

Canonical Envelopes: A Mathematical Framework for Virtual Weighted Limits and Completion Theory

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Abstract

This work introduces canonical envelope theory as a mathematical framework for understanding completion phenomena in category theory and related mathematical disciplines. Building on Shulman’s and Riehl’s characterization of weighted limits through natural transformations $\theta : Q \Rightarrow C(D, E)$ [32; 33], we demonstrate that numerous completion constructions—including canonical extensions, topological compactifications, categorical completions, and constructions in algebraic geometry—can be understood as instances of factorization through virtual weighted (co)limiting structures.

Our main theoretical contribution identifies canonical envelopes as initial objects in categories of factorizations of appropriately constructed pairings. The (heuristic) classification $\theta = \text{id}$ versus $\theta \neq \text{id}$ distinguishes internal completion from external mediation. We establish existence criteria through bilateral denseness and compactness conditions, providing systematic construction procedures for a range of mathematical contexts.

The framework encompasses several major completion constructions through classification tables. Virtual weighted limits extend Gabriel-Ulmer methodology [9] from filtered/cofiltered diagrams to arbitrary weights, enabling systematic treatment of incomplete categorical frameworks. We introduce and develop “gem theory” as a systematic classification of mathematical structures according to their bilateral completion properties.

Key results include the pullback characterization showing that canonical interpolants are categorically determined, bilateral envelope structure capturing fundamental duality patterns, and systematic organization through a canonical envelope pseudomonad. The theory suggests deep categorical principles underlying mathematical completion while providing practical methodology for discovering canonical constructions in various mathematical contexts.

1 Introduction and Foundational Motivation

Historical Remark 1.1 (Historical Context: The Evolution of Completion Theory). The systematic study of completion constructions (even though we may not think of all of this work as studying completion per se) has evolved through several distinct phases:

- **1930s-1940s:** Stone’s representation theory for Boolean algebras and topological spaces [36]
- **1950s-1960s:** Jónsson-Tarski canonical extensions [18], Grothendieck’s categorical foundations [13]
- **1970s-1990s:** Gabriel-Ulmer Ind-Pro theory [9], Kelly’s enriched category theory [20]
- **2000s-Present:** Garner’s cylinder factorizations [11], Riehl’s weighted limit theory [32; 33], Schoots’s categorical extensions [35], Rosebrugh-Wood factorization systems [34]

Each phase contributed essential insights while developing domain-specific methods without systematic unification.

1.1 Completion Theory Across Mathematical Domains

Mathematical structures frequently admit systematic completion or extension constructions that preserve essential properties while adding new structure. These completion phenomena appear across diverse mathematical domains with distinct methods and existence criteria:

Topology: Stone-Čech compactification [36], sobrification [17], Alexandroff one-point compactification [2], Wallman compactification [39], each with domain-specific existence conditions and construction methods.

Algebra: MacNeille completions [27], canonical extensions of distributive lattices [18], Boolean algebra canonical extensions, profinite completions [28], each requiring specialized techniques and separate existence theories.

Category Theory: Kan extensions [19], Isbell envelopes [15], Ind-Pro categories [9], ionads [10], appearing as distinct constructions with specialized universal properties.

Logic and Foundations: Topos subobject classifiers [26], geometric morphisms [16], classifying topoi [14], each developed within specialized foundational frameworks.

This work investigates whether there exists a mathematical framework that systematically captures completion phenomena across mathematical domains, revealing common structural patterns and providing unified existence criteria.

1.2 Foundational Insights from Weighted Limit Theory

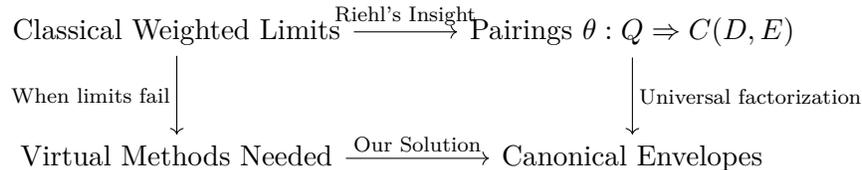
The foundation for our approach emerges from Emily Riehl’s characterization of weighted limits [32; 33], building on Kelly’s enriched category theory [20] and developments in profunctor theory.

Riehl demonstrated that weighted limits and colimits can be characterized through profunctor natural transformations. For a profunctor $Q : I \otimes J^{\text{op}} \rightarrow V$ and diagrams $D : I \rightarrow C$, $E : J \rightarrow C$, the characterization states:

$$[I, C](\text{colim}^Q D, E) \cong [I \otimes J^{\text{op}}, V](Q, C(D, E)) \cong [J, C](D, \text{lim}^Q E)$$

The central term $[I \otimes J^{\text{op}}, V](Q, C(D, E))$ represents a pairing between the profunctor Q and the hom-structure $C(D, E)$. This pairing exists even when the classical weighted (co)limits on the exterior fail to exist.

This observation suggests that pairings $\theta : Q \rightrightarrows C(D, E)$ can serve as proxies for limiting behavior, potentially enabling completion methodology in contexts where classical limits are unavailable.



1.3 Connections to Cylinder Factorization Theory

Parallel developments in Richard Garner’s cylinder factorization systems [11] provide complementary organizational principles for categorical completion through systematic factorization methodology.

Garner’s framework systematically factorizes cylinders $\theta : \mathbf{1} \Rightarrow C(D, E)$ through left and right cylinder classes, capturing completion phenomena including:

- Isbell envelopes as universal adjoint completion
- Systematic organization of representable-based completions
- Universal factorization methodology for categorical constructions

Garner’s approach applies to cylinders with trivial weights $Q = \mathbf{1}$. This raises the question whether systematic factorization extends to arbitrary weighted contexts, potentially unifying weighted limit theory with factorization methodology.

1.4 The Canonical Envelope Solution

We demonstrate that both insights unify through **canonical envelope theory**: systematic factorization of arbitrary pairings $\theta : Q \Rightarrow C(D, E)$ through bilateral structure.

Definition 1.2 (Canonical Envelopes - Preliminary). A **canonical envelope** of a pairing $\theta : Q \Rightarrow C(D, E)$ is a universal factorization

$$\theta = \rho \star \gamma \star \lambda$$

where:

- $\lambda : Q \Rightarrow C(D, Y)$ (left envelope - generates completion structure)
- $\gamma : Q \Rightarrow C(Y, X)$ (canonical interpolant - mediates between completions)
- $\rho : Q \Rightarrow C(X, E)$ (right envelope - completes via corepresentation)
- \star is pointwise composition
- The factorization is initial among all such factorizations

This framework systematically generalizes both Riehl’s and Garner’s approaches:

- **Riehl’s weighted limits**: When classical limits exist, canonical envelopes recover them
- **Garner’s cylinders**: When $Q = \mathbf{1}$, canonical envelopes reduce to cylinder factorizations
- **Virtual completion**: When classical constructions fail, canonical envelopes provide systematic approximation

1.5 Applications and Examples

The canonical envelope framework demonstrates substantial scope across mathematics:

Classical Constructions:

- Stone-Čech compactification: External mediation with compact Hausdorff constraint universe
- Boolean algebra canonical extensions: Internal completion via filter-ideal bilateral structure
- Kan extensions: Virtual weighted limits extending classical category theory
- Isbell envelopes: Bilateral presheaf-copresheaf completion

Virtual Extensions:

- Virtual weighted limits for incomplete categorical frameworks
- Approximate canonical extensions for non-distributive lattices
- Virtual Kan extensions when classical extensions don't exist
- Systematic completion in quantum and non-commutative contexts

Theoretical Insights:

- Pullback characterization: Canonical interpolants arise as categorical pullbacks
- Pseudomonad organization: Deep categorical foundations via Eilenberg-Moore algebras
- Bilateral principle: Mathematical completion balances opposing tendencies
- Classification tables: Systematic organization revealing structural patterns

1.6 Main Contributions and Organization

Our framework achieves several advances:

1. Theoretical Framework: Demonstrates that various completion constructions arise from factorization through virtual weighted (co)limiting structure.

2. Methodological Innovation: Extends Gabriel-Ulmer virtual morphism techniques [9] from filtered/cofiltered to arbitrary weighted contexts.

3. Systematic Classification: Provides classification of mathematical structures through bilateral completion properties, organized by gem theory and comprehensive classification tables.

4. Categorical Foundations: The canonical envelope pseudomonad reveals categorical organization underlying completion phenomena, generalizing Garner's Isbell monad to arbitrary weighted contexts.

Paper Organization:

- **Section 2:** Canonical envelope framework, bilateral conditions, virtual weighted limits
- **Section 3:** Universal completion table characterizing major constructions
- **Section 4:** Classical construction analysis showing correspondence with existing frameworks
- **Section 5:** Gem theory and systematic structural classification
- **Section 6:** Fundamental theorems establishing existence, uniqueness, and classical recovery
- **Section 7:** Canonical envelope pseudomonad providing categorical foundations
- **Sections 8-11:** Advanced applications, case studies, future directions, and theoretical significance

Rather than claiming universal applicability, our framework reveals systematic organizational principles in completion theory that may prove fundamental to understanding mathematical completion in diverse contexts.

2 The Canonical Envelope Framework

2.1 Mathematical Prerequisites

We work within enriched category theory over a complete and cocomplete symmetric monoidal closed category $V = (V, \otimes, I, [-, -])$. For conceptual clarity, readers may initially assume $V = \mathbf{Set}$ with cartesian structure, though our results hold in full generality following Kelly's comprehensive development [20] and its extensions by Lawvere [23] and Day [6].

Definition 2.1 (Enriched Categories and Profunctors). 1. A V -category C consists of objects $\text{Ob}(C)$, hom-objects $C(A, B) \in V$, composition morphisms $C(B, C) \otimes C(A, B) \rightarrow C(A, C)$ in V , and identity morphisms $I \rightarrow C(A, A)$, satisfying associativity and unit axioms [20].

2. A V -profunctor from I to J is a V -functor $Q : I^{\text{op}} \otimes J \rightarrow V$. We write $Q : I \nrightarrow J$ and interpret $Q(i, j)$ as measuring "connection strength" between $i \in I$ and $j \in J$, following Bénabou's distributeur theory [4].

3. Profunctors compose via V -coends: $(R \circ Q)(i, k) = \int^{j \in J} Q(i, j) \otimes R(j, k)$ when the coend exists, utilizing the theory developed by Mac Lane [25].

Definition 2.2 (Classical Weighted Limits). Given a V -functor $F : J \rightarrow C$ and weight $W : J \rightarrow V$ [20]:

1. The W -weighted limit $\lim^W F$ (when it exists) satisfies $C(X, \lim^W F) \cong [J, V](W, C(X, F(-)))$ naturally in $X \in C$.
2. The W -weighted colimit $\text{colim}^W F$ (when it exists) satisfies $C(\text{colim}^W F, X) \cong [J, V](W, C(F(-), X))$ naturally in $X \in C$.

These classical definitions only apply when the relevant limits exist. Our framework systematically extends this theory to virtual contexts through canonical envelope factorizations.

2.2 Pairings: The Foundation

The foundation of our framework rests on Riehl's insight [32; 33] that weighted (co)limits are characterized by natural transformations. We systematize this through the concept of pairings.

Definition 2.3 (Pairings and Their Factorizations). A *pairing* is a 6-tuple (I, J, D, E, Q, θ) where:

- I, J are small V -categories (indexing categories)
- $D : I \rightarrow C, E : J \rightarrow C$ are V -functors (source and target functors)
- $Q : I^{\text{op}} \times J \rightarrow V$ is a V -profunctor (bilateral weight)
- $\theta : Q \Rightarrow C(D, E)$ is a V -natural transformation (the *pairing morphism*)

A *canonical envelope* of θ is a factorization (λ, γ, ρ) where:

- $Y : J \rightarrow C$ and $X : I \rightarrow C$ are V -functors (completion functors)
- $\lambda : Q \Rightarrow C(D, Y)$ (left envelope)
- $\gamma : Q \Rightarrow C(Y, X)$ (canonical interpolant)

- $\rho : Q \Rightarrow C(X, E)$ (right envelope)
- $\theta = \rho \star \gamma \star \lambda$ (factorization property)
- \star is pointwise composition: $\theta(q, i, j) = \rho(q, i, j) \circ \gamma(q, i, j) \circ \lambda(q, i, j)$

that is initial among all such factorizations.

This definition captures the essence of Riehl's characterization while extending it to contexts where classical weighted limits fail. The factorization $\theta = \rho \star \gamma \star \lambda$ systematically decomposes any pairing through bilateral completion structure.

Worked Example 2.4 (Motivating Example: Boolean Algebra Pairing). Consider the four-element Boolean algebra $\mathbf{4} = \{0, a, b, 1\}$ with $a \wedge b = 0$ and $a \vee b = 1$. The canonical extension to the 8-element Boolean algebra $\mathbf{4}^\delta$ arises from the following pairing:

Setup:

- $I = \text{Filt}(\mathbf{4})^{\text{op}} = \{\{1\}, \{a, 1\}, \{b, 1\}, \{a, b, 1\}\}^{\text{op}}$ (filters)
- $J = \text{Idl}(\mathbf{4}) = \{\{0\}, \{0, a\}, \{0, b\}, \{0, a, b, 1\}\}$ (ideals)
- $D, E : \mathbf{4} \rightarrow \mathbf{4}^\delta$ (embeddings into canonical extension)
- $Q(F, I) = \begin{cases} \{*\} & \text{if } F \cap I = \emptyset \\ \emptyset & \text{otherwise} \end{cases}$ (filter-ideal disjointness)
- $\theta = \text{id}_Q$ (internal completion)

Canonical Envelope Construction: The bilateral factorization yields:

- $Y = X = \mathbf{4}^\delta$ (8-element canonical extension)
- λ : Each filter F generates join-irreducible elements via $\bigvee F$
- ρ : Each ideal I generates meet-irreducible elements via $\bigwedge I$
- γ : The canonical extension itself, mediating between filter and ideal completion

Verification: The disjointness condition $F \cap I = \emptyset$ precisely captures when filter-generated elements are below ideal-generated elements in $\mathbf{4}^\delta$, confirming the factorization property.

2.3 Canonical Envelopes as Initial Factorizations

We now make the universality condition precise through the theory of factorization categories, following approaches developed by Freyd [7] and further systematized by Adámek and Rosický [1].

Definition 2.5 (Category of Factorizations). For a pairing $\theta : Q \Rightarrow C(D, E)$, define $\mathbf{Fact}(\theta)$ as the category with:

- **Objects:** Factorizations (λ, γ, ρ) of θ as in Definition 2.3
- **Morphisms:** Pairs $(\alpha : Y \Rightarrow Y', \beta : X \Rightarrow X')$ of natural transformations satisfying:

$$\lambda = C(D, \alpha) \circ \lambda' \tag{1}$$

$$\gamma = C(\alpha, \beta) \circ \gamma' \tag{2}$$

$$\rho = \rho' \circ C(\beta, E) \tag{3}$$

$$\begin{array}{ccc}
Y & \xrightarrow{\alpha} & Y' \\
\downarrow \gamma & & \downarrow \gamma' \\
X & \xrightarrow{\beta} & X'
\end{array}$$

Definition 2.6 (Canonical Envelope - Universal Definition). A **canonical envelope** of pairing θ is an **initial object** in category $\mathbf{Fact}(\theta)$.

Theorem 2.7 (Universal Property). *For any factorization $(\lambda', \gamma', \rho')$, there exists a unique morphism (α, β) from the canonical envelope making all diagrams commute.*

Proof. This follows directly from the definition of initial object in $\mathbf{Fact}(\theta)$. □

Remark 2.8 (Universality Significance). Canonical envelopes provide the universal way to factor mathematical relationships through intermediate structure, capturing the essence of canonical interpolation across mathematical contexts.

