Wave Function of the Universe near Cosmological Singularity in three dimensions I

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Abstract

We investigate the wave function of the universe near the Big Bang Singularity in three dimensions. The Wheeler-DeWitt equation in the minisuperspace model is solved to obtain a wave function of the universe in Anti-deSitter space (cosmological constant :$\Lambda < 0$). In addition we speculate the case of TMG (Topologically Massive Gravity) Universe.

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Contents

1 Introduction

Space-time singularity (Black hole singularity, Big Bang singularity, Big Crunch singularity, etc)[1] is one of the most fascinating scientific problems[28][31]. Our world, namely our universe is explained to have begun around the past cosmological singularity (the Big Bang). If it is true, how did it begin? How our classical space-time (including "metric structure") was constructed through quantum gravitational process? (see for example[17][30][31]) Unfortunately we have not yet had a perfect quantum gravitational theory. Meanwhile quantum cosmology emerged as a simplified theory of quantum gravity. The creation of the universe can be described by the boundary condition of quantum cosmology. In quantum cosmology [1][2][7] the universe is described by a wave function which satisfies the Wheeler-DeWitt equation. The quantum cosmology is expected to be connected with inflation model smoothly around the Planck time. The key word of quantum cosmology is the boundary condition[29], since it is associated with the birth of the universe. We have no idea whether the Big Bang singularity represents the beginning of the universe.However it is believed to represent it. Vilenkin proposed the universe which is created by tunneling from nothing [1][3], on the other hand Hawking et al proposed the wave function which is represented by path integral over Euclidean manifolds with no boundary[2].Both are treated to be based on de Sitter Universe. In reality the observed cosmological constant is

[1]big bang singularity is different from black hole one fundamentally, since the latter is covered by a horizon, on the other hand the former is not.

positive , though nearly equal to zero. The geometrical structure of the universe is drastically changed according that cosmological constant is zero, positive and negative. The geometry corresponds to flat space, De Sitter space, and Anti de sitter space respectively. In addition,both models are dealt with well as if they escaped from the cosmological singularity by Wick rotation. The important point of quantum cosmology is the fact that it can treat the creation of the universe by the boundary condition while it deals with the singularity well. Hawking's model universe appears as a universe with a finite scale $(\frac{1}{\sqrt{\Lambda}})$ at initial point ($t = 0$). That scale is large enough compared with the Planck scale, since cosmological constant Λ is very small. Our real world is constructed of 4 dimensional spacetime. As we know, it is very difficult to deal with (3+1) gravity. So for simplicity we will focus on three dimensional spacetime (2+1) gravity[20] which has no local degree of freedom. We would like to study the quantum gravitational effect near cosmological singularity by the wave function of the universe.. If the wave function of the universe has the quantum gravitational aspect, it could represent the property of cosmological singularity even if not perfect. The wave function of the universe may have the information of the near Big Bang singularity which may be the beginning of the universe. (More precisely , whether the singularity is the beginning or not is not sure.) As we see, the wave function of the universe is the solutions of the Wheeler-DeWitt equation which is in a sense quantum gravitational equation. In such a meaning the wave function of the universe is a quantum gravitational wave function.

Big Bang singularity is the place where classical physics breaks down[4] and quantum gravity plays an important role. Especially the area within Planck scale from initial singularity is the most interesting . Now we call the region as the past singularity area (PSA). In PSA the space-time is not defined geometrically. The concept of space loses its meaning below the Planck length[31]. That region is the just quantum gravitational region. If the Wheeler-deWitt equation is a quantum gravitational equation, it could describe the wave function in PSA.We will investigate the wave function in this region.

In four dimensional case the wave function near singularity was studied with tunneling boundary condition [22]

The observed cosmological constant Λ is positive but very small ($\Lambda \cong 2 \times 10^{-123}$ in Planck units[24]) Recently observational accelerating universe may be the evidence of positive cosmological constant which means the repulsive force. However Hartle , Hawking, and Hertog presented the possibility that even theories with a negative cosmological constant can predict accelerating classical histories[23].

Geometrical structure of the universe depends on the cosmological constant signature though the observed value is almost zero.This fact is very strange.

Now our concentration is within Planck scale from the beginning of universe. The Friedmann equation is the classical equation. Quantum theory is represented by the Wheeler-DeWitt equation.

In addition we have a very useful conjecture based on holography, that is to say,

AdS/CFT correspondence which was induced from string theory. It is one of the most wonderful byproducts of string theory. By AdS/CFT correspondence, the wave function of AdS_3 universe is represented by partition function of CFT_2 on the boundary. The $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence states that AdS quantum gravity is holographically encoded in the boundary CFT_2 .
The cosmological singularity was studied by the standpoint of AdS/CFT [13][16][21] Originally, the concept of AdS/CFT correspondence, holography, was derived from the fact that black hole entropy is proportional not to volume but to area of horizon.

