Generalizing the Concept of Repetends

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Abstract

Here, we generalize the concept and notation of repetends, develop an algebra of rules for manipulation, and give two examples of how these can be used in mathematics.

INTRODUCTION

In 1872, when Dedekind, Cantor, and Heine each gave their formal definitions of real numbers, it was already well-known that irrational numbers had non-repeating decimal digits. Two years later, in 1874, the word "repetend" appeared in literature to denote a repeated word, phrase, or sound[1]. The word "repetend" also began to be used for repeated decimal characters.[2]

NOTATION AND RULES OF MANIPULATION

However, there is presently no consistent notation for decimal repetends throughout the world. Depending on the country, a repetend is represented by parentheses $(a\ldots b)$, an overlying line (called a vinculum) $\overline{a\ldots b}$, an overlying arc $\overset{\frown}{a\ldots b}$, one or two overlying single dots $\dot{a}$ or $\dot{a}\ldots\dot{b}$, or an (ambiguous) ellipsis $a.bc\ldots$.[3]

One notation that suits our purposes is the single or one overdot[4], as follows:

$$a_1a_2\ldots a_n\cdot b_1b_2\ldots\dot{b}_m = a_1a_2\ldots a_n\cdot b_1b_2\ldots b_mb_mb_m\ldots$$

and $$a_1a_2\ldots a_n\cdot b_1b_2\ldots\dot{b}_m\ldots\dot{b}_{m+r} = a_1a_2\ldots a_n\cdot b_1b_2\ldots b_m\ldots b_{m+r}b_m\ldots b_{m+r}b_m\ldots b_{m+r}\ldots\ .$$

This notation can also be applied to strings or lists of characters outside its use with decimals. To do this, we first introduce the following concepts and notation:

Write $A=\{a_{n=1},a_2,\ldots\}$ to denote a list or listing of possible expressions[5] of A, as opposed to a set. We will deal here exclusively with lists.

We have/define

$$\{\{a_{n=1},a_2,\ldots\}\}=\{a_{n=1},a_2,\ldots\}, \qquad \text{(R-1)}$$

a property that is not shared by sets. The list is ordered with respect to its indices. These indices will usually be non-negative integers, but need not be as long as they are ordered and we know what the index ordinal value is.[6] Unless otherwise

[1] According to the Merriam-Webster English dictionary.
[2] I have been unable so far to find out whether the literary or mathematical use came first.
[3] See Wikipedia: Repeating Decimals.
[4] We are going to use "overdot" instead of "overlying dot".
[5] We use "expressions" here in a generic sense, not a mathematical one. For precision in meaning, we could use "list" for the expression $\{a_{n=1},a_2,\ldots\}$ and the gerundive "listing" for the process of creating it, but this may be too pedantic.
[6] We must always be able to do a change of variables so we can list the elements using integral ordinal values when applying arithmetic rule (R-7) later. An index may then be negative, zero, or non-integral.

indicated, we will assume the indices are the integral ordinal values. The first index will usually be shown as $n=$ to establish that we are working with a list/listing and not a set, as well as to identify the index as n.[7] We can also use $\{a_{n=1}, a_{n=2}, \dots, a_{n=r}, \dots\}$ at any time. This helps avoid confusion, especially with more than one index variable. Any combination of notations that is not ambiguous is acceptable. For example,

$$\begin{Bmatrix} \{a_{n=1}, a_{n=2}, \dots\}_{m=1}, \\ \{b_{n=1}, b_{n=2}, \dots\}_{m=2}, \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \{a_{n=1}, a_2, \dots\}_{m=1}, \\ \{b_{n=1}, b_2, \dots\}_{m=2}, \\ \vdots \end{Bmatrix}.$$

Call the expressions the elements of the list/listing. The elements $a_1, a_2, \dots$ here may be anything, *e.g*., characters, numerical values, mathematical formulae, images, etc. Unless otherwise indicated, it is understood that the listing $\{a_{n=m}, a_{m+1}, \dots\}$ contains all the elements a_n with $n \ge m$, while $\{a_{n=m}, a_{m+1}, \dots, a_{m+r}\}$ is a finite list of $r+1$ elements.