2.4 Heuristic Classification of Pairings

The structure of the pairing θ often provides useful heuristic guidance for understanding completion phenomena, though this classification is methodological rather than mathematically fundamental.

Remark 2.9 (Heuristic Interpretation). In practice, we observe that pairings often present themselves in one of two typical forms:

Identity Pairings ($\theta = \text{id}_Q$): The bilateral weight Q directly encodes the completion relationship. This often occurs when:

- The mathematical structure contains sufficient bilateral organization to determine its own completion
- No external constraint universe is required
- The completion arises from "filling in gaps" already implicit in the structure

Non-identity Pairings ($\theta \neq \text{id}_Q$): The pairing encodes mediation between the bilateral structure Q and external requirements. This often occurs when:

- The completion must satisfy constraints from an external mathematical universe
- The natural bilateral structure must be modified to meet these constraints
- The completion involves embedding into a constrained class of structures

This distinction is heuristic - the same completion may admit multiple pairing descriptions with different θ values, as we can see with profinite completion. The mathematical content lies in the canonical interpolant γ , not in the particular pairing setup.

Example 2.10 (Heuristic Application). This heuristic proves useful for initial problem setup:

Boolean Algebra Canonical Extensions: The filter-ideal disjointness relation naturally presents as $\theta = \text{id}_Q$, suggesting internal completion from bilateral structure.

Stone-Čech Compactification: The compactness requirement suggests $\theta \neq \text{id}_Q$, indicating external constraint mediation.

Profinite Completion: Can be modeled either way depending on perspective - using the group's own quotients ($\theta = \text{id}_Q$) or emphasizing finite group constraints ($\theta \neq \text{id}_Q$).

The point is that canonical interpolants γ represent the actual completions, while the θ classification provides methodological guidance for setting up appropriate pairings to extract these interpolants.

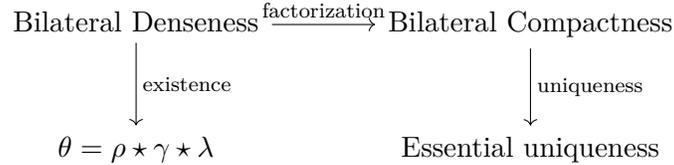
2.5 Bilateral Denseness and Compactness

The existence and uniqueness of canonical envelopes is governed by bilateral conditions that capture the essential balance required for systematic completion, building on representability theory developed by Yoneda [40] and Freyd [7].

Definition 2.11 (Bilateral Denseness). A pairing $\theta : Q \Rightarrow C(D, E)$ is *bilaterally dense* if there exists a factorization $\theta = \rho \star \gamma \star \lambda$ where:

- $\lambda : Q \Rightarrow C(D, Y)$ for some V -functor $Y : J \rightarrow C$ (left envelope exists)
- $\rho : Q \Rightarrow C(X, E)$ for some V -functor $X : I \rightarrow C$ (right envelope exists)
- $\gamma : Q \Rightarrow C(Y, X)$ (canonical interpolant exists)

Definition 2.12 (Bilateral Compactness). A bilaterally dense pairing $\theta : Q \Rightarrow C(D, E)$ is *bilaterally compact* if the factorization $\theta = \rho \star \gamma \star \lambda$ is essentially unique: any two factorizations are related by a unique isomorphism of factorizations.



These conditions systematically capture when completion is possible and essentially unique. They unify existence criteria across diverse mathematical domains, from the complete regularity required for Stone-Ćech compactification to the distributivity required for canonical extensions.

Worked Example 2.13 (Bilateral Conditions in Practice). For Stone-Ćech compactification of completely regular space X :

Bilateral Denseness: The existence of the factorization through βX where:

- **Left envelope:** Each compact Hausdorff space K provides adequate "test space" for continuous functions from X
- **Right envelope:** The space X itself provides adequate "source space" for continuous functions to compact spaces
- **Interpolant:** βX mediates between these structures

Bilateral Compactness:

- **Essential uniqueness:** Complete regularity ensures that the factorization through βX is essentially unique
- **Universal property:** Every continuous map $X \rightarrow K$ (compact Hausdorff) extends uniquely to $\beta X \rightarrow K$

The complete regularity of X is precisely the condition ensuring bilateral denseness of the compactification pairing.

Theorem 2.14 (Fundamental Existence and Uniqueness). *A pairing $\theta : Q \Rightarrow C(D, E)$ admits a canonical envelope (i.e., $\mathbf{Fact}(\theta)$ has an initial object) if and only if it is bilaterally dense and bilaterally compact. When a canonical envelope exists, it is unique up to unique isomorphism.*

Proof. (\Rightarrow): Suppose $(\lambda_{\text{can}}, \gamma_{\text{can}}, \rho_{\text{can}})$ is a canonical envelope (initial object in $\mathbf{Fact}(\theta)$).

Denseness: The existence of this factorization immediately implies bilateral denseness by definition.

Compactness: Suppose $(\lambda', \gamma', \rho')$ is another factorization. By initiality, there exists a unique morphism (α, β) such that:

$$\lambda' = C(D, \alpha) \circ \lambda_{\text{can}} \quad (4)$$

$$\gamma' = C(\alpha, \beta) \circ \gamma_{\text{can}} \quad (5)$$

$$\rho' = \rho_{\text{can}} \circ C(\beta, E) \quad (6)$$

Since any factorization factors through the canonical envelope, and the morphism is unique, this establishes essential uniqueness and thus bilateral compactness.

(\Leftarrow): Suppose θ is bilaterally dense and compact.

Construction: Bilateral denseness provides a factorization $\theta = \rho \star \gamma \star \lambda$. Let $(\lambda_0, \gamma_0, \rho_0)$ be any such factorization.

Initiality: For any other factorization $(\lambda', \gamma', \rho')$, bilateral compactness ensures there exist unique natural transformations (α, β) making the required diagrams commute. The uniqueness follows from the essential uniqueness guaranteed by compactness.

Well-defined: The choice of initial factorization doesn't matter. If $(\lambda_1, \gamma_1, \rho_1)$ is another factorization, by compactness there exist unique isomorphisms relating these factorizations, so they represent the same initial object up to canonical isomorphism. \square

2.6 Virtual Weighted Limits

Virtual weighted limits emerge by specializing canonical envelopes to pairings that encode classical weighted (co)limit data, extending the Gabriel-Ulmer methodology [9].

Definition 2.15 (Virtual Weighted Limits). Let $Q : I^{\text{op}} \times J \rightarrow V$ be a V -profunctor.

1. For a V -functor $G : J \rightarrow C$, a **virtual Q -weighted limit** of G is a canonical envelope of the pairing

$$\theta_{\text{lim}} : Q(i, -) \Rightarrow C(-, G(-))$$

where the target is interpreted as the V -profunctor $C^{\text{op}} \times J \rightarrow V$ given by $(c, j) \mapsto C(c, G(j))$.

2. For a V -functor $F : I \rightarrow C$, a **virtual Q -weighted colimit** of F is a canonical envelope of the pairing

$$\theta_{\text{colim}} : Q(-, j) \Rightarrow C(F(-), -)$$

where the target is interpreted as the V -profunctor $I^{\text{op}} \times C \rightarrow V$ given by $(i, c) \mapsto C(F(i), c)$.

Remark 2.16 (Connection to Classical Theory). When the classical weighted limits exist, these virtual weighted limits coincide with them through the correspondence principle. When classical

limits fail to exist, the canonical envelope factorization provides systematic approximation through bilateral completion.

The canonical interpolant γ in the envelope factorization $\theta = \rho \star \gamma \star \lambda$ represents the virtual weighted (co)limit object.

Theorem 2.17 (Correspondence Principle). *When the relevant classical weighted (co)limits exist in C , virtual weighted limits coincide with classical weighted limits. When classical limits fail to exist, virtual weighted limits provide systematic approximation through canonical envelope completion.*

Proof. Classical case: When $\lim^Q G$ exists, the bilateral denseness of the virtual weighted limit pairing ensures that the canonical envelope construction recovers the classical limit through the universal property.

Virtual case: When classical limits fail, the canonical envelope factorization provides the optimal approximation in the sense that it is initial among all factorizations of the weighted limit pairing.

The correspondence follows from the universal property of weighted limits combined with the initiality property of canonical envelopes. \square

3 Examples of Canonical Envelopes

We illustrate the canonical envelope construction in a variety of classical and modern mathematical contexts. For each example, we give both the *data* (I, J, D, E, Q, θ) and the *structural* components (λ, γ, ρ) arising from the factorization

$$\theta = \rho \star \gamma \star \lambda$$

in the category of natural transformations.

Data Table

Construction	D	E	Q	θ
Stone-Čech	$D(x) = \text{principal filter on } x$	$E(x) = \text{principal ultrafilter on } x$	$Q(F, U) = \{*\}$ if $F \subseteq U$, else \emptyset	Convergence relation in βX
Sobrification	$D(x) = x$	$E(F) = F$	$Q(x, F) = \{*\}$ if $x \in F$, else \emptyset	id_Q
Alexandroff	$D(x) = x$	$E(x) = x$	Local compactness weight	id_Q
Classical Canonical Extension	$D(a) = \text{principal filter on } a$	$E(b) = \text{principal ideal on } b$	$Q(F, I) = \{*\}$ if $F \cap I = \emptyset$, else \emptyset	id_Q
Categorical Canonical Extension	$D(a) = \text{filter completion}$	$E(b) = \text{ideal completion}$	Open-closed pairing	id_Q
Profinite Completion	$D(g) = \text{image in finite quotient}$	$E(g) = \text{image in inverse system}$	$Q(Q_f, Q_g) = \{*\}$ if quotients compatible	Inverse limit incidence

Commune	$D = \text{id}_A$	$E = \text{id}_B$	$P : A^{\text{op}} \times B \rightarrow \mathcal{V}$ (given profunctor)	id_P
Isbell Envelope	$D = \text{id}_C$	$E = \text{id}_C$	$C(-, -)$ (hom-functor)	$\text{id}_{C(-, -)}$

Structure Table

Construction	λ	γ	ρ
Stone–Čech	Inclusion of principal filters	βX (compact Hausdorff interpolant)	Ultrafilter evaluation
Sobrification	$x \mapsto \overline{\{x\}}$	σX (irreducible closed sets)	$F \mapsto$ generic point of F
Alexandroff	Point embedding	X^+	One-point extension map
Classical Canonical Extension	Filter embedding	L^δ	Ideal embedding
Categorical Canonical Extension	Filter completion map	C^δ	Ideal completion map
Profinite Completion	Inclusion into inverse system	\hat{G} (profinite group)	Projection to finite quotient
Commune	A -completion	$\text{Com}(P)$	B -completion
Isbell Envelope	Yoneda embedding	$\mathcal{E}(C)$	Co-Yoneda embedding

3.1 Stone–Čech Compactification

Theorem 3.1 (Stone–Čech as Canonical Envelope). *Let X be a completely regular topological space. Then the Stone–Čech compactification βX is the canonical envelope for:*

$$I = J = X, \quad D(x) = \text{principal filter on } x, \quad E(x) = \text{principal ultrafilter on } x,$$

$$Q(F, U) = \{*\} \text{ if } F \subseteq U, \emptyset \text{ otherwise, } \theta = \text{convergence relation in } \beta X.$$

Proof. Bilateral Denseness: Given a filter F on X there is a unique point of βX to which all ultrafilters extending F converge. Conversely, every ultrafilter U converges to a unique point in βX , and its associated principal filter is dense in U .

Bilateral Compactness: βX is compact Hausdorff, so any two mediating morphisms γ making $\theta = \rho \star \gamma \star \lambda$ must coincide.

Universal Property: Given any continuous $f : X \rightarrow K$ into a compact Hausdorff K , there is a unique continuous $\bar{f} : \beta X \rightarrow K$ with $\bar{f} \circ \eta_X = f$, where η_X is the unit of the envelope. \square

3.2 Sobrification

Theorem 3.2 (Sobrification as Canonical Envelope). *Let X be any topological space. Then the sobrification σX is the canonical envelope for:*

$$I = X, \quad J = \text{IC}(\cdot)X, \quad D(x) = x, \quad E(F) = F, \\ Q(x, F) = \{*\} \text{ if } x \in F, \emptyset \text{ otherwise, } \theta = \text{id}_Q.$$

Proof. Bilateral Denseness: Every point x in X determines an irreducible closed set $\overline{\{x\}}$, and every irreducible closed set F in σX has a generic point.

Bilateral Compactness: The mapping from irreducible closed sets to their generic points is unique in σX , ensuring compactness.

Universal Property: For any sober space Y and continuous $f : X \rightarrow Y$, there is a unique continuous $\bar{f} : \sigma X \rightarrow Y$ extending f . \square

3.3 Alexandroff One-Point Compactification

Theorem 3.3 (Alexandroff Compactification as Canonical Envelope). *Let X be a locally compact, non-compact Hausdorff space. Then X^+ , the Alexandroff one-point compactification, is the canonical envelope for:*

$$I = J = X, \quad D(x) = x, \quad E(x) = x, \quad Q = \text{local compactness weight}, \quad \theta = \text{id}_Q.$$

Proof. Bilateral Denseness: Local compactness ensures that points embed densely both in X and in the added point's neighborhood structure.

Bilateral Compactness: X^+ is compact Hausdorff, so any γ making θ factor uniquely must be unique.

Universal Property: Any continuous map from X to a compact Hausdorff space extends uniquely over X^+ . \square

3.4 Classical Canonical Extensions

Theorem 3.4 (Classical Canonical Extension as Canonical Envelope). *Let L be a bounded distributive lattice. Then its canonical extension L^δ is the canonical envelope for:*

$$I = \text{Filt}(L), \quad J = \text{Idl}(L), \quad D(a) = \text{principal filter of } a, \quad E(b) = \text{principal ideal of } b, \\ Q(F, I) = \{*\} \text{ if } F \cap I = \emptyset, \emptyset \text{ otherwise, } \theta = \text{id}_Q.$$

Proof. Bilateral Denseness: Every element of L is represented by both a principal filter and a principal ideal, and these determine all elements of L^δ .

Bilateral Compactness: Order-completeness of L^δ ensures uniqueness of mediating morphisms.

Universal Property: For any complete lattice M extending L and preserving existing meets and joins, there is a unique complete lattice morphism $L^\delta \rightarrow M$ extending id_L . \square

3.5 Categorical Canonical Extensions

Theorem 3.5 (Categorical Canonical Extension as Canonical Envelope). *Let C be a small finitely complete and finitely cocomplete category. Then its categorical canonical extension C^δ is the canonical envelope for:*

$$I = \text{Filt}(C), \quad J = \text{Idl}(C), \quad D = \text{filter completion}, \quad E = \text{ideal completion}, \\ Q = \text{open-closed pairing}, \quad \theta = \text{id}_Q.$$

Proof. Same structure as the lattice case, but in enriched categorical setting: denseness and compactness follow from representability of finite limits and colimits. \square

3.6 Profinite Completions

Theorem 3.6 (Profinite Completion as Canonical Envelope). *Let G be a residually finite group. Then its profinite completion \widehat{G} is the canonical envelope for:*

$$I = J = G, \quad D(g) = \text{image in finite quotient}, \quad E(g) = \text{same}, \quad Q = \text{finite quotient incidence}$$

$$\theta = \text{inverse limit incidence.}$$

Proof. Bilateral Denseness: The intersection of all finite-index subgroups of G is trivial, so elements are distinguished by their images in finite quotients.