We assume the initial spacetime of the universe is AdS_3 which boundary exists at $z = 0$ where z means Poincare Coordinates of AdS space (see Appendix), since AdS space has good affinity with quantum gravity, compared with flat space and deSitter space. In AdS the radial wave function is a CFT partition function[12]. Three dimensional gravity has been studied well by many researchers [11][12][25]. In the past, three dimensional cosmology were studied based on this conjecture.[5][6]. The Hartle-Hawking wave function was computed in three dimensions with a positive cosmological constant[26]. Though the wave function near singularity is represented by the dual CFT partition function. Quantum gravity is not established as yet . In such a situation three dimensional gravity is supposed to be investigated profoundly compared with other dimensional gravity.
In addition pure three dimensional gravity is represented by Chern-Simons gauge theory.[14][15] Namely three dimensional gravity corresponds to both Conformal Field theory on boundary and Chern-Simons Gauge theory. In this article we will not deal with Chern-Simons Gauge theory , but in next article we will deal with it.

The purpose of this paper is to investigate the wave function of the universe near Big bang singularity in three dimension. We will focus on Anti-deSitter case , since Anti-deSitter case may be better to apply AdS/CFT correspondence to SA. We assume that in SA region the space-time structure is not constructed as de-sitter space but as Anti-deSitter space though it sounds crazy. The boundary has two meanings, namely 1, the boundary condition of the wavefunction of the universe , namely the beginning of the universe and 2, the boundary of AdS space where CFT lives.
We will also study wave function of the universe in Topologically Massive Gravity , which is the three dimensional gravity with gravitational Chern-Simons term. This paper is organized as follows: In Section 2 we derive the Friedmann equation in three dimensions. In section 3 the Wheeler-deWitt equation is derived by canonical quantization . In section 4 the wave function of the universe is obtained in the case of negative cosmological constant by semiclassical approximation. In section 5 we study TMG cosmology roughly. Section 6 contains our discussion and speculation .

2 Classical cosmological model in three dimensions

We consider the cosmological model in three dimensions based on pure Einstein gravity with no matter. The action is

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R-2\Lambda) \tag{1}$$

where R is the scalar curvature and Λ is the cosmological constant. We first consider the equations governing homogeneous and isotropic universe in minisuperspace model. In terms of the (2+1) dimensional Robertson-Walker metric:

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right) \\ &= -dt^2 + a^2(t)d\sigma^2 \\ &= g_{\mu\nu}dx^\mu dx^\nu \end{aligned} \tag{2}$$

where t and r, θ are time and space spherical coordinates, the signature of the metric is $(-,+,+)$, and $a(t)$is the scale factor of the universe which describes the classical dynamics of the extension of the universe. Here we introduce new coordinate:

$$d\chi \equiv \frac{dr}{\sqrt{1-kr^2}} \tag{3}$$

Integral is performed as follows:

$$\chi = \int d\chi = \int \frac{dr}{\sqrt{1-kr^2}} = \sin^{-1} r : k = 1(\text{closed}) \tag{4}$$

$$\chi = \sinh^{-1} r : k = -1(\text{open}) \tag{5}$$

$$\chi = r : k = 0(\text{open}) \tag{6}$$

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\theta^2 = d\chi^2 + \sin^2\chi d\theta^2 : k = 1 \tag{7}$$

$$d\sigma^2 = d\chi^2 + \sinh^2\chi d\theta^2 : k = -1 \tag{8}$$

$$d\sigma^2 = d\chi^2 + \chi^2 d\theta^2 : k = 0 \tag{9}$$

The above metric has to be satisfied with Einstein equation (pure gravity):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R-2\Lambda) = 0 \tag{10}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, and Λ is cosmological constant. We consider pure gravity with no matter for simplicity. First of all, we will calculate $R_{\mu\nu}$

and R. From equation (1),

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 \\ 0 & 0 & a^2r^2 \end{pmatrix}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1-kr^2}{a^2} & 0 \\ 0 & 0 & \frac{1}{a^2r^2} \end{pmatrix}$$

Now we can calculate easily the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R .