If f is a function[8] on $A = \{a_{n=1}, a_2, \dots\}$, we define

$$f\left(\{a_{n=1}, a_2, \dots\}\right) = \left\{\left(f(a_{n=1})\right)_{n=1}, \left(f(a_{n=2})\right)_{n=2}, \dots\right\}. \qquad \text{(R-2)}$$

No matter what the value of $f(a_n)$ is, its ordinality in the list on the right is the same as that of a_n in the list on the left.

If $f(a_m) = a_m$, then $\left(f(a_m)\right)_m = (a_m)_m = a_m$, to agree with (R-2).

Now we can formally introduce our single or one overdot notation. For any $r \ge 1$, let

$$\begin{aligned} \{\dot{a}_{n=1}, a_2, \dots, \dot{a}_r, \dots\} &= \left\{\{a_{n=1}, a_2, \dots, a_r\}_{m=1}, \{a_{n=1}, a_2, \dots, a_r\}_{m=2}, \dots\right\} \\ &= \left\{(a_1)_{k=1}, (a_2)_{k=2}, \dots, (a_r)_{k=r}, (a_1)_{k=1+r}, \dots, (a_n)_{k=n+(m-1)r}, \dots\right\}, \end{aligned}$$

for all $m, n \ge 1$ such that $(a_n)_{k=n+(m-1)r}$ lies within $\{a_{n=1}, a_2, \dots, a_r, \dots\}$,

be a finite or infinite listing of a repeated list $\{a_{n=1}, a_2, \dots, a_r\}$. This gives

$$\{\dot{a}_{n=1}, a_2, \dots, \dot{a}_r\} = \left\{\{a_{n=1}, a_2, \dots, a_r\}\right\} = \{a_{n=1}, a_2, \dots, a_r\}.$$

In $\left\{\{a_{n=1}, a_2, \dots, a_r\}_{m=1}, \{a_{n=1}, a_2, \dots, a_r\}_{m=2}, \dots\right\}$, we call $a_1, a_2, \dots, a_r$ the repetend, or more specifically the n-repetend. We call r the repetend length, or more specifically the n-repetend length. For $r=1$, we have

$$\{\dot{a}_1, \dots\} = \left\{\{a_1\}_{m=1}, \{a_1\}_{m=2}, \dots\right\} = \{(a_1)_{m=1}, (a_1)_{m=2}, \dots\}.$$

From here on, we are concerned with elements that have quantifiable values[9]. We introduce double or two overdot notation in lists to denote the following property:

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[7] This is not necessary and may be dispensed with if there is no ambiguity.

[8] A function here need not produce numerical values.

[9] A function on these values, for our purposes now, must give numerical values, ones that can be used for numerical arithmetic.

for any $r \geq 1$,

$$A = \{a_{n=1}, a_2, \ldots\} = \{\ddot{a}_{n=1}, \ldots, \ddot{a}_{1+r}, \ldots\},$$

if and only if $d = a_{k+r} - a_k$ is a constant, for all $k \geq 1$. (R-3)

For every $k \geq 1$, unless $d = 0$, it is not the elements $a_{n=k}, a_{k+1}, \ldots, a_{k+r-1}$ that are repeated here but their (arithmetic) pattern in a step-wise manner.

For either single or double overdot notation the following hold:

(1) the elements may be real or complex;
(2) we will say that $A = \{a_{n=1}, a_2, \ldots\}$ is a repetend function of n, or an n-repetend function;
(3) for $A = \{a_{n=1}, a_2, \ldots\}$, we will call $a_k, \ldots, a_{k+r-1}$ for each $k \geq 1$ a repetend, or more precisely an n-repetend;
(4) we will call r the repetend length, or more precisely an n-repetend length; and
(5) we will call d the repetend difference, or more precisely the n-repetend difference.

Using the terminology "n-repetend" emphasizes which index variable is involved and is helpful when more than one index variable is involved.