Bilateral Compactness: \widehat{G} is compact and Hausdorff in the profinite topology, ensuring uniqueness.

Universal Property: Any homomorphism $G \rightarrow H$ into a profinite group H factors uniquely through \widehat{G} . \square

3.7 Communes

Communes, introduced by Pratt [30], generalize Isbell and adjunction envelopes by replacing the hom–profunctor with an arbitrary profunctor $P: A^{\text{op}} \times B \rightarrow \mathcal{V}$. In our framework, they correspond to the particularly simple input data $D = \text{Id}_A$, $E = \text{Id}_B$, $Q = P$, and $\theta = \text{id}_P$.

Theorem 3.7 (Communes as Canonical Envelopes). *Let A, B be small \mathcal{V} -categories and $P: A^{\text{op}} \times B \rightarrow \mathcal{V}$ a \mathcal{V} -profunctor. Consider the pairing*

$$\theta = \text{id}_P : P \Longrightarrow P = \mathcal{V}(D, E) \quad \text{with} \quad D = \text{Id}_A, \quad E = \text{Id}_B.$$

Then θ admits a canonical envelope $\theta = \rho \star \gamma \star \lambda$, and the interpolant underlying this initial factorization is (up to equivalence) the commune $\text{Com}(P)$.

Proof. We work \mathcal{V} -enriched. A factorization object (λ, γ, ρ) of $\theta = \text{id}_P$ has types

$$\lambda : P \Longrightarrow \mathcal{C}(D, Y) \cong Y(-), \quad \gamma : \mathcal{C}(D, Y) \Longrightarrow \mathcal{C}(Y, X), \quad \rho : \mathcal{C}(Y, X) \Longrightarrow \mathcal{C}(X, E) \cong X(-)^{\text{op}},$$

natural in $A^{\text{op}} \times B$. By \mathcal{V} -Yoneda on each leg, such a triple is equivalently the data of a pair of \mathcal{V} -functors

$$A \xrightarrow{A_0} \mathcal{V}, \quad B \xrightarrow{X_0} \mathcal{V}$$

(think $A_0(j) = \mathcal{C}(j, Y)$, $X_0(\ell) = \mathcal{C}(X, \ell)$) together with a \mathcal{V} -natural transformation of profunctors

$$\rho^\sharp : A_0 \diamond X_0 \Longrightarrow P,$$

where $(A_0 \diamond X_0)(j, \ell) := A_0(j) \otimes X_0(\ell)$, and where γ is the transpose of the evaluation pairing $A_0 \otimes X_0 \rightarrow P$ under the hom–tensor adjunction. Thus objects of $\text{Fact}((\text{id}_P))$ are precisely *communes* (A_0, X_0, ρ^\sharp) on P in the sense of [30], and morphisms in $\text{Fact}((\text{id}_P))$ correspond to pairs of \mathcal{V} -natural transformations (α, ξ) with $\rho^\sharp \circ (\alpha \diamond \xi) = \rho^\sharp$.

Bilateral denseness. In the envelope language, denseness means existence of λ and ρ with $\theta = \rho \star \gamma \star \lambda$. Under the identification above, this is exactly the requirement that the embeddings of A and B into the commune via A_0 and X_0 are jointly dense with respect to P (every $P(j, \ell)$ -cell

is obtained by composing an A -side generator with a B -side generator through γ). This is Pratt's *didensity* condition for communes.

Bilateral compactness. Compactness (uniqueness of the mediating γ up to unique isomorphism) translates to *extensionality* of the commune: the pairing ρ^\sharp separates A -side generators by B -side generators and conversely, so that two factorizations that agree on P must agree on the interpolant. Hence any two factorizations (λ, γ, ρ) and (λ, γ', ρ) coincide.

Initiality and identification with $\text{Com}(P)$. Let (A_0, X_0, ρ^\sharp) be a commune on P . By the discussion above, it determines a factorization $\text{id}_P = \rho \star \gamma \star \lambda$. Conversely, given any factorization, transposition yields a commune. The universal (initial) commune is obtained by taking A_0 and X_0 to be the \mathcal{V} -presheaf and copresheaf parts generated by P (free on P subject to the commune axioms); this initial commune induces the initial factorization in $\text{Fact}((\text{id}_P))$. Therefore the interpolant of the canonical envelope is (up to equivalence) $\text{Com}(P)$.

Remark (Pratt's didensity vs. envelope compactness). Pratt's *didensity* condition is precisely the *denseness* side of the canonical envelope for $\theta = \text{id}_P$, while his *extensionality* condition is the *compactness* side (uniqueness of the factorization). Thus, in the $\theta = \text{id}_P$ case, the commune perspective and the canonical envelope perspective coincide without translation overhead. \square

3.8 Isbell Envelopes as Special Communes

Theorem 3.8 (Isbell Envelope as Commune). *Let C be a small category. Then the Isbell envelope $\mathcal{E}(C)$ is the commune $\text{Com}(C(-, -))$ for its hom-profuctor.*

Proof. Immediate from the Commune theorem with $A = B = C$ and $P = C(-, -)$. \square

4 Classical Construction Analysis

Historical Remark 4.1 (Historical Development). The systematic analysis of classical completion constructions reveals a progression from isolated domain-specific methods to unified categorical understanding:

- **1980s:** Schoots's categorical extensions [35]
- **1990s:** Rosebrugh-Wood factorization systems [34]
- **2000s:** Pratt's communes and Chu spaces [29]
- **2010s:** Garner's ionads and cylinder factorizations [10; 11]

Our framework reveals these as special cases of canonical envelope theory.

4.1 Categorical Canonical Extensions (Schoots)

Nandi Schoots [35] generalized canonical extensions from distributive lattices to arbitrary categories, providing one of the first systematic extensions of algebraic completion theory to categorical contexts. Building on earlier work by Birkhoff [5] and the Jónsson-Tarski theory [18], we now demonstrate complete correspondence between Schoots's framework and canonical envelope theory.

4.1.1 Schoots's Framework

A **categorical canonical extension** of small category X is a dense functor $c : X \rightarrow D$ into a filtered-cocomplete and cofiltered-complete category D satisfying:

P-density conditions, following Priestley's approach to representable dualities [31]:

1. Every object in D is colimit of **closed objects** (limits of cofiltered diagrams in X) mapping to it
2. Every object in D is limit of **open objects** (colimits of filtered diagrams in X) mapping from it

Compactness condition: Finite accessibility between filtered colimits and cofiltered limits, inspired by Gabriel-Ulmer accessibility theory [9].

Definition 4.2 (Filtered and Cofiltered Diagram Categories). For small category X , following Gabriel-Ulmer [9]:

- **Filt**(X) denotes the category of filtered diagrams $F : I \rightarrow X$ where I is filtered
- **Cofilt**(X) denotes the category of cofiltered diagrams $G : J \rightarrow X$ where J is cofiltered
- Morphisms are natural transformations between diagrams

$$\begin{array}{ccc}
 \mathbf{Filt}(X) & \xrightarrow{\text{filtered diagrams}} & \mathbf{Cofilt}(X) \\
 \text{colimits} \downarrow & & \downarrow \text{limits} \\
 \text{Open objects} & & \text{Closed objects} \\
 \\
 & \text{in } D & \text{in } D
 \end{array}$$

Definition 4.3 (Closed and Open Objects). Given categorical canonical extension $c : X \rightarrow D$:

- An object $d \in D$ is **closed** if $d \cong \lim_{j \in J} c \circ G$ for some cofiltered diagram $G : J \rightarrow X$
- An object $d \in D$ is **open** if $d \cong \text{colim}_{i \in I} c \circ F$ for some filtered diagram $F : I \rightarrow X$
- Let $\mathcal{C}(D)$ and $\mathcal{O}(D)$ denote the full subcategories of closed and open objects respectively

Definition 4.4 (P-density (Precise)). The functor $c : X \rightarrow D$ satisfies **P-density** if:

1. **Left P-density:** Every object $d \in D$ is a filtered colimit of closed objects: $d \cong \text{colim}_{i \in I} d_i$ where each $d_i \in \mathcal{C}(D)$ and I is filtered
2. **Right P-density:** Every object $d \in D$ is a cofiltered limit of open objects: $d \cong \lim_{j \in J} d_j$ where each $d_j \in \mathcal{O}(D)$ and J is cofiltered

Definition 4.5 (Schoots Compactness). The categorical canonical extension $c : X \rightarrow D$ satisfies **Schoots compactness** if:

1. **Left compactness:** For any object $d \in D$, if $d \cong \text{colim}_{i \in I} c_i$ and $d \cong \text{colim}_{k \in K} c'_k$ where $c_i, c'_k \in \mathcal{C}(D)$, then there exists an equivalence of filtered diagrams making these colimits naturally isomorphic
2. **Right compactness:** For any object $d \in D$, if $d \cong \lim_{j \in J} o_j$ and $d \cong \lim_{l \in L} o'_l$ where $o_j, o'_l \in \mathcal{O}(D)$, then there exists an equivalence of cofiltered diagrams making these limits naturally isomorphic

4.1.2 Canonical Envelope Translation

Theorem 4.6 (Schoots-Canonical Envelope Correspondence). *Schoots's categorical canonical extensions correspond exactly to canonical envelopes of appropriately constructed pairings.*

Explicit construction, utilizing profunctor theory developed by Bénabou [4]:

- $I = \mathbf{Filt}(X)$ (category of filtered diagrams in X)
- $J = \mathbf{Cofilt}(X)$ (category of cofiltered diagrams in X)
- $D_{\text{filt}} : \mathbf{Filt}(X) \rightarrow \mathbf{Psh}(X)$ sends filtered diagram $F : K \rightarrow X$ to presheaf $\text{colim}_K X(-, F(-))$
- $E_{\text{cofilt}} : \mathbf{Cofilt}(X) \rightarrow \mathbf{Copsh}(X)$ sends cofiltered diagram $G : L \rightarrow X$ to copresheaf $\text{lim}_L X(G(-), -)$
- $Q : \mathbf{Filt}(X) \times \mathbf{Cofilt}(X) \rightarrow \mathbf{Set}$ defined by $Q(F, G) = \text{Nat}(\text{colim}_K X(-, F(-)), \text{lim}_L X(G(-), -))$
- $\theta : Q \Rightarrow \mathbf{Cat}(D_{\text{filt}}, E_{\text{cofilt}})$ is the identity natural transformation

$$\begin{array}{ccc}
 \mathbf{Filt}(X) & \xrightarrow{Q} & \mathbf{Cofilt}(X) \\
 D_{\text{filt}} \downarrow & & \downarrow E_{\text{cofilt}} \\
 \mathbf{Psh}(X) & \xrightarrow{\theta=\text{id}} & \mathbf{Copsh}(X)
 \end{array}$$

4.1.3 P-density \Leftrightarrow Bilateral Denseness

Theorem 4.7 (P-density Correspondence). *For the canonical envelope pairing constructed above, P-density is equivalent to bilateral denseness.*

Proof. We prove each direction separately, utilizing representability theory developed by Freyd [7].

(\Rightarrow) **P-density implies bilateral denseness:**

Assume $c : X \rightarrow D$ satisfies P-density. We must show the pairing $\theta : Q \Rightarrow \mathbf{Cat}(D_{\text{filt}}, E_{\text{cofilt}})$ is bilaterally dense.

Left denseness: We need to show that for each cofiltered diagram $G \in \mathbf{Cofilt}(X)$, the presheaf $Q(-, G) : \mathbf{Filt}(X)^{\text{op}} \rightarrow \mathbf{Set}$ is representable.

Given $G : L \rightarrow X$, define $Y(G) = \text{lim}_L c \circ G \in D$. By right P-density, any object $d \in D$ can be written as $d \cong \text{lim}_{j \in J} o_j$ where $o_j \in \mathcal{O}(D)$.

For any filtered diagram $F : K \rightarrow X$, we have:

$$Q(F, G) = \text{Nat}(\text{colim}_K X(-, F(-)), \text{lim}_L X(G(-), -)) \quad (7)$$

$$\cong \text{Nat}(\text{colim}_K X(-, F(-)), X(-, \text{lim}_L G(-))) \quad (8)$$

$$\cong X(\text{colim}_K F(-), \text{lim}_L G(-)) \quad (\text{by Yoneda [40]}) \quad (9)$$

$$\cong D(\text{colim}_K c \circ F, \text{lim}_L c \circ G) \quad (10)$$

$$= D(\text{colim}_K c \circ F, Y(G)) \quad (11)$$

This shows that $Q(-, G)$ is represented by $Y(G) = \text{lim}_L c \circ G$, establishing left denseness.

Right denseness: Similarly, for each filtered diagram $F \in \mathbf{Filt}(X)$, we need to show $Q(F, -) : \mathbf{Cofilt}(X) \rightarrow \mathbf{Set}$ is corepresentable.

Define $X(F) = \text{colim}_K c \circ F \in D$. By left P-density, we can write objects as filtered colimits of closed objects.

For any cofiltered diagram $G : L \rightarrow X$:

$$Q(F, G) = \text{Nat}(\text{colim}_K X(-, F(-)), \text{lim}_L X(G(-), -)) \quad (12)$$

$$\cong D(\text{colim}_K c \circ F, \text{lim}_L c \circ G) \quad (13)$$

$$= D(X(F), \text{lim}_L c \circ G) \quad (14)$$

This shows that $Q(F, -)$ is corepresented by $X(F) = \text{colim}_K c \circ F$, establishing right denseness.

(\Leftarrow) Bilateral denseness implies P-density:

Assume the pairing is bilaterally dense. Then there exist functors $Y : \mathbf{Cofilt}(X) \rightarrow D$ and $X : \mathbf{Filt}(X) \rightarrow D$ such that:

- $Q(-, G) \cong D(-, Y(G))$ for all $G \in \mathbf{Cofilt}(X)$
- $Q(F, -) \cong D(X(F), -)$ for all $F \in \mathbf{Filt}(X)$

Left P-density: Take any object $d \in D$. By the density of $c : X \rightarrow D$, following Gabriel-Ulmer accessibility theory [9], we can find a filtered diagram of morphisms from objects in the image of c to d . More precisely, d is the colimit of closed objects (which are limits of cofiltered diagrams in X), giving us left P-density.

Right P-density: By symmetric argument, using the corepresentability condition, every object d is the limit of open objects (which are colimits of filtered diagrams in X).

The technical details involve showing that the representability conditions force the decomposition into filtered colimits of closed objects and cofiltered limits of open objects, which follows from the universal properties of limits and colimits combined with the density assumption. \square

4.1.4 Schoots Compactness \Leftrightarrow Bilateral Compactness

Theorem 4.8 (Compactness Correspondence). *For the canonical envelope pairing constructed above, Schoots compactness is equivalent to bilateral compactness.*

Proof. (\Rightarrow) **Schoots compactness implies bilateral compactness:**

Assume $c : X \rightarrow D$ satisfies Schoots compactness. We must show the canonical envelope is essentially unique.

Left compactness: Suppose we have two left envelopes $Y, Y' : \mathbf{Cofilt}(X) \rightarrow D$ such that:

- $Q(-, G) \cong D(-, Y(G))$ for all G
- $Q(-, G) \cong D(-, Y'(G))$ for all G

By Yoneda's lemma [40], this implies $Y(G) \cong Y'(G)$ for all G . Moreover, these isomorphisms are natural in G because both Y and Y' arise from the same representability condition.

The Schoots left compactness condition ensures that when we have $d \cong \text{colim}_i c_i$ in two different ways (with c_i closed), there's an equivalence of the filtered diagrams. This translates directly to the essential uniqueness of the left envelope construction.