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\nu\mu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} + \Gamma^{\lambda}{}_{\lambda\kappa}\Gamma^{\kappa}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\nu\kappa}\Gamma^{\kappa}{}_{\lambda\mu} \tag{11}$$

and

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu} \tag{12}$$

where $\Gamma^{\lambda}{}_{\mu\nu}$ is the Christoffel Tensor:

$$\Gamma^{\kappa}{}_{\mu\nu} = \frac{1}{2}g^{\kappa\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \tag{13}$$

We can calculate $\Gamma^{\kappa}{}_{\mu\nu}$ easily as follows: for example:

$$\begin{aligned} \Gamma^{0}{}_{11} &= \frac{1}{2}g^{0\lambda}(\partial_1 g_{1\lambda} + \partial_1 g_{1\lambda} - \partial_\lambda g_{11}) \\ &= \frac{1}{2}g^{00}(\partial_1 g_{10} + \partial_1 g_{10} - \partial_0 g_{11}) \\ &= \frac{1}{2}\frac{d}{dt}\left(\frac{a^2}{1-kr^2}\right) \\ &= \frac{a\dot{a}}{1-kr^2} \end{aligned} \tag{14}$$

Other non-vanishing terms are

$$\begin{aligned} \Gamma^{0}{}_{22} &= r^2 a\dot{a} \\ \Gamma^{1}{}_{10} &= \Gamma^{1}{}_{01} = \frac{\dot{a}}{a} \\ \Gamma^{1}{}_{11} &= \frac{kr}{1-kr^2} \\ \Gamma^{1}{}_{22} &= -r(1-kr^2) \\ \Gamma^{2}{}_{02} &= \Gamma^{2}{}_{20} = \frac{\dot{a}}{a} \\ \Gamma^{2}{}_{12} &= \Gamma^{2}{}_{21} = \frac{1}{r} \end{aligned} \tag{15}$$

All other ${\Gamma^\lambda}_{\mu\nu} = 0$.
Then we can calculate R_{00}, R_{11},and R_{22} by use of the above results

$$\begin{aligned} R_{00} = {R^\lambda}_{0\lambda 0} &= \partial_\lambda {\Gamma^\lambda}_{00} - \partial_0 {\Gamma^\lambda}_{\lambda 0} + {\Gamma^\lambda}_{\lambda\kappa}{\Gamma^\kappa}_{00} - {\Gamma^\lambda}_{0\kappa}{\Gamma^\kappa}_{\lambda 0} \\ &= -\partial_0({\Gamma^1}_{10} + {\Gamma^2}_{20}) - ({\Gamma^1}_{01}{\Gamma^1}_{10} + {\Gamma^2}_{02}{\Gamma^2}_{20}) \\ &= -\frac{d}{dt}(\frac{\dot{a}}{a} + \frac{\dot{a}}{a}) - (\frac{\dot{a}}{a}\frac{\dot{a}}{a} + \frac{\dot{a}}{a}\frac{\dot{a}}{a}) \\ &= -2\frac{\ddot{a}}{a} \end{aligned} \tag{16}$$

Similarly

$$R_{11} = \frac{1}{1-kr^2}(\dot{a}^2 + a\ddot{a} + k) \tag{17}$$

$$R_{22} = r^2 a\ddot{a} + kr^2 + r^2\dot{a}^2 \tag{18}$$

Then the scalar carvature is

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} = 2\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} + 2\frac{\ddot{a}}{a}\right) \tag{19}$$

Substituting this R into action (1), we obtain

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R - 2\Lambda) \\ &= \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[2\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} + 2\frac{\ddot{a}}{a}\right) - 2\Lambda\right] \end{aligned} \tag{20}$$

Integrating the $\ddot{a}$ term in the resulting expression by parts with respect to t [3], we obtain the lagrangian:

$$L(a, \dot{a}) = \frac{1}{2}a(\dot{a}^2 - k + \Lambda a^2) \tag{21}$$

Substituting the components of the Ricci tensor $R_{\mu\nu}$ and the scalar curvature R into the equation (2) at $\mu = \nu = 0$. Namely, $(0,0)$ component,

$$R_{00} - \frac{1}{2}(R - 2\Lambda)g_{00} = 0 \tag{22}$$

We obtain the Freedman equation:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = 0 \tag{23}$$

From (i, i) components

$$R_{ii} - \frac{1}{2}(R - 2\Lambda)g_{ii} = 0 \tag{24}$$

second Freedman equation is obtained,

$$-\ddot{a} + \Lambda a = 0 \tag{25}$$

Einstein equation , in minisuperspace model , under the assumption that the universe is homogeneous and isotropic, corresponds to the Friedmann equation.

Mathematically (k, Λ) has nine cases :

$(1, \Lambda > 0), (-1, \Lambda > 0), (0, \Lambda > 0)$

$(1, \Lambda = 0), (-1, \Lambda = 0), (0, \Lambda = 0)$

$(1, \Lambda < 0), (-1, \Lambda < 0), (-0, \Lambda < 0)$

Almost all studies about wave function of the universe have dealt with the case $(k = 1, \Lambda > 0)$ which means closed de Sitter space . Here we will deal with the case $(k = -1, \Lambda < 0)$ which means open Anti de Sitter space, since AdS/CFT seems effective compared with dS/CFT and flat/CFT.