If we write $\{a_{n=1}, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\}$, for any $(m \geq 2;\ r \geq 1)$, the elements $a_1, a_2, \ldots, a_{m-1}$ may not satisfy (R-3). We still have $d = a_{m+k+r} - a_{m+k}$ is a constant, for all $k \geq 0$, and the first n-repetend may be $a_m, a_{m+1}, \ldots, a_{m+r-1}$. We may therefore move the double overdots to the right as follows:

Proposition 1. $\{a_1, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\} = \{a_1, \ldots, \ddot{a}_{m+k}, \ldots, \ddot{a}_{m+k+r}, \ldots\}$, for all $m, k, r \geq 1$. □

However, the converse is not true in general, although there are exceptions[10]. If we are given $\{a_{n=1}, a_2, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\}$ and the start of the first an n-repetend is at $n = m > 1$, we may not be able to move the double overdots to the left to get

$$\{a_1, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\} = \{a_1, \ldots, \ddot{a}_{m-k}, \ldots, \ddot{a}_{m-k+r}, \ldots\}, \text{ for any } k \geq 1.$$

For any $n \geq 1$,[11] given a single overdot notation listing, the single overdot notation can always be converted to the double overdot notation, as follows:

$$\{\dot{a}_{n=1}, \ldots, \dot{a}_r, \ldots\} = \{\ddot{a}_{n=1}, \ldots, \ddot{a}_{r+1}, \ldots\}, \text{ with } d = a_{k+r+1} - a_{k+1} = 0, \text{ for all } k \geq 0. \quad \text{(R-4)}$$

This allows us to call r in both the single and double overdot notations by the same name, n-repetend length.

Converting the double overdot notation to a single overdot notation is possible if and only if $d = a_{k+r+1} - a_{k+1} = 0$, for all $k \geq 0$.

[10] An obvious exception is one in which the overdots have already been moved to the right before a list is presented to you, but there are others. One that uses **Proposition 4** is given at the end of this appendix.

[11] The choice of $n \geq 1$ here is for convenience only. The indices may be any values that can be assigned ordinal values relative to each other.

As with the double overdots, the single overdots may also always be moved any distance to the right, but not always to the left.

In general,

$$\text{the smallest repetend length possible is } r = 1. \qquad \text{(R-5)}$$

$$\text{the smallest absolute repetend difference possible is } d = 0. \qquad \text{(R-6)}$$

Along with (R-2), we impose rules for arithmetic with lists. For elements in two lists with the same ordinal index values we use the following rule:

$$\{a_{n=1}, a_2, \ldots, a_m, \ldots\} \pm \{b_{n=1}, b_2, \ldots, b_m, \ldots\} = \{(a_{n=1} \pm b_{n=1})_{n=1}, (a_{n=2} \pm b_{n=2})_{n=2}, \ldots, (a_{n=m} \pm b_{n=m})_{n=m}, \ldots\}. \qquad \text{(R-7)}^{12}$$

For any c, we impose the following two rules:

$$c + \{a_{n=1}, a_2, \ldots, a_m, \ldots\} = \{(c + a_{n=1})_{n=1}, (c + a_2)_2, \ldots, (c + a_m)_m, \ldots\}, \qquad \text{(R-8)}$$

$$c\{a_{n=1}, a_2, \ldots, a_m, \ldots\} = \{(ca_{n=1})_{n=1}, (ca_2)_2, \ldots, (ca_m)_m, \ldots\}. \qquad \text{(R-9)}$$

We multiply a list by a list in the same way as we do a matrix by a matrix:

$$\{a_{i=1}, a_2, \ldots\}^{\mathrm{t}}\{b_{j=1}, b_2, \ldots\} = \{a_i b_j\} = \left\{\begin{matrix} a_{i=1}\{b_{j=1}, b_2, \ldots\}, \\ a_{i=2}\{b_{j=1}, b_2, \ldots\}, \\ \vdots \end{matrix}\right\}, \qquad \text{(R-10)}$$

where the superscript t indicates transpose: $\{a_{i=1}, a_2, \ldots\}^{\mathrm{t}} = \left\{\begin{matrix} a_{i=1}, \\ a_{i=2}, \\ \vdots \end{matrix}\right\}$.