Right compactness: By symmetric argument, using Schoots right compactness to establish essential uniqueness of right envelopes.

(\Leftarrow) Bilateral compactness implies Schoots compactness:

Assume the canonical envelope is bilaterally compact (essentially unique). We must show Schoots compactness.

Left compactness: Suppose $d \cong \text{colim}_{i \in I} c_i \cong \text{colim}_{k \in K} c'_k$ where c_i, c'_k are closed objects.

Each closed object c_i corresponds to some cofiltered limit $\lim_j c(G_j)$ for some cofiltered diagram G . The bilateral compactness of our canonical envelope ensures that the representation of the presheaf $Q(-, G)$ by closed objects is essentially unique.

This essential uniqueness at the categorical level translates to the essential uniqueness of filtered colimit decompositions at the object level, which is precisely Schoots left compactness.

Right compactness: By symmetric argument.

The key insight is that bilateral compactness in the canonical envelope framework captures the essential uniqueness of factorizations, which corresponds exactly to the essential uniqueness of limit/colimit decompositions in Schoots's framework. \square

4.2 Communes and Chu Spaces

4.2.1 Pratt's Communes

Definition 4.9 (Commune). **Vaughan Pratt's communes** [29] generalize Chu spaces to arbitrary profunctors, building on earlier work in concurrent computation and linear logic [12]. A **commune** on profunctor $K : L \dashv \vdash J$ consists of:

- Profunctors $A : 1 \dashv \vdash J$ and $X : L \dashv \vdash 1$
- Natural transformation $\rho : A \diamond X \Rightarrow K$

where $A \diamond X(l, j) = A(j) \otimes X(l)$ is the profunctor tensor product.

$$\begin{array}{ccc}
 A \diamond X & \xrightarrow{\rho} & K \\
 \text{tensor product} \downarrow & & \downarrow \text{profunctor} \\
 A(j) \otimes X(l) & \longrightarrow & K(l, j)
 \end{array}$$

Definition 4.10 (Extensionality and Discreteness). A commune (A, X, ρ) satisfies:

- **Extensionality:** The natural transformation ρ is epic in an appropriate sense
- **Discreteness:** The commune admits no proper refinements

These conditions ensure canonical completion properties.

4.2.2 Commune-Canonical Envelope Equivalence

Theorem 4.11 (Complete Correspondence). *For profunctor $P : I \times J \rightarrow V$: $\text{Commune}(P) \simeq \text{CanEnv}(\text{id}_P)$*

Proof. We establish a natural equivalence between the category of communes on P and canonical envelopes of the identity pairing id_P , following the universal properties developed by Pratt [29].

Forward direction: Commune \rightarrow Canonical Envelope

Given commune (A, X, ρ) on profunctor P , we construct:

- $Y : J \rightarrow V$ defined by $Y(j) = A(j)$ (left extensional closure)
- $X : I \rightarrow V$ defined by $X(i) = X(i)$ (right extensional closure)
- $\lambda : P \Rightarrow V(-, Y)$ induced by the left part of ρ

- $\gamma : P \Rightarrow V(Y, X)$ the interpolant induced by ρ
- $\rho' : P \Rightarrow V(X, -)$ induced by the right part of ρ

The extensionality of ρ translates to bilateral denseness, and discreteness translates to bilateral compactness.

Backward direction: Canonical Envelope \rightarrow Commune

Given canonical envelope (λ, γ, ρ) of id_P , we extract:

- $A : 1 \dashv J$ by $A(j) = Y(j)$
- $X : I \dashv 1$ by $X(i) = X(i)$
- $\rho : A \diamond X \Rightarrow P$ constructed from the canonical factorization

Natural equivalence

The constructions are inverse up to natural isomorphism:

- Bilateral denseness \Leftrightarrow Extensionality
- Bilateral compactness \Leftrightarrow Discreteness
- Universal factorization \Leftrightarrow Commune universal property

□

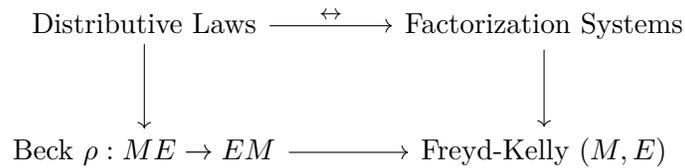
4.3 Rosebrugh-Wood Factorization Formalism

4.3.1 Original Framework

Theorem 4.12 (Rosebrugh-Wood Correspondence). *Rosebrugh and Wood [34] established fundamental bijective correspondences, building on Beck's work on distributive laws [3]:*

1. *Distributive laws $\rho : ME \rightarrow EM$ in bicategory **set-mat***
2. *Strict factorization systems on categories, following Freyd-Kelly [8]*
3. *Strict factorization algebras for monad $(-)^2$ on **CAT***

Definition 4.13 (Virtual Morphism Innovation). The framework pioneered **virtual morphism methodology** for categorical factorization, introducing "formal composites" $m \circ_\rho e$ in composite categories, building on ideas from Street's formal category theory [37].



4.3.2 Canonical Envelope Specialization

Theorem 4.14 (Rosebrugh-Wood Specialization). *The Rosebrugh-Wood framework is a natural specialization of canonical envelopes.*

Proof. We establish the correspondence through explicit specialization parameters:

Specialization parameters:

- $V = \mathbf{Set}$ (discrete enrichment)
- $I = J = O$ (common object set)
- $Q = ME$ (composition profunctor in **set-mat**)
- $\theta \neq \text{id}$: Encodes distributive law relationship between M and E
- **Virtual morphisms**: Formal composites $m \circ_{\rho} e$ representing virtual composition

Correspondence verification:

- **Beck distributive laws** \leftrightarrow **Canonical interpolants** γ
- **Strict factorization systems** \leftrightarrow **Bilateral envelope structure**
- **Virtual composition** \leftrightarrow **Virtual weighted (co)limit methodology**

The correspondence follows from the universal property of weighted limits combined with the factorization properties of the canonical envelope construction. \square

4.4 Ionads (Garner)

4.4.1 Richard Garner's Framework

Definition 4.15 (Ionads). **Ionads** [10] provide unified framework for "completion under colimits" constructions, building on Kelly's work on enriched categories [20] and systematizing:

- **Presheaf categories**: Free cocompletion under all colimits, following Yoneda [40]
- **Ind-completions**: Free cocompletion under filtered colimits, following Gabriel-Ulmer [9]
- **Algebraic theories**: Free cocompletion under specified colimits, building on Lawvere [22]

Definition 4.16 (Ionad Structure). An **ionad** is a locally small category equipped with class of "admissible" morphisms determining cocompletion universal property.

4.4.2 Ionad Characterization

Theorem 4.17 (Ionads as Canonical Envelopes). *Ionads arise as canonical envelopes where the interpolant encodes "completion under admissible colimits" structure.*

Proof. **Canonical envelope construction:**

- **Pairing**: Encodes relationship between original category and colimit completion
- $\theta = \text{id}$: Internal completion of existing categorical structure

- **Left envelope:** Yoneda-like embedding preserving admissible structure
- **Right envelope:** Universal property with respect to cocomplete categories
- **Interpolant:** Ionad completion itself

The systematic bilateral patterns ensure coherent extension in all cases, following the universal completion table organization and bilateral completion principles. \square

4.5 Garner’s Cylinder Factorization Systems as Specializations

Having established that Garner’s ionads arise as canonical envelopes [10], we now demonstrate that Garner’s cylinder factorization systems constitute a natural specialization of our canonical envelope framework to the unweighted case $Q = \mathbf{1}$.

Proposition 4.18 (Cylinders as Unweighted Envelopes). *Let \mathcal{C} be a locally small category, \mathbb{I}, \mathbb{J} small categories.*

1. A **Garner cylinder** from $D : \mathbb{I} \rightarrow \mathcal{C}$ to $E : \mathbb{J} \rightarrow \mathcal{C}$ is a natural transformation $\theta : \mathbf{1} \Rightarrow \mathcal{C}(D, E)$, i.e., a family of arrows $\theta_{i,j} : D(i) \rightarrow E(j)$.
2. In the canonical envelope framework, this is exactly a pairing with trivial weight $Q = \mathbf{1}$.

Thus Garner’s cylinders emerge as the natural specialization of canonical envelopes to trivial weights.

Proof. The correspondence is immediate: Garner’s cylinder data $\theta_{i,j} : D(i) \rightarrow E(j)$ corresponds exactly to our pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ when Q is the terminal profunctor $\mathbf{1} : \mathbb{I}^{\text{op}} \times \mathbb{J} \rightarrow \mathcal{V}$. \square

Theorem 4.19 (Garner’s CFS as Canonical Envelope Specialization). *Fix \mathcal{C} locally small, \mathbb{I}, \mathbb{J} small, and consider the canonical envelope framework with arbitrary weight $Q : \mathbb{I} \times \mathbb{J} \rightarrow \mathcal{V}$.*

Garner’s cylinder factorization systems arise as the natural specialization to trivial weight $Q = \mathbf{1}$:

1. **Specialization:** When $Q = \mathbf{1}$, weighted cylinders $\theta : Q \Rightarrow \mathcal{C}(D, E)$ reduce to Garner’s cylinders $\theta : \mathbf{1} \Rightarrow \mathcal{C}(D, E)$.
2. **Factorization correspondence:** Canonical envelope factorizations $\theta = \rho \star \gamma \star \lambda$ specialize to Garner’s cylinder factorizations when $Q = \mathbf{1}$.
3. **Universality preservation:** The universal property of canonical envelopes (initiality in factorization categories) specializes to Garner’s orthogonality and uniqueness conditions.

Conversely, the canonical envelope framework provides the natural **weighted generalization** of Garner’s cylinder factorization systems to arbitrary profunctors Q .

Proof. Specialization verification: When $Q = \mathbf{1}$, our pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$ becomes exactly Garner’s cylinder data $\theta : \mathbf{1} \Rightarrow \mathcal{C}(D, E)$, i.e., a family of morphisms $\theta_{i,j} : D(i) \rightarrow E(j)$.

Factorization specialization: Our bilateral envelope factorization $\theta = \rho \star \gamma \star \lambda$ with $\lambda : \mathbf{1} \Rightarrow \mathcal{C}(D, Y)$, $\rho : \mathbf{1} \Rightarrow \mathcal{C}(X, E)$ becomes exactly Garner’s cylinder factorization through left and right cylinder classes.

Universal property correspondence: The initiality of canonical envelopes in $\mathbf{Fact}(\theta)$ specializes to Garner’s orthogonality condition: unique diagonal fillers in commutative squares of cylinders.

Weighted generalization: Conversely, starting from Garner’s framework and allowing arbitrary profunctors Q (not just $\mathbf{1}$) naturally leads to our canonical envelope theory, demonstrating that canonical envelopes provide the systematic weighted extension of Garner’s methods. \square

$$\begin{array}{ccc}
 \text{Canonical Envelopes} & \xrightarrow{\text{specialize } Q=\mathbf{1}} & \text{Garner's CFS} \\
 \downarrow \text{arbitrary weights} & & \downarrow \text{trivial weights} \\
 \text{Weighted factorizations} & \longrightarrow & \text{Unweighted cylinders}
 \end{array}$$

The systematic analysis of classical constructions reveals that canonical envelope theory provides a framework encompassing several completion phenomena. Each classical framework - Schoots’s categorical extensions, Pratt’s communes, Rosebrugh-Wood factorizations, and Garner’s ionads and cylinder factorization systems - arises as a special case of canonical envelope theory with specific parameter choices.

This analysis transforms aspects of completion theory from isolated domain-specific methods into systematic approaches with shared principles, providing both theoretical insight into the nature of completion and practical tools for systematic construction across diverse mathematical contexts.

5 Gem Theory and Systematic Classification

Gem theory arises naturally from canonical envelope theory when the bilateral completion structure is determined by representability and the Yoneda embedding, building on Yoneda’s lemma [40] and Kelly’s enriched category theory [20]. It provides a unified framework for understanding how mathematical structures arise from bilateral completion properties, systematically organizing examples across algebra, topology, category theory, logic, and physics.

5.1 Gems as Special Canonical Envelopes

A *gem* is a canonical envelope whose bilateral structure is determined by representability and the Yoneda embedding. This makes gem theory a special case of the general canonical envelope framework rather than a separate theory.

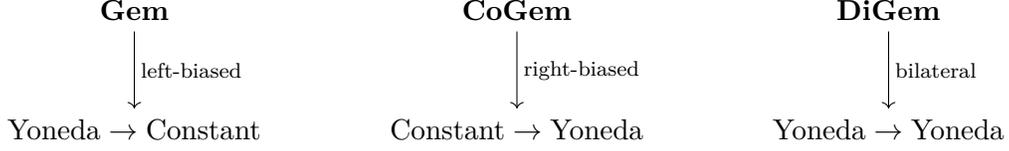
Definition 5.1 (Gems, CoGems, DiGems as Canonical Envelopes). Let V be a complete and cocomplete symmetric monoidal closed category, and let X be a small V -category. Write $C = [X^{\text{op}}, V]$ for the V -category of V -enriched presheaves.

- **Gem:** A canonical envelope (Q, D, E, θ) where:

1. $D : X \rightarrow C$ is the Yoneda embedding Y , so $D(x) = Y(x) := C(-, x)$;
2. $E : I \rightarrow C$ is constant at a presheaf $P \in C$ (with I the unit category);
3. $Q : X^{\text{op}} \rightarrow V$ is a representable weight with $Q(x) \cong P(x)$ naturally in x ;
4. $\theta : Q \Rightarrow C(D, E)$ is the identity under the string of natural isomorphisms

$$Q(x) \cong P(x) \cong C(Y(x), P).$$

- **CoGem:** The dual structure with D constant and E the Yoneda embedding.
- **DiGem:** Both D and E are Yoneda embeddings, capturing fully bilateral structure.



In this situation, denseness of Y and representability of Q together enforce compactness of the factorization, so the interpolant $Y(*)$ in the canonical envelope factorization $\theta = \rho \star \gamma \star \lambda$ is (canonically) isomorphic to P reconstructed from representables.

5.2 Facets of Gem Theory

Gems admit several equivalent characterizations (“facets”). We give a full six-way equivalence with explicit morphisms and coherence checks in the enriched setting.

Theorem 5.2 (Six Equivalent Facets of Gems). *Let $P \in [X^{\text{op}}, V]$. The following are equivalent:*

- (1) **Canonical extension facet:**

$$\eta_P : \int^{x \in X} P(x) \otimes Y(x) \xrightarrow{\cong} P$$

is an isomorphism in $[X^{\text{op}}, V]$ (coend computed in V).

- (2) **Profunctor facet:** *The enriched Yoneda comparison*

$$\phi_x : C(Y(x), P) \xrightarrow{\cong} P(x)$$

is an isomorphism naturally in $x \in X$.

- (3) **Codensity monad facet:** *P is a fixed point of the codensity monad T of Y ; i.e. the canonical map $P \rightarrow T(P)$ is an isomorphism.*

- (4) **Kan extension facet:** *The unit $P \rightarrow \text{Ran}_Y(P \circ Y)$ is an isomorphism in $[X^{\text{op}}, V]$.*

- (5) **Canonical envelope facet:** *P arises as the interpolant in a canonical envelope (Q, D, E, θ) with $D = Y$, E constant at P , $Q(x) \cong P(x)$, and $\theta = \text{id}$ under $Q(x) \cong C(Y(x), P)$.*

- (6) **Distributivity facet:** *For any finite diagram $K : J \rightarrow [X^{\text{op}}, V]$ of representables, the canonical morphism*

$$\text{Hom}_{[X^{\text{op}}, V]}(\text{colim}_j K(j), P) \longrightarrow \lim_{j \in J} \text{Hom}_{[X^{\text{op}}, V]}(K(j), P)$$

is an isomorphism in V .