From now on we will concentrate on the case : $\Lambda = -|\Lambda| < 0, k = -1$, since we are interested in Anti-deSitter case. In addition , the wave function of Anti de-Sitter Universe is expected to be represented by Conformal boundary partition function by AdS/CFT correspondence.
Friedman equation is:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} + |\Lambda| = 0 \tag{26}$$

Considering initial condition $a(0) = 0$ and expanding universe , solutions are:

$$a = \frac{1}{\sqrt{|\Lambda|}} \sin |\Lambda| t \tag{27}$$

The above solution represents periodic expanding and contracting universe. Of course the Friedmann equation is the one of classical cosmology. We are interested in only PSA (Past

Singularity Area). This solution is simple but the solution of Einstein equation in the homogeneous, isotropic , and mini-superspace model. but classical Anti deSitter solution. We are interested in only Anti deSitter space , since AdS/CFT seems better compared with dS/CFT and flat/CFT..

3 Wave Function of the Universe

Next we will go to quantum area.

3.1 canonical quantization

We find the conjugated momentum from the Lagrangian:

$$p = \frac{\partial L}{\partial \dot{a}} = a\dot{a} \tag{28}$$

Hamiltonian is obtained as follows.

$$H = p\dot{a} - L = \frac{1}{2a}(p^2 + ka^2 - \Lambda a^4) \tag{29}$$

Here we can check the value of Hamiltonian is equal to zero.

$$\begin{aligned} H &= \frac{1}{2a}(p^2 + ka^2 - \Lambda a^4) \\ &= \frac{1}{2a}(a^2\dot{a}^2 + ka^2 - \Lambda a^4) \\ &= \frac{1}{2a^5}\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda\right) \\ &= 0 \end{aligned} \tag{30}$$

Friedman equation is the Einstein equation on the assumption of homogeneneous and isotropic universe.

Now we shall be looking for the wave function of the universe which is the result of quantization. we have two alternative methods of quantizing a system:

- quantization via path integral
- canonical quantization

In quantum cosmology we describe the universe by a wave function which satisfies the Wheeler-De Witt equation. We call it the wave function of the universe. We will arrive at the Wheeler-De Witt equation by conventional canonical quantization. Namely it is obtained by replacing $p \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial a}$.

$$\left(-\hbar^2 \frac{d^2}{da^2} + U(a)\right)\psi(a) = 0 \tag{31}$$

$$U(a) = -\Lambda a^4 + k a^2 \tag{32}$$

Wheeler DeWitt equation is the equation of quantum gravity and the Scrodinger equation of the universe. In this simple model we will obtain the wave function of the universe by WKB method. Our main purpose is to speculate the wave function near Big Bang singularity. In other words we will picture the wave function within Planck time from big bang. The usual wave function of the universe is constructed on De Sitter space. It is reasonable, since our observed cosmological constant is positive. However within Planck scale space-time it is not so concreted, so we would like to assume that the Planck scale wave function is composed of three parts, that is to say, De Sitter part wave function, flat part wave function , and Anti De Sitter part wave function. In other case we can assume the wave function in SA is composed of AdS wave function, since we know well that AdS wave function is represented by CFT partition function from AdS/CFT correspondence. Probably the total wave function there may be superposition of these three wave functions. From the standpoint of AdS/CFT correspondence, AdS_3 quantum gravity is represented by Conformal Field Theory on the boundary. More concretely the wave function of AdS_3 universe may be represented by CFT partition function on the boundary even around the region of Big Bang singularity.

$$\Psi_{tot} = \Psi_{dS} + \Psi_{flat} + \Psi_{AdS} \approx \Psi_{AdS} = Z_{CFT} \tag{33}$$

4 Semiclassical approximation

We have already obtained the equation of the universe - Wheeler-De Witt equation. Now we will resolve it to obtain the wave function of the universe. However it is very difficult to resolve it with no approximation. Here we use WKB approximation. The Wheeler-DeWitt equation is again:

$$\left(-\hbar^2 \frac{d^2}{da^2} + U(a)\right)\psi(a) = 0 \tag{34}$$

$$U(a) = -\Lambda a^4 + k a^2 = |\Lambda| a^4 - a^2 \tag{35}$$

The equation(34) has the same form as a zero energy , time-independent Schrodinger equation with a ptential that diverges like $|\Lambda| a^4$ at large a. Now we will solve the above equation by the usual WKB method (turning point : $a_0 = \frac{1}{\sqrt{|\Lambda|}}$)

