \right).$$

Multiplying the two series using the Cauchy product gives:

$$\sum_{r=0}^{\infty} \left(\frac{1}{r!} \sum_{m=0}^{r} (-1)^{m-r} \binom{r}{m} m^n \right) x^r.$$

Comparing this with

$$\sum_{k=0}^{n} S(n,k)x^k,$$

and noting that the right-hand side is a polynomial of degree n, we identify:

$$S(n,r) = \frac{1}{r!}\sum_{m=0}^{r}(-1)^{m-r}\binom{r}{m}m^n, \quad r = 0, 1, \ldots, n.$$

2.2 Power Function

Consider $Y = x^y$. Using Grunert's formula,

$$\vartheta^n x^y = \sum_{k=0}^{n} S(n,k)x^k \frac{d^k}{dx^k}x^y,$$

and observing that

$$\frac{d^k}{dx^k}x^y = (y)_k x^{y-k},$$

where $(y)_k = y(y-1)\cdots(y-k+1)$ is the falling factorial, we obtain

$$\vartheta^n x^y = x^y \sum_{k=0}^{n} S(n,k)(y)_k.$$

Since $\vartheta x^y = yx^y$ implies $\vartheta^n x^y = y^n x^y$, it follows that

$$y^n = \sum_{k=0}^{n} S(n,k)(y)_k. \tag{2}$$

3 Sum of Powers and Falling Factorials

Summing both sides of (2) over $y = 0, 1, \ldots, m-1$ yields

$$\sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S(n,k)\sum_{y=0}^{m-1}(y)_k.$$

The inner sum can be evaluated using the telescoping property of falling factorials:

$$(y+1)_{k+1} - (y)_{k+1} = (k+1)(y)_k,$$

which leads to

$$\sum_{y=0}^{m-1}(y)_k = \frac{(m)_{k+1}}{k+1}.$$

Therefore,

$$\sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S(n,k)\frac{(m)_{k+1}}{k+1}. \tag{3}$$

4 Bernoulli Polynomials and Numbers

Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi.$$

Setting $x = 0$ yields the Bernoulli numbers $B_n = B_n(0)$:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

A key property of Bernoulli polynomials is the finite difference relation

$$B_{n+1}(y + 1) - B_{n+1}(y) = (n + 1)y^n.$$

Summing this relation over $x = 0, 1, \ldots, m - 1$, we obtain

$$B_{n+1}(m) - B_{n+1} = (n + 1)\sum_{y=0}^{m-1} y^n,$$

where $B_{n+1} = B_{n+1}(0)$.

Using (3) we rewrite the above as

$$B_{n+1}(m) - B_{n+1} = (n + 1)\sum_{k=0}^{n} S(n, k)\frac{(m)_{k+1}}{k + 1}. \tag{4}$$

5 Closed Formula for Bernoulli Numbers

Express the falling factorials $(m)_{k+1}$ as polynomials in m:

$$(m)_{k+1} = \sum_{j=0}^{k} c_j m^{j+1},$$

where the coefficients c_j are known integers depending on k.

Substituting into (4) and expanding the Bernoulli polynomials in powers of m,

$$B_{n+1}(m) = \sum_{r=0}^{n+1} \binom{n+1}{r} B_{n+1-r} m^r,$$

we compare coefficients of m^r on both sides to get the system

$$\binom{n+1}{r} B_{n+1-r} = (n + 1)\sum_{k=r-1}^{n} S(n, k)\frac{c_r}{k + 1},$$

For $r = 1$, this yields the explicit formula for the Bernoulli numbers:

$$B_n = \sum_{r=0}^{n} (-1)^r \frac{r!}{r+1} S(n,r),$$

where the coefficient $c_0 = (-1)^k k!$ arise from the falling factorial expansion.

Using the explicit formula for the Stirling numbers of the second kind, we rewrite the Bernoulli numbers in the double-sum closed form

$$B_n = \sum_{r=0}^{n} \frac{1}{r+1} \sum_{m=0}^{r} (-1)^m \binom{r}{m} m^n.$$

6 Conclusion

This paper demonstrated how the operator $\vartheta = x\frac{d}{dx}$ and Grunert's operational formula provide a powerful tool to unify the treatment of Stirling numbers, Bernoulli numbers, and sums of powers. The explicit connections obtained highlight the rich interplay between combinatorics, analysis, and number theory.

References

[1] Johann A. Grünert, *Über die Summation der Reihen ...*, Journal für die reine und angewandte Mathematik, vol. 25, 1843, pp. 240–279.

[2] Maurice d'Ocagne, *Sur Les Nombres de Bernoulli*, Bulletin de la Société Mathématique de France, 1889, pp. 107–109.

[3] J. Worpitzky, *Studien über die Bernoullischen und Eulerschen Zahlen*, Journal für die reine und angewandte Mathematik, vol. 94, 1883, pp. 203–232.