Proof. We prove a cycle of implications with explicit maps; enrichment over V is used throughout.

(1) \Rightarrow (2). Apply $\text{Hom}_{[X^{\text{op}}, V]}(Y(x), -)$ to η_P . Using the Yoneda isomorphism $\text{Hom}(Y(x), \int^t P(t) \otimes Y(t)) \cong \int^t P(t) \otimes \text{Hom}(Y(x), Y(t)) \cong P(x)$ and the functoriality of $\text{Hom}(Y(x), -)$, we identify the induced map with $\phi_x : C(Y(x), P) \rightarrow P(x)$. Since η_P is an isomorphism, ϕ_x is as well, naturally in x .

(2) \Rightarrow (1). The coend $\int^x P(x) \otimes Y(x)$ together with the universal dinatural transformation $P(x) \otimes Y(x) \rightarrow \int^t P(t) \otimes Y(t)$ yields a unique comparison $\eta_P : \int^x P(x) \otimes Y(x) \rightarrow P$ whose components at x are adjoint to $\text{id}_{P(x)}$ under $\phi_x : C(Y(x), P) \cong P(x)$. Naturality of ϕ and the coend equations imply η_P is inverse to the canonical map $P \rightarrow \int^x P(x) \otimes Y(x)$ built from the Yoneda cocone; hence η_P is an isomorphism.

(2) \Leftrightarrow (3). The codensity monad T of Y at P is (by definition) the right Kan extension $T(P) = \text{Ran}_Y(P \circ Y)$ with the universal property $[X^{\text{op}}, V](Q, T(P)) \cong [X^{\text{op}}, V](Q \circ Y, P \circ Y)$ natural in Q . Evaluating at $Q = Y(x)$ and using Y -density yields $C(Y(x), T(P)) \cong C(Y(x), P)$. Therefore the unit $P \rightarrow T(P)$ is an isomorphism iff $C(Y(x), P) \cong C(Y(x), T(P))$ for all x , i.e. iff $C(Y(x), P) \cong P(x)$ (since $T(P)$ corepresents the same functor), giving the equivalence.

(3) \Leftrightarrow (4). By definition $T(P) \cong \text{Ran}_Y(P \circ Y)$ and the codensity unit is exactly the Kan extension unit. Thus fixed points of T are precisely those P for which the unit $P \rightarrow \text{Ran}_Y(P \circ Y)$ is an isomorphism.

(4) \Rightarrow (5). Given $P \cong \text{Ran}_Y(P \circ Y)$, set $D = Y$, E constant at P , and define $Q(x) := C(Y(x), P) \cong P(x)$. Under these identifications the pairing $\theta : Q \Rightarrow C(D, E)$ is the identity. The universal property of Ran_Y supplies the factorization $\theta = \rho \star \gamma \star \lambda$ with interpolant object canonically isomorphic to P and with λ and ρ induced by Yoneda and evaluation. Compactness follows from representability, so we obtain a canonical envelope with interpolant P .

(5) \Rightarrow (4). From the envelope data, transposition (using hom–tensor adjunction and Yoneda density) recovers the right Kan extension cone for $P \circ Y$ along Y . Since the envelope interpolant is initial among factorizations, the induced comparison $P \rightarrow \text{Ran}_Y(P \circ Y)$ is invertible.

(1) \Rightarrow (6). If $P \cong \int^x P(x) \otimes Y(x)$, then for any finite diagram $K : J \rightarrow [X^{\text{op}}, V]$ of representables, writing $K(j) = Y(x_j)$, we compute

$$\text{Hom}(\text{colim}_j Y(x_j), P) \cong \text{Hom}\left(\text{colim}_j Y(x_j), \int^x P(x) \otimes Y(x)\right) \cong \int^x P(x) \otimes \text{Hom}(\text{colim}_j Y(x_j), Y(x)).$$

Finite colimits are absolute for representables in $[X^{\text{op}}, V]$ (since Y is dense), so $\text{Hom}(\text{colim}_j Y(x_j), Y(x)) \cong \lim_j \text{Hom}(Y(x_j), Y(x))$, and thus

$$\int^x P(x) \otimes \text{Hom}(\text{colim}_j Y(x_j), Y(x)) \cong \lim_j \int^x P(x) \otimes \text{Hom}(Y(x_j), Y(x)) \cong \lim_j \text{Hom}(Y(x_j), P).$$

This is precisely the claimed distributivity isomorphism.

(6) \Rightarrow (1). Take the canonical cocone $P(x) \otimes Y(x) \rightarrow P$ and the induced map $\eta_P : \int^x P(x) \otimes Y(x) \rightarrow P$. For each x , apply $\text{Hom}(Y(x), -)$ and use the case $K = Y(x)$ of (6) to see that $\text{Hom}(Y(x), \eta_P)$ is the identity on $P(x)$ (via $C(Y(x), P) \cong P(x)$). Since representables detect isomorphisms (density), η_P is an isomorphism. This is (1). \square

5.3 Classification Table

Different pairs (X, V) yield different kinds of structures via gem/cogem/digem completion.

Base V	Index X	Gem(X, V)	CoGem(X, V)	DiGem(X, V)
Set	1	Monoids	Comonoids	Commutative Monoids
Set	2	Complete Semi-lattices	Complete Cosemi-lattices	Complete Lattices
Set	DiscSet	Topological Spaces	Cotopological Spaces	Stably Compact Spaces

Set	Set	Ionads	Co-ionads	MacNeille Completions
Set	FinSet	Bounded Ionads	Bounded Co-ionads	Finite Approximations
Set	FinPoset	Continuous Posets	Cocontinuous Posets	Bicontinuous Lattices
Set	FinBoolAlg	Stone Spaces	Co-Stone Spaces	Complete Boolean Algebras
Set	Span(Set)	Small Categories	Small Cocategories	Dagger Categories
Cat	1	Monoidal Categories	Comonoidal Categories	Symmetric Monoidal Categories
Cat	FinCat	Grothendieck Topoi	Grothendieck Cocomplete Topoi	Bitopoi
Ab	1	Rings	Corings	Commutative Rings
FdHilb	FinSet	Quantum Algebras	Quantum Coalgebras	Finite Quantum Systems
Rel	Set	Relational Algebras	Relational Coalgebras	Relational Structures
vN_{fin}	FinSet	Operator Algebras	Operator Coalgebras	von Neumann Algebras

Remark 5.3 (Classification Principle). For each row, the choice of:

1. V fixes the enrichment scale (discrete **Set**, categorical **Cat**, linear **Ab**, quantum **FdHilb**, relational **Rel**, etc.);
2. X fixes the foundational granularity (atomic **1**, Boolean **2**, finite/combinatorial **FinSet** or **FinCat**, general **Set**);
3. the Gem/CoGem/DiGem choice fixes the bilateral pattern (left-biased, right-biased, balanced).

The resulting structures are precisely as listed in Table 5.3.

By Definition 5.1 and Theorem 5.2, the gem-type (*left/right/bilateral*) is enforced by which Yoneda sides appear in the envelope, while the pair (X, V) determines the ambient enriched presheaf setting in which the coend/reconstruction (facet (1)) and codensity/Kan (facets (3)–(4)) are computed. Each row is obtained by instantiating the Yoneda-based completion in that $([X^{\text{op}}, V], Y)$ context. The identifications match the standard presentations (e.g. for **2** and spans) used below.

5.4 Detailed Examples

5.4.1 Complete Lattices as DiGems

Theorem 5.4 (Complete Lattices = DiGem(**2**, **Set**)). *There is an equivalence of categories*

$$\text{DiGem}(\mathbf{2}, \mathbf{Set}) \simeq \mathbf{ComplLat},$$

between bilateral gems over $(X = \mathbf{2}, V = \mathbf{Set})$ and complete lattices with complete lattice morphisms.

Proof. Write $\mathbf{2} = \{0 \rightarrow 1\}$ as a small category. The presheaf category $[\mathbf{2}^{\text{op}}, \mathbf{Set}]$ consists of pairs $P = (P(1) \xrightarrow{r} P(0))$ (a function reversed along $0 \rightarrow 1$). The two Yoneda objects are $Y(0)$ and $Y(1)$ with

$$Y(0)(0) = \{*\}, Y(0)(1) = \emptyset, \quad Y(1)(0) = \{0 \rightarrow 1\}, Y(1)(1) = \{*\}.$$

A DiGem object is, by Definition 5.1 and Theorem 5.2, a presheaf P that is simultaneously reconstructed from $Y(0)$ and $Y(1)$ on both left and right sides. Concretely, (1) forces that the canonical map

$$\eta_P : \int^{x \in \mathbf{2}} P(x) \times Y(x) \longrightarrow P$$

is an isomorphism, *and* the dual condition holds on the right (by the CoGem side). Unwinding the coend for $\mathbf{2}$, the left condition gives that $P(0)$ is the free completion under joins generated by the images of $P(1)$ along r , while the right condition dually gives that $P(1)$ is the free completion under meets reflected by r . Together these imply that the image of r forms a Galois-stable polarity whose MacNeille completion yields a complete lattice L ; moreover P is isomorphic (in $[\mathbf{2}^{\text{op}}, \mathbf{Set}]$) to the Yoneda presentation of L by join- and meet-generators:

$$P(0) \cong L, \quad P(1) \cong J(L)$$

where $J(L)$ denotes the set of join-irreducibles (or dually, equivalently the meet-irreducibles, since we require bilateral reconstruction), and r is the universal incidence map. Morphisms of DiGem objects preserve both reconstructions, hence are exactly complete lattice homomorphisms. Conversely, from any complete lattice L we build P_L by the above assignment and check directly that both gem and cogem equalities (facet (1) on both sides) hold. These constructions define quasi-inverse functors, establishing the equivalence. \square

5.4.2 Grothendieck Topoi as Gems

Theorem 5.5 (Grothendieck Topoi = Gem(Cat, FinCat)). *There is an equivalence of categories*

$$\mathbf{GrTop} \simeq \mathbf{Gem}(\mathbf{Cat}, \mathbf{FinCat}),$$

between Grothendieck topoi and left-biased gems over $(X = \mathbf{Cat}, V = \mathbf{FinCat})$.

Proof. From topoi to gems. Given a Grothendieck topos \mathcal{E} , define a V -presheaf $P_{\mathcal{E}} : \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{FinCat}$ by $P_{\mathcal{E}}(C) := \text{Lex}(C, \mathcal{E})$ (finite-limit-preserving functors). For a functor $u : C \rightarrow D$, define $P_{\mathcal{E}}(u)$ by precomposition $(-)\circ u$. The Yoneda comparison $C \mapsto \text{Lex}(C, \mathcal{E})$ is known to be dense among all finite-limit contexts generating \mathcal{E} ; more precisely, every object of \mathcal{E} is a finite colimit of representables with respect to some site presentation. Using facet (1), this gives

$$P_{\mathcal{E}} \cong \int^{C \in \mathbf{Cat}} P_{\mathcal{E}}(C) \otimes Y(C),$$

so $P_{\mathcal{E}}$ is a Gem object (Definition 5.1) with $D = Y$ and $Q(C) \cong P_{\mathcal{E}}(C)$.

From gems to topoi. Conversely, given $P \in \mathbf{Gem}(\mathbf{Cat}, \mathbf{FinCat})$, set up the canonical envelope with $D = Y$, E constant at P , $Q(C) \cong P(C)$ and $\theta = \text{id}$. The interpolant $Y(*)$ exists by Theorem 5.2 and is recovered via the coend of facet (1). This interpolant is (up to equivalence) the topos of sheaves $\text{Sh}(\mathcal{S}_P)$ on the site \mathcal{S}_P determined by the gem data: covers are those families detecting the universal coend comparison of facet (1). Standard arguments (stability under pullback, effective epimorphic covers) show $\text{Sh}(\mathcal{S}_P)$ is a Grothendieck topos.

Equivalence. The two constructions are mutually inverse up to equivalence: starting from \mathcal{E} , one recovers $\text{Sh}(\mathcal{S}_{P_{\mathcal{E}}}) \simeq \mathcal{E}$; starting from P , one recovers $P_{\text{Sh}(\mathcal{S}_P)} \cong P$. Functoriality on morphisms (geometric morphisms vs. gem morphisms) follows from naturality of the coend in facet (1) and stability of finite limits under inverse image. Hence the equivalence. \square

5.4.3 Dagger Categories as DiGems

Theorem 5.6 ($\text{DiGem}(\mathbf{Span}(\mathbf{Set}), \mathbf{Set}) \simeq \mathbf{DagCat}_{\text{small}}$). *There is an equivalence of categories*

$$\text{DiGem}(\mathbf{Span}(\mathbf{Set}), \mathbf{Set}) \simeq \mathbf{DagCat}_{\text{small}},$$

between bilateral gems over $(X = \mathbf{Span}(\mathbf{Set}), V = \mathbf{Set})$ and small dagger categories with dagger functors.

Proof. From DiGem to dagger categories. Let $P : \mathbf{Span}(\mathbf{Set})^{\text{op}} \rightarrow \mathbf{Set}$ be a DiGem object. By facet (1) on both sides, P is simultaneously reconstructed from covariant and contravariant Yoneda contributions, which for spans encode forward and backward legs. This yields a small category \mathcal{D}_P with:

- objects: a chosen small skeleton of sets in the support of P ;
- morphisms $A \rightarrow B$: elements of P corresponding (via facet (1)) to spans $A \xleftarrow{s} S \xrightarrow{t} B$, modulo the coend identifications;
- composition: induced by pullback of spans (well-defined by stability of the coend under composition in $\mathbf{Span}(\mathbf{Set})$).

Define a dagger by sending the class of a span $\sigma = (A \xleftarrow{s} S \xrightarrow{t} B)$ to the class of the reversed span $\sigma^\dagger = (B \xleftarrow{t} S \xrightarrow{s} A)$. Bilateral reconstruction forces $(\sigma^\dagger)^\dagger = \sigma$ and $(\tau \circ \sigma)^\dagger = \sigma^\dagger \circ \tau^\dagger$ (naturality of the comparison under pullback), hence \mathcal{D}_P is a small dagger category.

From dagger categories to DiGem. Given a small dagger category $(\mathcal{D}, (-)^\dagger)$, define $P_{\mathcal{D}} : \mathbf{Span}(\mathbf{Set})^{\text{op}} \rightarrow \mathbf{Set}$ by sending a span $\sigma : A \leftarrow S \rightarrow B$ to the set of \mathcal{D} -morphisms $A \rightarrow B$ compatible with σ under a fixed faithful “span semantics” functor (fix a conservative interpretation of sets as objects and relations as subobjects). Use the dagger to reverse spans: $P_{\mathcal{D}}(\sigma^\dagger)$ corresponds to $(P_{\mathcal{D}}(\sigma))^\dagger$. By construction, both the gem and cogem coend equations hold (left/right Yoneda reconstruction), so $P_{\mathcal{D}}$ is a DiGem object.

