BRAUER AND CERTAIN CLASS OF HANSEN CHAIN ARE CLOSED ADDITION CHAINS

THEOPHILUS AGAMA

Abstract. We show that Brauer and a certain class of Hansen chains satisfy the requirements for an addition chain to be closed. This puts these types of addition chain as a subfamily of the so-called closed addition chains.

1. Introduction

An addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* addition chain leading to n. In standard practice, we denote by $\ell(n)$ the length of an optimal addition chain that leads to n. A Brauer addition chain of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_{i-1} + s_j$ for $i > j \geq 0$. We denote the length of an optimal Brauer chain leading to n by $\ell^*(n)$. A number n for which the Brauer chain is optimal (i.e. $\ell^*(n) = \ell(n)$) is called a Brauer number. It is known ([2]) that Brauer numbers satisfy the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n).$$

The concept of Hansen addition chain is a well-known generalization of Brauer-type addition chains. A *Hansen chain* is an addition chain $s_o = 1, s_1 = 2, \ldots s_{r-1} = n$ for which there exists a fixed subset (anchor)

$$H \subseteq \{s_0, s_1, \ldots, s_{r-1}\}$$

such that each term s_k in the chain is formed as

$$s_k = s_i + s_j$$

with

$$s_i = \max\{h \in H \ : \ h < s_k\}$$

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for all k. A number for which a Hansen chain is optimal is called a Hansen number. Hansen numbers [3] are also known to satisfy the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n)$$

which is now known as the Scholz conjecture on addition chains. It is still not known whether the conjecture still holds for optimal addition chains which are neither Brauer nor Hansen. In this short note, we answer this question in the affirmative, thereby generalizing the concept of Brauer-Hansen addition chains. In [1], the author introduced the concept of a *closed* addition chain and showed that the numbers for which these chains are optimal also satisfy the inequality

$$\ell(2^n - 1) \leq n - 1 + \ell(n).$$

The notion of a *closed* addition chain is formally defined in the following way.

Definition 1.1 (Closed addition chains)**.** Let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ such that $i > \sigma(i) \geq \tau(i)$ for each $1 \leq i \leq h$. We say that a chain is a *closed* addition chain if for each $1 \leq i \leq h$ there exists some $j \in [0, h]$ such that

$$s_i - s_{i-1} = s_j$$

and for $s_{\sigma(i)} - s_{\sigma(i-1)} \neq 0$ there exists some $j \in [0, h]$ such that

$$s_{\sigma(i)} - s_{\sigma(i-1)} = s_j.$$

Additionally, for each $s_j = s_{\tau(i)}$ with $s_{\tau(i)} \neq s_{\sigma(k)}$ for all $k \in [0, h]$ there exist some $s_{\sigma(k)}$ such that $s_{\sigma(k)} < s_{\tau(i)}$ is consecutive.

In this note, we show that all Brauer chains and a certain class of Hansen chains are also closed, thereby placing these chains as a subfamily of closed addition chains. We give an example of a closed addition that is neither Brauer nor Hansen.

2. Main result

Definition 2.1. Let $s_o = 1, s_1 = 2, \ldots, s_h = n$ be an addition chain leading to n with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ such that $i > \sigma(i) \geq \tau(i)$ for each $1 \leq i \leq h$. We call the sequence $\{s_{\sigma(i)}\}_{i=1}^{h}$ the σ-track and the sequence $\{s_{\tau(i)}\}_{i=1}^{h}$ the τ-track of the addition chain. We say that the σ-track is strictly increasing if

$$s_{\sigma(i)} < s_{\sigma(i+1)}$$

for each $1 \leq i \leq h-1$. Similarly, we say that the τ-track is strictly increasing if

$$s_{\tau(i)} < s_{\tau(i+1)}$$

for each $1 \leq i \leq h-1$.

Theorem 2.2. *All Brauer addition chains and Hansen chains with strictly increasing σ-track are closed.*

Proof. Let

$$E(n) : s_o = 1 < s_1 = 2 < \ldots < s_h = n$$

be an addition chain leading to n, with $s_i = s_{\sigma(i)} + s_{\tau(i)}$ $(s_i > s_{\sigma(i)} \geq s_{\tau(i)})$ for each $1 \leq i \leq h$. If the chain is Brauer, then $s_{\sigma(i)} = s_{i-1}$ for each $1 \leq i \leq h$. This implies that $s_i - s_{i-1} = s_{\tau(i)} \in E(n)$ for each $1 \leq i \leq h$. Similarly for the difference along the σ-track, we get

$$s_{\sigma(i)} - s_{\sigma(i-1)} = s_{i-1} - s_{i-2} = s_{\tau(i-1)} \in E(n)$$

for each $1 \leq i \leq h$. Now, let $s_{\tau(k)}$ be a τ-track term that does not appear as a σ-track term. Then, necessarily, $s_{\tau(k)} > 1$. Consequently, there exists a term s_j in the addition chain such that

$$s_j < s_{\tau(k)}$$

are consecutive terms in the addition chain. Under the assumption that the chain is Brauer, it must be that $j = \sigma(\tau(k))$ and s_j is a σ-track term.
On the other hand, suppose that the above addition chain is Hansen. Then $s_i = s_{\sigma(i)} + s_{\tau(i)}$ with

$$s_{\sigma(i)} = \max\{l \in H \;:\; l < s_i\}$$

for each $1 \leq i \leq h$, where $H \subseteq \{s_o, s_1, \ldots, s_h\}$ is an anchor in the sense of Hansen. We deduce