$$\begin{aligned} U(a) &= -\Lambda a^4 - a^2 \\ &= |\Lambda| a^4 - a^2 \\ &= a^2(|\Lambda| a^2 - 1) \end{aligned}$$

$$\Psi_1(a) = \frac{2C}{\sqrt{p}} \cos\left(\frac{1}{\hbar}\int_a^{\frac{1}{\sqrt{|\Lambda|}}} p(a)da - \frac{\pi}{4}\right) \tag{36}$$

$$= \frac{2C}{\sqrt{p}} \cos\left(\frac{(-|\Lambda|a^2+1)^{\frac{3}{2}}}{3\hbar|\Lambda|} - \frac{\pi}{4}\right) \tag{37}$$

$$\Psi_2(a) = \frac{C}{\sqrt{\rho}} \exp\left(-\frac{1}{\hbar}\int_{\frac{1}{\sqrt{|\Lambda|}}}^{a} \rho(a)da\right) \tag{38}$$

$$= \frac{C}{\sqrt{\rho}} \exp\left(-\frac{(|\Lambda|a^2-1)^{\frac{3}{2}}}{3\hbar|\Lambda|}\right) \tag{39}$$

where

$$\begin{aligned} p^2 &= -a^4|\Lambda| + a^2 \\ \rho^2 &= |\Lambda|a^4 - a^2 \end{aligned}$$

If a tends to zero, wave function of the universe diverges.

$$\lim_{a\to 0} \Psi_1 \to \infty \tag{40}$$

In the above case the cosmological Singularity is the place where wave function of the universe divergent.

So the WKB method is not so good to investigate the behavior of the wave function near singularity here. As we would like to know the wave function around $a = 0$, let's do Taylor expansion of $U(a)$ around $a = 0$.

$$U(a) = |\Lambda|a^4 - a^2 \tag{41}$$

$$= U(0) + U'(0)a + \frac{1}{2!}U''(0)a^2 + \frac{1}{3!}U'''(0)a^3 + \frac{1}{4!}U^{(4)}(0)a^4 + \tag{42}$$

$$= -a^2 + |\Lambda|a^4 + \tag{43}$$

$$\approx -a^2 \tag{44}$$

where we assume $|\Lambda|$ is very small. Then the Wheeler-DeWitt equation is

$$\left(-\hbar^2 \frac{d^2}{da^2} - a^2\right)\psi(a) = 0 \tag{45}$$

The solutions of the above equation are represented as parabolic cylinder functions. We define x as:

$$a \equiv \sqrt{\frac{\hbar}{2}}x$$

Then we obtain the Weber differential equation:

$$\frac{d^2\Psi}{dx^2} + \frac{1}{4}x^2\Psi = 0 \tag{46}$$

Solutions are

$$\Psi_1 = D_{-\frac{1}{2}}(ix), \quad \Psi_2 = D_{-\frac{1}{2}}(-x) \tag{47}$$

where $D_\nu(z)$ is Weber function

$$D_\nu(z) \equiv 2^{\frac{\nu}{2}+\frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{\nu}{2}+\frac{1}{4},-\frac{1}{4}}\left(\frac{1}{2}z^2\right) \tag{48}$$

$$= 2^{\frac{\nu}{2}}\sqrt{\pi}e^{-z^2/4}\left[\frac{1}{\Gamma(\frac{1-\nu}{2})}F\left(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\right) - \frac{\sqrt{2}z}{\Gamma(-\frac{\nu}{2})}F\left(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\right)\right] \tag{49}$$

$$= \frac{2^{\nu/2}e^{-z^2/4}(-iz)^{1/4}(iz)^{1/4}}{\sqrt{z}}U\left(-\frac{1}{2}\nu,\frac{1}{2},\frac{1}{2}z^2\right) \tag{50}$$

$W_{k,m}(z)$: Whittaker function:

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}M_{k,-m}(z)$$

where $M_{k,m}(z)$[2] :

$$\begin{aligned} M_{k,m}(z) &= z^{m+1/2}e^{-z/2}F(m-k+\frac{1}{2}, 2m+1 : z) \\ &= z^{m+1/2}e^{-z/2}\sum_{n=0}^{\infty}\frac{\Gamma(2m+1)\Gamma(m-k+n+\frac{1}{2})}{\Gamma(2m+n+1)\Gamma(m-k+\frac{1}{2})}\frac{z^n}{n!} \end{aligned}$$