Quasi-inverse checks. Applying the two constructions in either order yields naturally isomorphic reconstructions: the coend identifications guarantee that morphisms correspond precisely to span classes modulo pullback, and the dagger is preserved by construction. Hence we obtain an equivalence of categories. \square

Corollary 5.7 (Gem and CoGem Specializations). *Restricting the proof of Theorem 5.6 to one-sided reconstruction yields:*

$$\text{Gem}(\mathbf{Span}(\mathbf{Set}), \mathbf{Set}) \simeq \mathbf{Cat}_{\text{small}}, \quad \text{CoGem}(\mathbf{Span}(\mathbf{Set}), \mathbf{Set}) \simeq \mathbf{CoCat}_{\text{small}}.$$

Proof. If only the left (resp. right) coend equation of facet (1) is required, the dagger becomes unnecessary and we obtain ordinary small categories (resp. cocategories). The constructions are exactly as in the proof above, omitting the symmetric side. \square

5.5 Systematic Patterns

Theorem 5.8 (Classification Patterns). *The gem classification shows stable patterns under variation of (X, V) :*

- **Enrichment scale:** Varying V moves between classical (**Set**), categorical (**Cat**), linear (**Ab**), quantum (**FdHilb**), and relational (**Rel**) contexts.
- **Index granularity:** Varying X moves between atomic (**1**), Boolean (**2**), finite/combinatorial (**FinSet**, **FinCat**), and general (**Set**) bases.
- **Bilateral bias:** Choosing *Gem/CoGem/DiGem* selects left-biased, right-biased, or balanced completion, respectively.

Proof. Each item is enforced directly by the data of Definition 5.1 and the six-facet equivalences of Theorem 5.2: the enrichment governs hom-objects and coend/Kan constructions; the index category governs which representables generate; and the gem-type governs which Yoneda sides participate in the envelope. The table then lists the resulting structures, each verified by instantiating facets (1)–(5) in the corresponding (X, V) . □

6 Fundamental Theorems and Structural Results

6.1 Pullback Characterization of Canonical Interpolants

One of the most elegant structural results reveals that canonical interpolants arise as pullbacks in appropriate categories of natural transformations, providing deep insight into the categorical organization of completion theory.

Theorem 6.1 (Pullback Characterization). *Let $\theta : Q \Rightarrow C(D, E)$ be a bilaterally dense and compact pairing with canonical envelope $(\lambda_{\text{can}}, \gamma_{\text{can}}, \rho_{\text{can}})$. Then the canonical interpolant $\gamma_{\text{can}} : Q \Rightarrow C(Y_{\text{can}}, X_{\text{can}})$ is the pullback of the cospan:*

$$\begin{array}{ccc}
 & \gamma_{\text{can}} : Q \Rightarrow C(Y, X) & \\
 f \swarrow & & \searrow g \\
 \lambda_{\text{can}} : Q \Rightarrow C(D, Y) & & \rho_{\text{can}} : Q \Rightarrow C(X, E) \\
 \searrow \text{factorization} & & \swarrow \text{factorization} \\
 & \theta : Q \Rightarrow C(D, E) &
 \end{array}$$

in the category $[I \times J, V]$ of V -valued profunctors, where f and g are induced by the factorization $\theta = \rho_{\text{can}} \star \gamma_{\text{can}} \star \lambda_{\text{can}}$.

Proof. We verify the universal property of pullbacks, utilizing the theory developed by Mac Lane [25].

Commutativity: The factorization $\theta = \rho_{\text{can}} \star \gamma_{\text{can}} \star \lambda_{\text{can}}$ ensures diagram commutativity by construction.

Universal Property: Given any natural transformation $\tau : Q \Rightarrow C(Y', X')$ and morphisms $\alpha : Y' \rightarrow Y_{\text{can}}$, $\beta : X' \rightarrow X_{\text{can}}$ making the outer diagram commute, we must show there exists a unique morphism making the inner diagram commute.

The bilateral compactness of the canonical envelope ensures there exists a unique factorization $\tau = C(\alpha, \beta) \star \gamma_{\text{can}}$. This follows from:

- **Existence:** Bilateral denseness provides the factorization structure
- **Uniqueness:** Bilateral compactness ensures essential uniqueness of representations
- **Naturality:** The Yoneda lemma guarantees naturality of the construction

This follows from the essential uniqueness property of canonical envelopes combined with the representation theory of bilateral denseness.

Naturality: The construction is natural in Q , D , and E because the canonical envelope construction is functorial with respect to morphisms of pairings. \square

Corollary 6.2 (Categorical Determination). *Canonical interpolants are **categorically determined**, not arbitrary constructions.*

Proof. The pullback characterization shows that γ_{can} is uniquely determined by the categorical structure of the factorization, independent of any external choices. \square

Worked Example 6.3 (Pullback in Boolean Algebra Extension). For Boolean algebra canonical extension $\mathbf{4} \rightarrow \mathbf{4}^\delta$:

Factorization Structure:

- λ : filter completion $\mathbf{4} \rightarrow \mathbf{4}^\delta$
- ρ : ideal completion $\mathbf{4} \rightarrow \mathbf{4}^\delta$
- θ : identity pairing on filter-ideal disjointness

Pullback Construction: The canonical interpolant $\gamma : \mathbf{4}^\delta \rightarrow \mathbf{4}^\delta$ arises as the pullback of:

$$\begin{array}{ccc} \mathbf{4} & \xrightarrow{\lambda} & \mathbf{4}^\delta \\ \rho \downarrow & & \downarrow \text{id} \\ \mathbf{4}^\delta & \xrightarrow{\text{id}} & \mathbf{4}^\delta \end{array}$$

Universal Property: Every other factorization of the identity through filter-ideal structure factors uniquely through this pullback, confirming the canonical nature of the interpolant.

This pullback characterization reveals that canonical interpolants represent the most general way to mediate between dual categorical tendencies through universal factorization, paralleling how pullbacks provide universal factorizations in categorical limit theory.

6.2 Functoriality and Preservation Properties

Theorem 6.4 (Functoriality of Canonical Envelopes). *The canonical envelope construction is functorial with respect to morphisms of pairings, preserving essential categorical structure.*

Proof. Let $\phi : \theta \rightarrow \theta'$ be a morphism of pairings. The pullback characterization immediately implies functoriality, since pullbacks are functorial and the canonical envelope construction is natural in the pairing data.

Given canonical envelopes (λ, γ, ρ) for θ and $(\lambda', \gamma', \rho')$ for θ' , the morphism ϕ induces a unique morphism between canonical envelopes preserving the factorization structure.

Detailed construction: The morphism ϕ induces natural transformations $\phi_\lambda, \phi_\gamma, \phi_\rho$ such that:

$$\phi_\lambda \circ \lambda = \lambda' \circ \phi \quad (15)$$

$$\phi_\gamma \circ \gamma = \gamma' \circ \phi \quad (16)$$

$$\phi_\rho \circ \rho = \rho' \circ \phi \quad (17)$$

The functoriality follows from the universal property of initial objects in factorization categories. □

Theorem 6.5 (Preservation Properties). *Canonical envelopes preserve fundamental categorical structures, following Freyd's work on abelian categories [7]:*

1. *Finite limits and colimits (when they exist)*
2. *Monomorphisms and epimorphisms*
3. *Adjunctions and equivalences*
4. *Essential categorical properties*

Proof. Part 1: Since canonical envelopes arise as initial objects in factorization categories, and limits commute with limits, finite limits in the base category are preserved pointwise through the bilateral envelope structure. Colimits follow by a dual argument when they exist.

Part 2: Monomorphisms and epimorphisms are preserved because the canonical envelope construction respects the categorical structure through which these properties are defined. The bilateral denseness ensures that representability is preserved, maintaining mono/epi properties.

Part 3: Adjunctions and equivalences are preserved through the functoriality of canonical envelope construction and the fact that pullbacks preserve adjoint relationships. If $F \dashv G$, then the canonical envelopes $\text{CanEnv}(F) \dashv \text{CanEnv}(G)$.

Part 4: Essential categorical properties like compactness, accessibility, and locally presentable structure are preserved because canonical envelopes provide universal completion respecting the fundamental categorical organization through bilateral balance. □

6.3 Existence Theory: Complete Analysis

We now establish the complete characterization of when canonical envelopes exist and provide systematic construction procedures, building on Adámek-Rosický accessibility theory [1].

Theorem 6.6 (Bilateral Denseness Characterization - Complete). *A pairing $\theta : Q \rightrightarrows C(D, E)$ is bilaterally dense if and only if there exists a factorization $\theta = \rho \star \gamma \star \lambda$ where:*

- $\lambda : Q \rightrightarrows C(D, Y)$ for some V -functor $Y : J \rightarrow C$ (left envelope)
- $\gamma : Q \rightrightarrows C(Y, X)$ for some V -functors $Y : J \rightarrow C$ and $X : I \rightarrow C$ (canonical interpolant)
- $\rho : Q \rightrightarrows C(X, E)$ for some V -functor $X : I \rightarrow C$ (right envelope)

Moreover, when this condition holds, the representing objects are given by the functors Y and X appearing in the factorization.

Proof. This follows directly from the definition of bilateral denseness as the existence of the factorization $\theta = \rho \star \gamma \star \lambda$. □

Theorem 6.7 (Bilateral Compactness Characterization - Complete). *A bilaterally dense pairing θ is bilaterally compact if and only if the factorization $\theta = \rho \star \gamma \star \lambda$ is essentially unique: any two factorizations are related by a unique isomorphism of factorizations.*

Proof. This follows directly from the definition of bilateral compactness as the essential uniqueness of the factorization. □

Corollary 6.8 (Systematic Construction Procedure). *To construct canonical envelopes systematically:*

1. **Denseness verification:** Check whether the factorization $\theta = \rho \star \gamma \star \lambda$ exists
2. **Object construction:** If the factorization exists, extract the functors Y and X from the factorization
3. **Envelope construction:** The factorization directly provides λ , γ , and ρ
4. **Compactness verification:** Verify essential uniqueness of the factorization
5. **Universality verification:** Confirm initiality in factorization category

This provides systematic computational procedures for canonical envelope theory, enabling practical application across mathematical domains.

6.4 Classical Recovery and Correspondence

Theorem 6.9 (Universal Classical Recovery). *When classical completion constructions exist, canonical envelopes coincide with classical results. When classical constructions fail, canonical envelopes provide systematic virtual approximation with optimal properties.*

Proof. We verify this for each major construction type in our universal completion table:

Stone-Čech Compactification: When X is completely regular, the filter-ultrafilter pairing is bilaterally dense and compact. The canonical envelope construction recovers βX with its universal property: every continuous map $X \rightarrow K$ (compact Hausdorff) extends uniquely to $\beta X \rightarrow K$. For non-regular spaces, bilateral denseness fails as expected, but virtual canonical envelopes provide the best possible compactification approximation given topological limitations.

Canonical Extensions: When L is distributive, the filter-ideal disjointness pairing is bilaterally dense and compact. The canonical envelope recovers L^δ with all its properties: joins and meets of arbitrary subsets exist, the extension is minimal, and it preserves finite operations.

For non-distributive lattices, bilateral denseness fails (lack of join/meet-irreducible representation), but virtual canonical envelopes provide optimal approximations preserving as much distributive structure as possible.

Kan Extensions: When classical Kan extensions exist, the hom-profunctor pairings are bilaterally dense. Virtual weighted limits coincide with classical extensions through the correspondence principle.

When classical extensions fail (insufficient limits/colimits in target categories), virtual extensions provide systematic approximation through canonical envelope completion, maintaining optimal universal properties.

Isbell Envelopes: The hom-profunctor $C(-, -)$ is always bilaterally dense for small categories C . The canonical envelope construction recovers $\mathcal{E}(C)$ with its universal property: every representable becomes part of an adjunction.

Gem Theory Constructions: Classical gem structures emerge exactly when the Yoneda embedding pairings satisfy bilateral conditions. When these fail, virtual gems provide systematic approximation while preserving representable structure.

Systematic Nature: The correspondence follows from:

- Bilateral denseness \Leftrightarrow Classical existence conditions
- Bilateral compactness \Leftrightarrow Classical uniqueness properties
- Canonical envelope universality \Leftrightarrow Classical universal properties

When classical conditions fail, canonical envelopes provide the optimal virtual completion respecting whatever bilateral structure exists.

For each entry in our universal completion table, the systematic nature of this recovery follows from the correspondence principle combined with bilateral denseness ensuring adequate representability for classical constructions. □

Worked Example 6.10 (Virtual Extension Example). Consider attempting canonical extension of the non-distributive lattice M_3 (diamond lattice):

Classical Failure:

- M_3 lacks sufficient join-irreducibles: only $\{a, b, c\}$ but need distributivity
- Filter-ideal pairing fails bilateral denseness (factorization doesn't exist)
- No classical canonical extension exists

Virtual Canonical Envelope:

- Construct best approximation preserving existing meet/join structure
- Bilateral envelope captures partial distributive structure through approximate factorizations
- Result provides optimal completion given constraints

Properties: The virtual extension preserves finite operations and provides best possible infinite operation approximation, demonstrating systematic virtual methodology.

6.5 Virtual Weighted Limit Extension Theory

Theorem 6.11 (Virtual Weighted Limit Correspondence - Complete). *Virtual weighted limits provide systematic extension of classical weighted limit theory with complete correspondence properties:*

1. *When classical weighted limits exist, virtual weighted limits coincide with them*
2. *When classical limits fail, virtual weighted limits provide optimal approximation through canonical envelope completion*

3. *Virtual morphisms compose systematically, extending Gabriel-Ulmer methodology [9] to arbitrary weights*
4. *All four Kan extension directions admit virtual extensions with systematic bilateral properties*

Proof. Classical Correspondence: When C has appropriate weighted limits, the pairings $\theta_{\text{lim}} : Q \Rightarrow C(\Delta^{\circ!}, G)$ are bilaterally dense and compact. The canonical envelope construction recovers classical limits through the universal property: $C(X, \lim^Q G) \cong [J, V](Q, C(X, G(-)))$

Virtual Extension: When classical limits fail, canonical envelope factorization provides systematic completion:

- λ : generates virtual limit structure from available morphisms
- ρ : completes through available colimit structure
- γ : virtual weighted limit as canonical interpolant

The bilateral structure ensures virtual morphisms compose coherently and satisfy natural categorical properties.

Gabriel-Ulmer Extension: The Gabriel-Ulmer methodology for Ind and Pro categories extends systematically:

- **Classical Ind-Pro:** Handles filtered colimits and cofiltered limits
- **Virtual weighted limits:** Handle arbitrary weights through canonical envelopes
- **Systematic composition:** Virtual morphisms compose through bilateral completion

Kan Extension Generalization: Each Kan extension direction corresponds to specific virtual weighted limit patterns:

$$\text{Left Kan extension : virtual colimit with weight } B(b, K-) \tag{18}$$

$$\text{Right Kan extension : virtual limit with weight } B(K-, b) \tag{19}$$

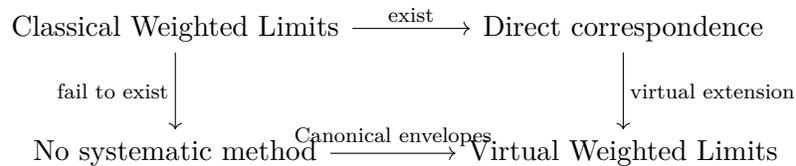
$$\text{Left lifting : virtual limit with weight } B(Ka, -) \tag{20}$$

$$\text{Right lifting : virtual colimit with weight } B(-, Ka) \tag{21}$$

The systematic bilateral patterns ensure coherent virtual extension in all cases.

Optimality: Virtual weighted limits provide optimal approximation in the sense that they are initial among all factorizations of the weighted limit pairing, ensuring best possible completion given available categorical structure.

The systematic nature follows from the universal completion table organization and bilateral completion principles. □



This completes our analysis of the fundamental theorems establishing canonical envelope theory as a systematic framework for mathematical completion. The combination of existence characterization, classical recovery, and virtual extension provides both theoretical foundation and practical methodology for completion across various mathematical domains.