Each element of one list can be treated independently from the other elements of that list for addition and subtraction with the elements of another list as follows:

$$\{a_{i=1}, a_2, \ldots\}^{\mathrm{t}} + \{b_{j=1}, b_2, \ldots\} = \left\{\begin{matrix} a_{i=1}, \\ a_{i=2}, \\ \vdots \end{matrix}\right\} + \{b_{j=1}, b_2, \ldots\} = \left\{\begin{matrix} \{a_{i=1} + \{b_{j=1}, b_2, \ldots\}\}_{i=1}, \\ \{a_{i=2} + \{b_{j=1}, b_2, \ldots\}\}_{i=2}, \\ \vdots \end{matrix}\right\}. \qquad \text{(R-11)}$$

In (R-7) it is the index ordinal values that determine which elements are added together or subtracted, not what the indices are labeled as or what the index values are. If we restrict ourselves to all the elements of the lists here being real, we must then also have c in (RT-8) and (R-9) being real.

If a repetend element a_m consists of more than one character, it is often convenient to write $\ddot{a}_m = \ddot{(a)}_m$ or $\ddot{a}_m = \ddot{(a_m)}$, whichever is appropriate under the circumstances.

When two index variables are involved, each independently associated with repetends, we can write

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[12] We will not concern ourselves here with one list containing index ordinal values that do not occur in the other list. In general, what is done in such situations depends on the context.

$$\left\{\begin{array}{l}\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r_n},\ldots\ddot{\}}_{m=1}, \\ \vdots \\ \{\ddot{b}_{n=1},\ldots,\ddot{b}_{1+r_n},\ldots\ddot{\}}_{m=1+r_m}, \\ \vdots\end{array}\right\}.$$

This can be extended to any number of index variables, similar to matrix notation. In such a case the double overdot relation (R-3) for any variable applies with the other variables being kept constant. When possible, different index variables can be displayed in different index rows associated with the elements.

We say $\{a_{n=1},\ldots,a_m\} \geq \{b_{n=1},\ldots,b_m\}$ if and only if the relation $a_n \geq b_n$ holds term by term, where m need not be finite, and similarly for $=,\leq,>,<$.

USEFUL RELATIONS

With these definitions, we can now start to do manipulations and calculations. If a proof is not given for any of the following relations, it is left to the reader.

Proposition 2. *If f is a linear function on $A = \{a_{n=1}, a_2, \ldots\}$, then*

$$f\left(\{\ddot{a}_{n=1}, a_2, \ldots, \ddot{a}_{1+r}, \ldots\}\right) = \left\{\left(f(a_{n=1})\ddot{\right)}_{n=1}, \left(f(a_2)\right)_2, \ldots, \left(f(a_{1+r})\ddot{\right)}_{1+r}, \ldots\right\}.^{13} \qquad \text{(P2-0)}$$

Proof: For some constants B and C, we have

$$f(a_n) = Ba_n + C.$$

By (R-3), we have $a_{k+r} - a_k = d_A$, for all $k \geq 1$, where d_A is a constant.
Therefore, for any $k \geq 1$, we have

$$\begin{aligned}\left(f(a_{k+r})\right)_{k+r} - \left(f(a_k)\right)_k &= (Ba_{k+r} + C)_{k+r} - (Ba_k + C)_k \\ &= B\left((a_{k+r})_{k+r} - (a_k)_k\right), \text{ by (R-9)} \\ &= B(a_{k+r} - a_k) \\ &= Bd_A, \text{ a constant.}\end{aligned}$$

Therefore, by (R-2) and (R-3), we have (P2-0) holds. □

Proposition 3. *If in $\{\ddot{a}_{n=1},\ldots,\ddot{a}_m,\ldots\}$ and $\{\ddot{b}_{n=1},\ldots,\ddot{b}_m,\ldots\}$ we have $\{a_{n=1},\ldots,a_m\} \geq \{b_{n=1},\ldots,b_m\}$ and the respective n-repetend differences $d_a \geq d_b$, then*

$$\{\ddot{a}_{n=1},\ldots,\ddot{a}_n,\ldots\} \geq \{\ddot{b}_{n=1},\ldots,\ddot{b}_n,\ldots\}. \ □$$

The statements of **Proposition 3** for the relations $=,\leq,>,<$ are left to the reader.