$U(a,b,z)$: confluent hypergeometric function of the first kind

$$D_{-\frac{1}{2}}(ix) = (ix)^{-\frac{1}{2}}W_{0,-\frac{1}{4}}(-\frac{1}{2}x^2) \tag{51}$$

$$D_{-\frac{1}{2}}(x) = x^{-\frac{1}{2}}W_{0,-\frac{1}{4}}(\frac{1}{2}x^2) \tag{52}$$

[2] $M_{k,m}(z)$ is also sometimes called Whittaker function

4.1 Interpretation of the wave function of the universe

In quantum mechanics, the state of a particle is described by its wavefunction $\Psi(x,t)$. The probability of finding the particle at (x,t) is represented by $|\Psi(x,t)|^2$. In quantum cosmology, the state of the universe is represented by the wave function of the universe. In principle, the wavefunction has to contain all information about the universe including the past, present , and future of the universe [2] although it is hard to extract all the information from it [1]. The wavefunction of the universe in minisuperspace model could be determined by the scale factor a, as $\Psi(a)$. $|\Psi(a)|^2$ may represent the probability of the universe staying in the state a during its evolution [10].
Behaviour of the wavefunction near singularity:

$$\Psi_1(a \to 0) \to \infty$$
$$\Psi_2(a \to 0) \to \infty$$

The wavefunction of the universe becomes divergent at cosmological singularity.
What does it mean? In normal way it means the breakdown of the theory at singularity. It has been said that quantum gravity will overcome the difficulty of the space-time singularity. Is it true, even if simplified Wheeler de Witt equation is applied? The above result may tell us that if simplified quantum gravity is applied to the cosmological singularity, it could not overcome the singularity problem. The wave function at initial singularity may include all informations of the universe. All may mean divergence. It is hoped that a complete theory of quantum gravity will overcome the singularity problem. We quantized the classical Friedmann equation to obtain the quantum Wheeler-de Witt equation. But the wavefunction which is the solution of Wheeler de Witt equation becomes divergent at singularity point. This may be the limitation of quantum cosmology though suitable boundary conditions (no boundary or tunneling boundary) may avoid the singularity well[1][2]. Multiverse model exists which avoids the big bang singularity well[27]

As a result, straight forward quantum cosmological approach seems difficult to investigate the cosmological singularity. in the next report, we will use the partition function on AdS_3 boundary as the wave function of the universe in order to investigate the cosmological singularity.

5 Cosmology based on TMG (Topologically Massive Gravity)

Topologically Massive Gravity (TMG) is a three dimensional general relativity with the gravitational Chern-Simons term. The action in Lorentzian signature is

$$S = \frac{1}{16\pi G}\left(\int d^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{\mu}S_{cs}\right) \tag{53}$$

where S_{cs} is the gravitational Chern-Simons term

$$S_{cs} = \frac{1}{2}\int d^3x\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\tau}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}_{\tau\nu} + \frac{2}{3}\Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\nu\tau}\right) \tag{54}$$

where $\epsilon^{\lambda\mu\nu}$ is Levi-Civita symbol. The above TMG action yields Einstein-Cotton equation through variational method

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0 \tag{55}$$

where Cotton tensor $C_{\mu\nu}$ is defined as(in detail [8]:

$$C_{\mu\nu} = \varepsilon_{\mu}^{\alpha\beta}\nabla_\alpha\left(R_{\nu\beta} - \frac{1}{4}Rg_{\nu\beta}\right) \tag{56}$$

where $\varepsilon_{\mu}^{\alpha\beta} = \frac{1}{\sqrt{-g}}\epsilon_{\mu}^{\alpha\beta}$ is Levi-Civita tensor.

5.1 TMG cosmology

If we deal with homogeneous and isotropic universe, Cotton tensor's contribution to the Friedmann equation disappears.

$$\begin{aligned}
C_{\mu\nu} &= \epsilon_{\mu}^{\alpha\beta}\nabla_\alpha\left(R_{\nu\beta} - \frac{1}{4}Rg_{\nu\beta}\right) & (57)\\
&= \epsilon_{\mu}^{\alpha\beta}\nabla_\alpha R_{\beta\nu} - \epsilon_{\mu}^{\alpha\beta}\nabla_\alpha(\frac{R}{4}g_{\beta\nu}) & (58)\\
&= \epsilon_{\mu}^{\alpha\beta}\nabla_\alpha R_{\beta\nu} - \epsilon_{\mu}^{\alpha\beta}\left[\nabla_\alpha(\frac{R}{4})g_{\beta\nu} + \frac{R}{4}\nabla_\alpha g_{\beta\nu}\right] & (59)\\
&= \epsilon_{\mu}^{\alpha\beta}\left(\partial_\alpha R_{\beta\nu} - \Gamma^{\lambda}_{\alpha\beta}R_{\lambda\nu} - \Gamma^{\lambda}_{\alpha\nu}R_{\beta\lambda}\right) - \epsilon_{\mu}^{\alpha\beta}g_{\beta\nu}\left(\partial_\alpha\frac{R}{4}\right) & (60)
\end{aligned}$$

where we used

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= 0 & (61)\\
\nabla_\alpha R &= \partial_\alpha R & (62)
\end{aligned}$$