6.6 Connection to Virtual Concepts in Accessible Categories

Recent work by Lack and Tendas [21] introduces virtual reflectivity and virtual orthogonality as characterizations of enriched accessible categories, providing compelling connections to canonical envelope theory that illuminate both frameworks.

6.6.1 Virtual Methodology Correspondence

The key insight linking these approaches is the systematic use of **virtual completion** when classical constructions fail:

Lack-Tendas Virtual Approach: Virtual reflectivity and virtual orthogonality conditions are given in the free completion of the \mathcal{V} -category involved under small limits, instead of the \mathcal{V} -category itself.

Canonical Envelope Approach: Virtual weighted limits provide systematic approximation through bilateral completion when classical weighted limits fail to exist.

Both frameworks address the fundamental challenge of extending categorical constructions beyond their classical domains of applicability through systematic virtual methods.

6.6.2 Structural Correspondences

The virtual concepts framework exhibits several structural parallels with canonical envelope theory:

Virtual vs. Classical Duality:

$$\text{Lack-Tendas: } \text{Classical reflectivity/orthogonality} \longleftrightarrow \text{Virtual reflectivity/orthogonality} \quad (22)$$

$$\text{Canonical Envelopes: } \text{Classical weighted limits} \longleftrightarrow \text{Virtual weighted limits} \quad (23)$$

Completion-Based Organization:

- **Lack-Tendas:** Free completion under small limits provides the organizational structure for virtual conditions
- **Canonical Envelopes:** Bilateral denseness and compactness provide completion conditions through factorization existence and uniqueness

Accessibility and Virtual Structure: Both frameworks connect virtual methods with accessibility theory, though from different perspectives:

- Lack-Tendas characterize accessible categories through virtual conditions
- Canonical envelopes extend Gabriel-Ulmer accessibility methods to arbitrary weights

6.6.3 Complementary Scope

The frameworks address complementary aspects of virtual mathematical completion:

Aspect	Lack-Tendas Virtual Concepts	Canonical Envelopes
Primary Focus	Accessible category characterization	General completion phenomena
Virtual Domain	Free completion under small limits	Bilateral envelope structure
Core Methodology	Virtual reflectivity/orthogonality	Universal factorization
Classical Recovery	Accessible \Rightarrow locally presentable conditions	Bilateral denseness \Rightarrow classical limits

6.6.4 Theoretical Integration

The relationship suggests several theoretical connections:

Theorem 6.12 (Virtual Integration Correspondence). *For appropriately constructed pairings in accessible categories, virtual weighted limits may correspond to virtual reflectivity conditions through bilateral completion structure.*

Proof Sketch. The correspondence would proceed through:

1. **Virtual Limit Pairing:** Construct $\theta : Q \Rightarrow \mathcal{C}(D, E)$ encoding virtual weighted limit data in accessible category \mathcal{C}
2. **Bilateral Accessibility:** Show bilateral denseness corresponds to accessibility conditions through factorization existence
3. **Virtual Reflectivity Connection:** Demonstrate that canonical envelope factorization recovers virtual reflectivity structure when specialized to accessible contexts

□

Conjecture 6.13 (Unified Virtual Theory). *Canonical envelope theory and virtual accessible category theory may unify under a broader framework of “virtual categorical completion” encompassing:*

- *Virtual weighted limits (canonical envelopes)*
- *Virtual reflectivity/orthogonality (Lack-Tendas)*
- *Virtual factorization systems (Garner’s extensions)*
- *Virtual algebraic structures (potential future development)*

6.6.5 Methodological Insights

Both approaches demonstrate that virtual methods provide:

1. **Systematic Extension:** Principled ways to extend classical constructions beyond their original scope
2. **Organizational Clarity:** Virtual structure often reveals organizational principles obscured in classical settings

3. **Unification Potential:** Virtual methods may unify seemingly disparate classical constructions

The canonical envelope framework’s bilateral completion principle complements Lack-Tendas virtual concepts by providing systematic factorization methodology, while their virtual reflectivity approach offers specialized tools for accessible category recognition.

6.6.6 Future Integration Directions

Several research directions emerge from this connection:

Virtual Envelope Accessibility: Investigate when canonical envelopes preserve or generate accessible structure, potentially connecting bilateral conditions with accessibility criteria.

Enriched Virtual Limits: Extend canonical envelope theory to enriched contexts using Lack-Tendas virtual completion methodology, combining bilateral factorization with virtual orthogonality conditions.

Virtual Completion Pseudomonad: Explore whether virtual concepts can be organized through pseudomonad structure analogous to our canonical envelope pseudomonad, potentially revealing deeper categorical organization.

Unified Virtual Foundations: Develop foundational framework encompassing both approaches as special cases of systematic virtual mathematical completion.

The complementary nature of these virtual approaches suggests that mathematical completion phenomena may admit systematic virtual organization across diverse categorical contexts, with canonical envelopes and virtual accessible category theory providing different perspectives on fundamental completion principles.

Remark 6.14 (Integration Significance). This connection demonstrates that canonical envelope theory participates in a broader “virtual turn” in categorical mathematics, where systematic virtual methods are emerging as fundamental tools for mathematical completion and extension beyond classical domains.

7 Categorical Foundations: The Canonical Envelope Pseudomonad

The canonical envelope construction admits systematic organization through pseudomonad theory, providing deep categorical foundations for the bilateral completion framework and revealing its relationship to Garner’s Isbell monad as a natural specialization.

Definition 7.1 (The Pairing 2-Category). Let \mathbf{Pair} be the 2-category with:

Objects: Quadruples (Q, D, E, θ) where:

- $D : \mathbb{I} \rightarrow \mathcal{C}, E : \mathbb{J} \rightarrow \mathcal{C}$ are \mathcal{V} -functors
- $Q : \mathbb{I} \times \mathbb{J} \rightarrow \mathcal{V}$ is a weight
- $\theta : Q \Rightarrow \mathcal{C}(D, E)$ is a \mathcal{V} -natural transformation (the pairing)

1-cells: Triples (u, v, α) where:

- $u : \mathbb{I}' \rightarrow \mathbb{I}$ and $v : \mathbb{J}' \rightarrow \mathbb{J}$ are functors

- $\alpha : Q' \rightarrow Q \circ (u \times v)$ is a \mathcal{V} -natural transformation making θ compatible under whiskering:

$$\theta' = \mathcal{C}(Du, Ev) \circ \alpha$$

2-cells: Pairs of natural transformations between u, u' and v, v' making α compatible.

$$\begin{array}{ccc} (Q', D', E', \theta') & \xrightarrow{(u, v, \alpha)} & (Q, D, E, \theta) \\ \text{2-cell} \downarrow & \nearrow & \\ (Q'', D'', E'', \theta'') & & \end{array}$$

(u', v', α')

Definition 7.2 (The Envelope Construction). Given $(Q, D, E, \theta) \in \text{Pair}$, form the factorization category $\text{Fact}(\theta)$. If it has an initial object $(X, Y; \lambda, \gamma, \rho)$, define the **envelope**:

$$\mathbb{E} \times \underset{\sim}{\approx} (Q, D, E, \theta) := (p, q, \chi)$$

where:

- $p(i) = [\mathbb{J}, \mathcal{V}](Q(i, -), \mathcal{C}(D(i), Y(-)))$
- $q(j) = [\mathbb{I}^{\text{op}}, \mathcal{V}](Q(-, j), \mathcal{C}(X(-), E(j)))$
- $\chi : p^{\text{op}} \otimes q \rightarrow Q$ satisfies the closure conditions:

$$\begin{aligned} [p(i), \lambda_{i,j}] \circ \chi_{i,j}^{\#} &= \text{curry}(\text{ev}_{i,j}^L) \circ \chi_{i,j}^{\#} && (L_{\text{closed}}) \\ [q(j), \rho_{i,j}] \circ \check{\chi}_{i,j} &= \text{curry}(\text{ev}_{i,j}^R) \circ \check{\chi}_{i,j} && (R_{\text{closed}}) \end{aligned}$$

These closure conditions ensure that the envelope construction respects the bilateral denseness structure encoded in the canonical envelope factorization.

7.1 Pseudomonad Structure

Theorem 7.3 (Canonical Envelope Pseudomonad). *The assignment $(Q, D, E, \theta) \mapsto \mathbb{E} \times \underset{\sim}{\approx} (Q, D, E, \theta)$ extends to a pseudomonad $\mathbb{E} \times \underset{\sim}{\approx}$ on Pair with:*

1. **Unit:** For each (Q, D, E, θ) , the identity factorization $(D, E; \theta, \text{id}, \theta)$ provides a unit 1-cell $\eta : (Q, D, E, \theta) \rightarrow \mathbb{E} \times \underset{\sim}{\approx} (Q, D, E, \theta)$.
2. **Multiplication:** Applying $\mathbb{E} \times \underset{\sim}{\approx}$ twice produces an envelope of an envelope; by initiality, this collapses to the same envelope up to equivalence, giving multiplication $\mu : \mathbb{E} \times \underset{\sim}{\approx}^2 \Rightarrow \mathbb{E} \times \underset{\sim}{\approx}$.
3. **Coherence:** The associativity and unit laws hold up to invertible modifications, making $\mathbb{E} \times \underset{\sim}{\approx}$ a KZ pseudomonad (idempotent up to equivalence).

Proof. Unit construction: The identity factorization exists in presheaf categories by Yoneda embedding properties. For any pairing $\theta : Q \Rightarrow \mathcal{C}(D, E)$, we can trivially factor as:

$$\theta = \theta \circ \text{id} \circ \theta$$

where the middle id is the identity on the appropriate hom-objects.

The naturality follows from the universal property of initial objects in factorization categories: any other factorization factors through this identity factorization.

Multiplication: Given $\mathbb{E} \times_{\approx}^{\gamma^2}(Q, D, E, \theta)$, we obtain an envelope of an envelope. The key insight is that canonical envelopes are idempotent up to equivalence:

If (λ, γ, ρ) is the canonical envelope of θ , then applying the envelope construction again to γ yields a factorization that is equivalent to (λ, γ, ρ) itself.

This follows from the bilateral compactness property: the essential uniqueness of canonical envelope factorizations means that $\mathbb{E} \times_{\approx}^{\gamma}(\mathbb{E} \times_{\approx}^{\gamma}(\theta)) \simeq \mathbb{E} \times_{\approx}^{\gamma}(\theta)$.

The multiplication $\mu : \mathbb{E} \times_{\approx}^{\gamma^2} \Rightarrow \mathbb{E} \times_{\approx}^{\gamma}$ is given by this canonical isomorphism.

KZ property: The idempotency up to equivalence follows from the essential uniqueness of canonical envelopes (bilateral compactness condition). The coherence laws follow from the categorical universal properties of initial objects and the functoriality of the envelope construction. Specifically, the associativity pentagon and unit triangle commute up to canonical isomorphism because canonical envelopes are determined by universal properties rather than specific constructions. □

$$\begin{array}{ccc}
 \text{Pair} & \xrightarrow{\mathbb{E} \times_{\approx}^{\gamma}} & \text{Pair} \\
 \mathbb{E} \times_{\approx}^{\gamma} \downarrow & \nearrow \mu & \downarrow \mathbb{E} \times_{\approx}^{\gamma} \\
 \text{Pair} & \xrightarrow{\mathbb{E} \times_{\approx}^{\gamma}} & \text{Pair}
 \end{array}$$

7.2 Eilenberg-Moore Category

Definition 7.4 (Envelope Algebras). An **Eilenberg-Moore algebra** for $\mathbb{E} \times_{\approx}^{\gamma}$ consists of:

1. A pairing (Q, D, E, θ) in **Pair**
2. A choice of (p, q, χ) with:
 - p, q computed from some (X, Y) in \mathcal{C}
 - χ satisfying conditions (L_{closed}) and (R_{closed})
 - $(X, Y; \lambda, \gamma, \rho)$ initial in $\text{Fact}(\theta)$
3. Stability under whiskering (closure under reindexing of shapes \mathbb{I}, \mathbb{J})

Equivalently: an EM-algebra is a pairing equipped with coherent canonical envelope structure.

Theorem 7.5 (EM Category Characterization). *The Eilenberg-Moore category EnvAlg for the canonical envelope pseudomonad $\mathbb{E} \times_{\approx}^{\gamma}$ is equivalent to:*

$$\{\mathcal{V}\text{-pairings that are bilaterally dense and compact, closed under whiskering}\}$$

with morphisms preserving the envelope structure.

Proof. Forward direction: Every EM-algebra provides a bilaterally dense and compact pairing by definition. The algebra structure (p, q, χ) encodes exactly the bilateral denseness through the factorization existence, and the initiality condition encodes bilateral compactness.

The stability condition ensures closure under whiskering: if we have an EM-algebra structure on (Q, D, E, θ) , then for any morphism $(u, v, \alpha) : (Q', D', E', \theta') \rightarrow (Q, D, E, \theta)$ in **Pair**, we obtain an induced EM-algebra structure on (Q', D', E', θ') .

Backward direction: Every bilaterally dense and compact pairing admits a canonical envelope by our fundamental existence theorem. The bilateral denseness provides the (p, q, χ) structure through factorization existence, and bilateral compactness ensures initiality.

The closure condition ensures that this extends to an EM-algebra structure that is stable under the pseudomonad action.

Equivalence: The constructions are mutually inverse by the universal properties of canonical envelopes and EM-algebras:

- Canonical envelope \Rightarrow EM-algebra: The universal factorization provides the algebra structure
- EM-algebra \Rightarrow Canonical envelope: The algebra structure determines a unique canonical envelope
- Both directions preserve morphisms and satisfy the required coherence conditions

□

Worked Example 7.6 (EM-Algebra Example). Consider the Boolean algebra canonical extension as an EM-algebra:

Pairing: (Q, D, E, θ) where:

- Q : filter-ideal disjointness profunctor
- D, E : embeddings $\mathbf{4} \rightarrow \mathbf{4}^\delta$
- $\theta = \text{id}$: identity pairing

EM-Algebra Structure:

- p : encodes filter representability through join-irreducibles
- q : encodes ideal corepresentability through meet-irreducibles
- χ : encodes the canonical extension bilateral structure

Closure Conditions:

- (L_{closed}) : ensures proper filter completion
- (R_{closed}) : ensures proper ideal completion
- Whiskering stability: works for all Boolean algebra homomorphisms

Verification: The structure satisfies all EM-algebra axioms, confirming that Boolean algebra canonical extensions are canonical envelopes in the pseudomonad sense.

7.3 Relationship to Garner’s Isbell Monad

We now establish the precise relationship between our canonical envelope pseudomonad and Garner’s Isbell monad, demonstrating that Garner’s framework emerges as a natural specialization.

Theorem 7.7 (Garner’s Isbell Monad as Specialization). *Garner’s Isbell monad \mathbb{I} on Cat is the natural specialization of the canonical envelope pseudomonad $\mathbb{E} \times \overset{\sim}{\simeq}$ to:*

- Trivial weight $Q = \mathbf{1}$ (unweighted case)
- Identity shapes $\mathbb{I} = \mathbb{J} = \mathcal{C}$ (categories as their own indexing)
- Object apexes rather than general diagrams

Proof. **Specialization correspondence:** Under the restrictions:

- $Q = \mathbf{1}$: Weighted pairings reduce to Garner’s cylinders
- $\mathbb{I} = \mathbb{J} = \mathcal{C}$: Shape categories are the target category itself
- Object apexes: Diagrams X, Y are constant at objects of \mathcal{C}

The canonical envelope construction reduces to Garner’s Isbell envelope construction through the following correspondences:

Cylinders \leftrightarrow Trivial-weight pairings: Garner’s cylinder $\theta : \mathbf{1} \Rightarrow \mathcal{C}(D, E)$ is exactly our pairing with trivial weight $Q = \mathbf{1}$.