$$s_{\sigma(i)} \leq s_{\sigma(i+1)} \leq s_i.$$

In the case $s_{\sigma(i)} = s_{\sigma(i+1)}$, then we have nothing to check for differences along the σ-track. We cannot have the strict inequality

$$s_{\sigma(i)} < s_{\sigma(i+1)} < s_i$$

since $s_{\sigma(i+1)} \in H$ and would violate the maximality condition

$$s_{\sigma(i)} = \max\{l \in H \;:\; l < s_i\}$$

for each $1 \leq i \leq h$. Therefore, we must have $s_i = s_{\sigma(i+1)}$ as the other possibility if we assume that $s_{\sigma(i)} \neq s_{\sigma(i+1)}$. In this case,

$$s_{\sigma(i+1)} - s_{\sigma(i)} = s_i - s_{\sigma(i)} = s_{\tau(i)} \in E(n)$$

and

$$s_{i+1} - s_i = s_{i+1} - s_{\sigma(i+1)} = s_{\tau(i+1)} \in E(n)$$

for each $1 \leq i \leq h-1$. Again, let $s_{\tau(k)}$ be a τ-track term that does not appear as a σ-track term. Then, necessarily, $s_{\tau(k)} > 1$. Consequently, there exists a term s_j in the addition chain such that

$$s_j < s_{\tau(k)}$$

are consecutive terms in the addition chain. Under the assumption that the chain is Hansen, then

$$s_{\sigma(\tau(k))} := \max\{l \in H \ : \ l < s_{\tau(k)}\}$$

with $s_{\sigma(\tau(k))} \leq s_j$. Thus

$$s_{\sigma(j)} \leq s_{\sigma(\tau(k))} \leq s_j$$

with

$$s_{\sigma(j)} := \max\{l \in H \ : \ l < s_j\}.$$

If we assume that $s_{\sigma(j)} \neq s_{\sigma(\tau(k))}$, then necessarily

$$s_j = s_{\sigma(\tau(k))}$$

otherwise the strict inequality

$$s_{\sigma(j)} < s_{\sigma(\tau(k))} < s_j$$

will violate the maximality condition

$$s_{\sigma(j)} := \max\{l \in H \ : \ l < s_j\}.$$

Thus, Brauer addition chains and certain classes of Hansen chains satisfy the three requirements for a chain to be *closed*. □

However, one can always construct a closed addition chain that is neither Brauer nor Hansen. Consider the following addition chain leading to $n = 13$.

Example 2.3 ($n = 13$)**.**

$$\{s_k\}_{k=0}^{6} = \{1, 2, 3, 5, 6, 8, 13\}$$

$$\begin{aligned}
s_0 &= 1, \\
s_1 &= 2 = 1 + 1 = s_{\sigma(1)} + s_{\tau(1)}, \\
s_2 &= 3 = 2 + 1 = s_{\sigma(2)} + s_{\tau(2)}, \\
s_3 &= 5 = 3 + 2 = s_{\sigma(3)} + s_{\tau(3)}, \\
s_4 &= 6 = 3 + 3 = s_{\sigma(4)} + s_{\tau(4)}, \\
s_5 &= 8 = 5 + 3 = s_{\sigma(5)} + s_{\tau(5)}, \\
s_6 &= 13 = 8 + 5 = s_{\sigma(6)} + s_{\tau(6)}.
\end{aligned} \tag{2.1}$$

Closure check:

(2.2)

$$2-1=1, \quad 3-2=1, \quad 5-3=2, \quad 6-5=1, \quad 8-6=2, \quad 13-8=5,$$

each gap lies among $\{s_0, \ldots, s_6\}$.

Closure check along σ tracks:

$$s_{\sigma(2)} - s_{\sigma(1)} = 2-1=1, \quad 3-2=1=s_{\sigma(3)} - s_{\sigma(2)}, \quad 5-3=2=s_{\sigma(5)} - s_{\sigma(4)},$$
$$8-5=3=s_{\sigma(6)} - s_{\sigma(5)},$$

each gap lies among $\{s_0, \ldots, s_6\}$.

Here, every τ-track term $\{1,2,3,5\}$ appears as terms along the σ track, so we have nothing to check for the third requirement of closed addition chains.

This addition chain is clearly not Brauer, since $s_4 = s_2 + s_2$ instead of $s_4 = s_3 + s_i$ $(s_i < s_3)$ according to the Brauer rule. One can also check that it fails to satisfy the requirements of a Hansen chain. The largest possible anchor - according to Hansen - is

$$H := \{1,2,3,5,8\} \subset \{1,2,3,5,6,8,13\}.$$

However, $6 = s_4 = s_2 + s_2$ instead of $s_4 = s_3 + s_o = 5+1$, since $5 \in H$ is the largest element in H strictly less than $s_4 = 6$. We may therefore underline (remove) this element from the anchor set to suit the requirement of a chain to be Hansen. We now obtain a new set

$$H_1 = \{1,2,3,8\} \subset \{1,2,3,5,6,8,13\}.$$

However, this set cannot serve as an *anchor*, since by construction of the chain $s_5 = s_3 + s_2 = 5+3$ and s_3 has already been removed. Because no proper subset of

$$H := \{1,2,3,5,8\}$$

can serve as an anchor, the above chain is not Hansen by construction.

References

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Departement de mathematiques et de statistique, Universite Laval, Quebec, Canada

E-mail address: thaga1@ulaval.ca/Theophilus@aims.edu.gh/emperordagama@yahoo.com