Now we calculate C_{00} and C_{ii} to obtain Friedmann equations which corresponds to Einstein-Cotton equation.

$$\begin{aligned}
C_{00} &= \varepsilon_{0}^{\alpha\beta}(\partial_\alpha R_{\beta 0} - \Gamma^{\lambda}_{\alpha\beta}R_{\lambda 0} - \Gamma^{\lambda}_{\alpha 0}R_{\beta\lambda}) - \frac{1}{4}\varepsilon_{0}^{\alpha\beta}g_{\beta 0}(\partial_\alpha R)\\
&= \varepsilon_{0}^{12}(\partial_1 R_{20} - \Gamma^{\lambda}_{12}R_{\lambda 0} - \Gamma^{\lambda}_{10}R_{2\lambda}) + \varepsilon_{0}^{21}(\partial_2 R_{10} - \Gamma^{\lambda}_{21}R_{\lambda 0} - \Gamma^{\lambda}_{20}R_{1\lambda}) - 0\\
&= \varepsilon_{0}^{12}(0 - \Gamma^{0}_{12}R_{00} - \Gamma^{2}_{10}R_{22}) + \varepsilon_{0}^{21}(0 - \Gamma^{0}_{21}R_{00} - \Gamma^{1}_{20}R_{11})\\
&= 0 & (63)
\end{aligned}$$

Similarly

$$
\begin{aligned}
C_{11} &= \varepsilon_1^{\alpha\beta}(\partial_\alpha R_{\beta 1} - \Gamma^\lambda_{\alpha\beta} R_{\lambda 1} - \Gamma^\lambda_{\alpha 1} R_{\beta\lambda}) - \frac{1}{4}\varepsilon_1^{\alpha\beta} g_{\beta 1}(\partial_\alpha R) \\
&= \varepsilon_1^{02}(\partial_0 R_{21} - \Gamma^\lambda_{02} R_{\lambda 1} - \Gamma^\lambda_{01} R_{2\lambda}) + \varepsilon_1^{20}(\partial_2 R_{01} - \Gamma^\lambda_{20} R_{\lambda 1} - \Gamma^\lambda_{21} R_{0\lambda}) - 0 \\
&= 0 \qquad (64)
\end{aligned}
$$

Similarly $C_{22} = 0$
Namely, Cotton tensor has no contribution to the Friedmann Equations , on the assumption of homogeneous and isotropic minisuperspace model. In this case we obtain the same Wheeler-DeWitt equation as section 3 through canonical quantization, therefore the same wavefunction as section 4. In Classical Friedmann equation →Quantum Wheeler-DeWitt equation →Wavefunction, the gravitational Chern-Simons term has no contribution in homogeneous and isotropic minisuperspace universe model.

5.2 TMG partion function

On the other hand we know the partition of three dimensional gravity with gravitational Chern-Simins term explicitly[9].

$$
\begin{aligned}
Z_{tot} &= Z_{classical} Z^{graviton}_{1-loop} Z_M \\
&= q^{-l/8G} \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}} \ for \ \ \mu l = 1 \qquad (65)
\end{aligned}
$$

where,

$$
\begin{aligned}
Z_{classical} &= q^{-l/8G} \\
Z^{graviton}_{1-loop} &= \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \\
Z_M &= \prod_{m=2}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1-q^m \bar{q}^{\bar{m}}} \qquad (66)
\end{aligned}
$$

Z_M means massive graviton fluctuations. Of course this partition function includes the contribution from the gravitational Chern-Simons term. In AdS space the radial wave function corresponds to the partition function on the suitable boundary condition , from the standpoint of AdS/CFT correspondence. However as we know from subsection 5.1, the contribution of the Cotton tensor disappears in homogeneous and isotropic minisuperspace model. As the result the wave function of the universe has no effect from the gravitational Chern-Simons term. However AdS/CFT whispers that the wave function of the universe may be equal to the partition function on the suitable boundary. if we deform the Robertson-Walker metric , for example not isotropic etc, the wavefunction will include the contribution from the Cotton tensor which corresponds to the partition function from massive graviton fluctuation. Anyway the three dimensional partition function is surely to be the most useful information to access to quantum gravity at the present stage.