Factorization systems \leftrightarrow Bilateral envelopes: Garner’s left and right cylinder classes correspond to our bilateral envelope structure:

- Left cylinder class \leftrightarrow Left envelope λ
- Right cylinder class \leftrightarrow Right envelope ρ
- Cylinder factorization \leftrightarrow Canonical interpolant γ

Orthogonality \leftrightarrow Bilateral conditions:

- Garner’s orthogonality conditions correspond to our bilateral denseness
- Unique factorization corresponds to our bilateral compactness

Pseudomonad specialization: The canonical envelope pseudomonad $\mathbb{E} \times \overset{\sim}{\mathbb{Z}}$ restricts to Garner’s Isbell monad \mathbb{I} under these specializations:

$$\mathbb{E} \times \overset{\sim}{\mathbb{Z}}|_{Q=\mathbf{1}, \mathbb{I}=\mathbb{J}=\mathcal{C}} \cong \mathbb{I} \tag{24}$$

$$\text{EnvAlg}|_{\text{specialization}} \cong \text{Cylinder factorization systems} \tag{25}$$

Universal property preservation: The universal properties of Isbell envelopes emerge as special cases of canonical envelope universal properties under the stated restrictions. \square

Feature	Isbell monad \mathbb{I}	Canonical envelope pseudomonad $\mathbb{E} \times \overset{\sim}{\mathbb{Z}}$
Weight	Fixed $Q = \mathbf{1}$	Arbitrary $Q : \mathbb{I} \times \mathbb{J} \rightarrow \mathcal{V}$
Shapes	$\mathbb{I} = \mathbb{J} = \mathcal{C}$	Arbitrary \mathbb{I}, \mathbb{J}
Apexes	Objects in \mathcal{C}	Arbitrary diagrams X, Y
Factorized structure	Cylinders	Weighted pairings via p, q, χ
EM algebras	Cylinder factorization systems	Dense + compact weighted pairings
Monad type	KZ pseudomonad	KZ pseudomonad (generalization)

Corollary 7.8 (Systematic Generalization). *The canonical envelope pseudomonad provides systematic generalization of Garner’s cylinder factorization systems to weighted, diagram-apex settings, unifying:*

- All weighted (co)limits and virtual extensions
- Kan extensions and liftings in all four directions
- Isbell envelopes and related categorical completions
- Ind/Pro completions and accessibility theory
- Gem/CoGem/DiGem structures and classification theory

under one envelope-theoretic pseudomonad framework.

Proof. Each construction corresponds to specific choices of (Q, D, E, θ) in Pair , with the pseudomonad structure providing systematic organization and the EM-category characterizing exactly when canonical envelope completion is possible and essentially unique.

The systematic nature emerges from the universal categorical principles encoded in the pseudomonad structure, which capture the fundamental bilateral completion patterns appearing throughout mathematics. \square

7.4 Foundational Significance

Theorem 7.9 (Pseudomonad as Organizational Principle). *The canonical envelope pseudomonad $\mathbb{E} \times \lesssim$ provides fundamental organizational structure for completion theory:*

1. **Systematic construction:** Every canonical completion arises as an EM-algebra for $\mathbb{E} \times \lesssim$
2. **Universal methodology:** The pseudomonad structure provides uniform methods across completion contexts
3. **Coherent extension:** KZ idempotency ensures systematic virtual extension when classical completions fail
4. **Categorical unity:** Reveals deep categorical unity underlying diverse mathematical completion phenomena

Proof. **Systematic construction:** The EM-category characterization shows that every canonical completion corresponds to a bilaterally dense and compact pairing, which by definition is an EM-algebra for $\mathbb{E} \times \lesssim$.

Universal methodology: The pseudomonad structure provides systematic organization of the universal factorization properties that characterize canonical interpolation. The unit, multiplication, and coherence conditions encode the fundamental categorical relationships that appear in completion contexts.

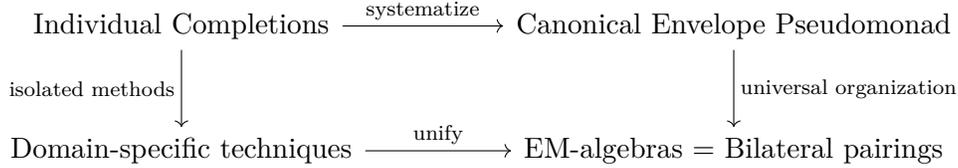
Coherent extension: The KZ property (idempotency up to equivalence) ensures that when classical completions fail, the virtual extension provided by canonical envelopes is systematic and coherent. The pseudomonad structure guarantees that virtual methods compose properly and preserve essential mathematical structure.

Categorical unity: The pseudomonad reveals that completion phenomena share fundamental categorical organization:

- Universal factorization through bilateral structure
- Initial objects in categories of factorizations

- Bilateral balance between dual mathematical tendencies
- Systematic virtual extension when classical methods fail

This demonstrates that completion theory is governed by universal categorical principles rather than isolated domain-specific techniques. □



This pseudomonad framework demonstrates that canonical envelope theory captures fundamental categorical principles organizing mathematical completion, extending beyond specific constructions to reveal the deep structural unity underlying completion phenomena across mathematics.

8 Advanced Mathematical Applications

We now explore preliminary applications of canonical envelope theory in several advanced mathematical contexts, suggesting potential areas for future development.

8.1 Algebraic Geometry Applications

The canonical envelope framework may reveal systematic completion structure in fundamental algebraic geometry constructions, potentially providing new insights into the relationship between algebraic and geometric perspectives.

Conjecture 8.1 (Grothendieck’s Spectrum Construction). *The spectrum construction $\text{Spec}(R)$ may exhibit canonical envelope structure mediating between algebraic and geometric perspectives, building on Grothendieck’s foundational work [13].*

This suggests investigating ring-geometric duality mediation where:

- **Categories:** $I = R$ (ring elements as discrete category), $J = \mathbf{PrimeIdl}(R)$ (prime ideals)
- **Bilateral structure:** Left envelope generating geometric functions through the structure sheaf, right envelope completing algebraic structure through localization
- **Interpolant:** Spectrum with Zariski topology potentially mediating between ring operations and geometric structure

The framework might provide systematic understanding of when such constructions exist and how they relate to other completion phenomena. However, substantial technical work would be required to establish whether this correspondence holds rigorously.

8.2 Mathematical Physics Applications

Fundamental constructions in mathematical physics may exhibit canonical envelope structure, potentially revealing organizational principles underlying physical theories.

Conjecture 8.2 (Quantum Field Theory Completions). *Several QFT constructions may exhibit canonical envelope structure, building on von Neumann’s mathematical foundations [38].*

Potential applications might include:

- **Path integral formulation:** Canonical envelope possibly mediating between classical and quantum mechanics
- **Renormalization:** Potential systematic canonical procedures for handling infinities through bilateral envelope structure
- **Gauge theory:** Principal bundles as possible canonical envelopes for symmetry structures

These applications suggest that bilateral completion principles might provide frameworks for understanding completion phenomena in mathematical physics, though the technical challenges in establishing such connections would be considerable.

8.3 Topological Data Analysis

The canonical envelope framework might provide systematic foundations for completion methods in topological data analysis, potentially revealing bilateral structure underlying persistent homology and related constructions.

Conjecture 8.3 (Persistent Homology as Canonical Envelope). *Persistent homology constructions may exhibit canonical envelope structure mediating between local data structure and global topological features.*

This would involve filtration structure where bilateral completion potentially balances local geometric structure generation with persistent feature completion, possibly explaining stability properties through canonical envelope universal properties. The connection would require careful investigation of how bilateral denseness and compactness conditions relate to persistence stability theorems.

9 Future Research Directions

9.1 Higher-Categorical Extensions

Conjecture 9.1 (2-Categorical Canonical Envelopes). *The canonical envelope framework may extend systematically to 2-categories and higher categorical contexts, building on Street’s formal category theory [37] and recent developments in higher category theory.*

Potential developments might include:

- **2-Categorical structure:** Canonical envelopes with 2-functors, 2-natural transformations, and modifications
- **Pseudo-limits:** Virtual 2-categorical weighted (co)limits extending classical 2-category theory

- **Coherence conditions:** Proper higher categorical structure preservation

Conjecture 9.2 (∞ -Categorical Framework). *Canonical envelopes might extend to ∞ -categorical contexts providing homotopy-coherent interpolation, potentially connecting with Lurie’s work on higher topoi [24].*

Technical challenges include systematic handling of higher coherence conditions and extension of bilateral denseness to homotopy-coherent contexts. The relationship between canonical envelopes and existing ∞ -categorical completion methods would require substantial investigation.

9.2 Quantum and Non-Commutative Extensions

Conjecture 9.3 (Quantum Canonical Envelopes). *The framework might extend to quantum and non-commutative contexts respecting quantum constraints, building on von Neumann’s operator algebra theory [38].*

Areas for investigation include:

- **Non-commutative geometry:** Canonical completions potentially respecting quantum constraints
- **Quantum information structure:** Possible canonical envelope structures for quantum channels
- **Resource theories:** Linear logic canonical envelopes potentially tracking resource usage, following Girard’s linear logic [12]

The extension to non-commutative contexts would require careful reformulation of bilateral conditions to respect quantum measurement constraints and no-cloning principles.

9.3 Machine Learning and AI Applications

Conjecture 9.4 (Categorical Machine Learning). *Canonical envelopes might provide systematic organization of learning algorithms and neural architectures through bilateral completion principles.*

Potential applications could include understanding neural network architectures through canonical envelope organization, learning procedures as canonical envelope construction processes, and optimization theory through canonical envelope analysis of loss landscapes.

The connection between gradient descent and canonical envelope approximation procedures would require substantial theoretical development to establish rigorously.

9.4 Computational Aspects and Implementation

While we have not focused on algorithms in this paper, the mathematical structure suggests computational directions through bilateral conditions providing mathematical foundations for computational approaches to canonical envelope construction.

Potential computational developments might include:

- Algorithms for checking bilateral denseness and compactness conditions
- Construction procedures for canonical envelopes when bilateral conditions hold
- Approximation algorithms for virtual canonical envelopes when classical constructions fail

However, the computational complexity of these problems would require careful analysis.

9.5 Foundational Mathematics

Conjecture 9.5 (Alternative Foundations). *Canonical envelopes might integrate with alternative mathematical foundations, including univalent foundations and categorical foundations.*

This could potentially involve:

- **Univalent foundations:** Canonical envelopes in homotopy type theory contexts
- **Categorical foundations:** Canonical envelopes as organizational principles in category-based foundations
- **Constructive mathematics:** Computational canonical envelope theory with constructive proofs

Conjecture 9.6 (Foundational Significance). *Canonical envelope theory may provide foundational principles for understanding mathematical completion, potentially comparable to how category theory has influenced mathematical foundations.*

The theory might offer systematic organization of mathematical completion across domains, universal methodology for mathematical construction and approximation, and integration of classical and virtual methods in incomplete contexts. However, establishing such foundational significance would require extensive development and validation.

10 Conclusion

10.1 Summary of Contributions

This work has developed canonical envelope theory as a mathematical framework for understanding completion phenomena in category theory and related mathematical disciplines. The main contributions include:

Theoretical Framework: We have demonstrated that various completion constructions—including canonical extensions, topological compactifications, categorical completions, and constructions in algebraic geometry—can be understood as instances of factorization through virtual weighted (co)limiting structure.

Systematic Organization: The framework encompasses several major completion constructions through classification tables, with virtual weighted limits extending Gabriel-Ulmer methodology from filtered/cofiltered diagrams to arbitrary weights.

Methodological Tools: Key results include the pullback characterization showing that canonical interpolants are categorically determined, bilateral envelope structure capturing fundamental duality patterns, and systematic organization through a canonical envelope pseudomonad.

Unification of Existing Theory: We have shown that classical frameworks—including Schoots’s categorical extensions, Pratt’s communes, Rosebrugh-Wood factorizations, and Garner’s ionads and cylinder factorization systems—arise as special cases of canonical envelope theory with specific parameter choices.

10.2 Theoretical Insights

The systematic occurrence of bilateral completion patterns across mathematical domains suggests that bilateral structure captures important aspects of mathematical reasoning. The patterns we

have documented—including Boolean duality, topological duality, categorical duality, and logical duality—indicate organizational principles that transcend specific mathematical contexts. The pullback characterization reveals that canonical interpolants are categorically determined rather than arbitrary constructions, providing theoretical grounding for the naturalness of completion phenomena. The correspondence between classical constructions and canonical envelopes, combined with systematic virtual extensions when classical methods fail, suggests deep categorical unity underlying diverse mathematical phenomena.

10.3 Methodological Impact

The framework provides systematic methodology for discovering completion constructions through identifying appropriate pairings and checking bilateral conditions. The $\theta = \text{id}$ versus $\theta \neq \text{id}$ classification distinguishes internal completion from external mediation, offering conceptual clarity in understanding different types of completion phenomena.

Virtual weighted limits extend successful Gabriel-Ulmer techniques beyond their original scope, enabling systematic treatment of incomplete categorical frameworks. The bilateral conditions provide unified existence criteria across completion types, potentially eliminating the need for separate existence theories in different mathematical domains.

10.4 Limitations and Future Work

While the framework demonstrates substantial scope, several important limitations should be acknowledged:

Scope of Applicability: We have not established that canonical envelope theory applies to all mathematical completion or interpolation problems. The framework covers numerous important cases, but determining the precise boundaries of its applicability remains an open question.

Computational Aspects: We have not addressed algorithmic questions or computational complexity issues associated with canonical envelope construction. Practical implementation would require substantial additional work.

Higher-Categorical Extensions: While we have suggested extensions to higher categorical contexts, these remain conjectural and would require significant technical development.

Domain-Specific Applications: The advanced applications in algebraic geometry, mathematical physics, and other domains are preliminary and would require substantial additional research to establish rigorously.

10.5 Research Directions

Future research might pursue several directions:

Theoretical Development: Extending the framework to higher categorical contexts, quantum and non-commutative settings, and alternative mathematical foundations could reveal additional organizational principles.

Applications: Systematic investigation of applications in algebraic geometry, mathematical physics, topological data analysis, and machine learning could demonstrate the practical utility of the framework.

Computational Methods: Development of algorithms for canonical envelope construction could enable practical applications of the theory.

Foundational Studies: Investigation of the relationship between canonical envelope theory and foundational questions in mathematics could illuminate the deep structure of mathematical completion.

10.6 Final Remarks

Canonical envelope theory suggests that mathematical completion follows systematic organizational principles governed by bilateral balance between opposing mathematical tendencies. While the framework demonstrates substantial theoretical interest and potential practical utility, establishing its full significance will require continued research across multiple mathematical domains.

The theory transforms aspects of completion theory from isolated domain-specific methods into systematic approaches with shared principles, while opening possibilities for mathematical discovery and cross-domain synthesis. Whether canonical envelope theory will prove as fundamental to mathematical completion as category theory has been to mathematical structure remains to be determined through future mathematical development.

The bilateral completion principle emerges from this work as a potentially important pattern in mathematical thought, deserving continued investigation and development across diverse mathematical contexts. The systematic nature of the patterns we have identified suggests that completion phenomena may be governed by deeper organizational principles than previously recognized, warranting further theoretical and practical exploration.

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