[13] This is an example in which placing the index values of n outside the parentheses (on the right side of the equation here) would be completely wrong, since $f(a)$ is undefined.

A word of caution in **Proposition 3** is needed here. If we have lists with different repetend lengths, say $\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r_a},\ldots\}$ and $\{\ddot{b}_{n=1},\ldots,\ddot{b}_{1+r_b},\ldots\}$, with $r_a \neq r_b$, then the inequality comparison must be shown out to index $1+\mathrm{lcm}(r_a, r_b)$ [14] here, and we must have $\frac{d_a}{r_a} \geq \frac{d_b}{r_b}$ (see why in **Theorem 9**) before we can say

$$\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r_a},\ldots\} \geq \{\ddot{b}_{n=1},\ldots,\ddot{b}_{1+r_b},\ldots\}.$$

Statements for similar situations for the other relations $=, \leq, >, <$ are left to the reader.

Proposition 4. *If, for a given double overdot notation listing, r is a repetend length and d the repetend difference for that r, then, for any integral $k \geq 2$, we have kr is also a repetend length, with corresponding repetend difference kd.* □

Proposition 5. *If, for a given double overdot notation listing, there are two different repetend lengths r_1 and r_2 with respective corresponding repetend differences d_1 and d_2, then $r_2 d_1 = r_1 d_2$.*

Proof: By **Proposition 1**, we may assume that there is a $k \geq 1$ such that

$$\{a_{n=1}, a_2,\ldots\} = \{a_{n=1},\ldots,\ddot{a}_k,\ldots,\ddot{a}_{k+r_1},\ldots\} = \{a_{n=1},\ldots,\ddot{a}_k,\ldots,\ddot{a}_{k+r_2},\ldots\}.$$

Therefore, by **Proposition 4**, we also have

$$\{a_{n=1},\ldots,\ddot{a}_k,\ldots,\ddot{a}_{k+r_1},\ldots\} = \{a_{n=1},\ldots,\ddot{a}_k,\ldots,\ddot{a}_{k+r_1 r_2},\ldots\} \qquad \text{(P5-1)}$$

and $$\{a_{n=1},\ldots,\ddot{a}_k,\ldots,\ddot{a}_{k+r_2},\ldots\} = \{a_{n=1},\ldots,\ddot{a}_k,\ldots,\ddot{a}_{k+r_2 r_1},\ldots\}. \qquad \text{(P5-2)}$$

Since the RHS[15] of both listings (P5-1) and (P5-2) are the same, we have the repetend differences in the RHS of (P5-1) and (P5-2) are the same, *i.e.*,

$r_2 d_1 = r_1 d_2$, by **Proposition 4**. □

The result in **Proposition 5** can also be stated $\frac{r_2}{r_1} = \frac{d_2}{d_1}$. As an immediate corollary we have the following:

Corollary 6. *If, for a given double overdot notation listing, there are two different repetend lengths r_1 and r_2 with respective repetend differences $d_1 < d_2$, then $r_1 < r_2$. Conversely, if $r_1 < r_2$, then $d_1 < d_2$.* □

We now have the following converse of **Proposition 4**:

Proposition 7. *If, for a given double overdot notation listing, r is a repetend length, then any lesser repetend length must be a factor[16] of r.* □

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[14] As usual, "lcm" stands for least common multiple.
[15] "RHS" stands for right hand side.
[16] Including the possibility 1.

Proof: By *reductio ad absurdum*:
For a given listing with two different repetends, if the two different first repetends start with different indices, shift one set of the repetend overdots to the right so the first elements of the two repetends coincide. Then relabel the indices so the first index of each repetend is 1. Let $A = \{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_1}, \ldots\} = \{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_2}, \ldots\}$ be the resultant listings starting with index 1, where r_1, r_2 are the respective repetend lengths. Without loss of generality, we may let

$$r_1 < r_2 . \quad \text{(P7-1)}$$

Suppose, to the contrary of the proposition statement, that

$$r_1 \nmid r_2 . \quad \text{(P7-2)}$$

Let the respective minimum repetend differences be d_1, d_2.
By (P7-1) and **Corollary 6**, we have