5.3 TMG cosmology : homogeneous and anisotropic universe?

The contribution of the Cotton Tensor to the Friedmann equation may not disappear if Robertson - Walker metric is deformed well. If partition function corresponds to wavefunction in AdS space, the above partition function - wavefunction - should include the contribution from the gravitational Chern-Simons term. On the other hand , in the case of homogeneous and isotropic universe the contribution of the Cotton tensor to Friedmann equation disappears as we pointed out. So my speculation is that if we deal with the deformation of the Robertson- Walker metric well, the contribution of the Cotton tensor to the Friedmann equation may not disappear. Anisotropic metric is known as the Kasner metric. We may be able to find out the corresponding part of the TMG partition function(65).

6 Discussion and Speculation

We obtained the Wheeler-DeWitt equation by canonical quantization from the Friedmann equation in three dimensions. We solved the Wheeler-DeWitt equation to obtain the wave function of the universe near cosmological singularity in AdS_3 space, which is represented by hypergiometric function. In principle quantum gravitational wave function of the universe could represent the cosmological singularity well. However , in our case the wave function is divergent at the point ($a = 0$). The big bang singularity is the place where the classical physics is break down. Our result shows the quantum cosmological wave function breaks down there as well as the solution of classical Friedmann equation. Quantum cosmology is the theory which deal with the singularity well by considering boundary condition(no boundary condition or tunneling condition). Namely it does not do with the singularity directly. Namely quantum cosmology is not a perfect quantum gravity.
In this paper we speculated , in order to investigate the wave function of the three dimensional universe near the past cosmological singularity, we could use the three dimensional gravity partition function which is one of the best studied results of quantum gravity based on AdS/CFT correspondence at the moment [9][12]. By the use of the result we may arrive at the clue of the Big Bang singularity. The singularity is the best place where quantum gravity plays a good role. Of course our speculation is not so confirmed.Recently Horowitz and coworkers studied cosmological singularity [13][16] which do with anisotropic case and calculated two point correlation function.
In next report we will find the corresponding wave function to the partition function by the use of the deformed Robertson-Walker metric.
My question ; holography can apply to the cosmological singularity? Holography was induced from the fact that black hole entropy is proportional to the area of the horizon,

not to the volume inside the horizon. But the situation of cosmological singularity is very different from the one of black hole singularity . The latter is covered with event horizon , but the former is not. Some study dealt with the wave function of the anti de Sitter space[19] where the cosmological constant is quantized.

On the other hand it was reported that asymptotic AdS gives possibly well-defined boundary condition for quantum gravity[18]. As AdS/CFT is well defined compared with dS/CFT, AdS may have a good affinity with quantum gravity. In this report we treated pure gravity case. Next we will treat the case including matter (scalar field).

This report is incomplete and preliminary. Next report will be improved better.

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Appendix AdS_3 space

1+2 dimensional Anti de-Sitter space(AdS_3) with length scale R is defined as the set of all points (X_0, X_1, X_2, X_3) in 4 dimensional spacetime $R^{2,2}$

$$ds^2 = -dX_0^2 - dX_3^2 + dX_1^2 + dX_2^2 \tag{67}$$

satisfying

$$X_0^2 + X_3^2 - X_1^2 - X_2^2 = R^2$$

Namely AdS_3 is embedded in 4 dimensional spacetime $R^{2,2}$. The isometry group for AdS_3 is $SO(2,2)$.

<u>Global coordinates</u>

AdS_3 is parameterized in global coordinates by the parameter (t, ρ, θ)

$$\begin{aligned} X_0 &= R\cosh\rho\cos t \\ X_1 &= R\cosh\rho\sin t \\ X_2 &= R\sinh\rho\cos\theta \\ X_3 &= R\sinh\rho\sin\theta \end{aligned}$$

The AdS_3 metric in these coordinates is

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2) \tag{68}$$

Poincaré coordinates
These coordinates are popular in the context of AdS/CFT.
AdS_3 is described as follows

$$\begin{aligned} X_0 &= \frac{z}{2}\left(1+\frac{R^2+x_1^2-x_0^2}{z^2}\right) \\ X_1 &= \frac{R}{2}x_1 \\ X_2 &= \frac{z}{2}\left(1-\frac{R^2-x_1^2+x_0^2}{z^2}\right) \\ X_3 &= \frac{Rx_0}{z} \end{aligned}$$

The AdS_3 metric in Poincaré coordinates is

$$ds^2 = \frac{R^2}{z^2}(-dx_0^2 + dx_1^2 + dz^2) \tag{69}$$

From (69) the boundary exists at $z = 0$. the boundary is $R^{1,1}$. Holographycally CFT_2 exists on $R^{1,1}$ as boundary condition.

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