$$d_1 < d_2 . \quad \text{(P7-3)}$$

Then, by **Proposition 1**, we have

$$\begin{aligned}
\{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_1}, \ldots\} &= \{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_1}, a_{2+r_1}, a_{3+r_1}, \ldots, a_{1+r_2}, \ldots\} \\
&= \{a_1, a_2, \ldots, \ddot{a}_{1+r_2-r_1}, \ldots, \ddot{a}_{1+r_2}, \ldots\} \\
&= \{a_1, a_2, \ldots, a_{1+r_2-r_1}, \ddot{a}_{2+r_2-r_1} \ldots, a_{2+r_2}, \ddot{a}_{3+r_2}, \ldots\} \\
&\vdots \\
&= \{a_1, a_2, \ldots, \ddot{a}_{1+r_2}, \ldots, \ddot{a}_{1+r_1+r_2}, \ldots\} \\
&\vdots
\end{aligned}$$

$$\begin{array}{lcl}
a_{1+r_2} - a_{1+r_2-r_1} = d_1 & \text{and} & a_{1+r_2} - a_1 = d_2 \\
a_{2+r_2} - a_{2+r_2-r_1} = d_1 & & a_{2+r_2} - a_2 = d_2 \\
\vdots & & \vdots \\
a_{2+r_1+r_2} - a_{2+r_2} = d_1 & & a_{2+r_1+r_2} - a_{2+r_1} = d_2 \\
\vdots & & \vdots
\end{array}$$

as well as

$$\begin{aligned}
a_{1+r_2-r_1} - a_1 &= d_2 - d_1 \\
a_{2+r_2-r_1} - a_2 &= d_2 - d_1 \\
&\vdots \\
a_{2+r_2} - a_{2+r_1} &= d_2 - d_1 \\
&\vdots
\end{aligned}$$

Therefore, there is a third repetend difference $d_3 = d_2 - d_1$ with its first repetend starting at a_1 also and with corresponding repetend length

$$\begin{aligned}
r_3 &= r_2 - r_1 \\
&< r_2 \text{, by (P7-1).}
\end{aligned}$$

Since we have $d_1 < d_2$, by (P7-3), **Corollary 6** gives

$$d_3 < d_2 . \quad \text{(P7-4)}$$

Suppose $d_2 = bd_1$, for some integer $b > 1$.

Then $\frac{r_2}{r_1} = \frac{d_2}{d_1} = b$, by **Proposition 5**, contradicting (P7-1).

Therefore, we also have

$$d_1 \nmid d_2$$
$$d_1 \nmid d_3$$
$$r_1 \nmid r_3$$
$$\text{and} \quad r_3 \nmid r_2 .$$

This now sets up an iteration creating infinitely nested listings, all starting with index 1 and having corresponding repetend lengths $r_3, r_4, r_5, \cdots$ satisfying the condition $r_2 > r_3 > r_4 > r_5 > \cdots > r_1$, an impossibility for finite integers.

Therefore, supposition (P7-2) is false and we can only have r_1 divides r_2. □

Proposition 8. *For any constants* B, C, D, *we have*

$$B\{\ddot{a}_{n=1}, \ldots, \ddot{a}_{1+r}, \ldots\} + Cn + D = \{(Ba_{n=1} + C + D)^{\cdot\cdot}_{n=1}, (Ba_2 + 2C + D)_2, \ldots, (Ba_{1+r} + (1+r)C + D)^{\cdot\cdot}_{1+r}, \ldots\} . \square$$

More general than **Proposition 8**, we have the following theorem:

Theorem 9. *Let* f *and* g *each be functions of integral* $n \geq 1$, *with* n*-repetends and with* n*-repetend lengths* r_f *and* r_g, *respectively. Then any linear combination of* $f(n)$, $g(n)$, *and* n *has* n*-repetends and has* $r = \text{lcm}(r_f, r_g)$ *as an* n*-repetend length.*

Proof: Consider first case 1 in which the first n-repetends start at $n = 1$, as follows:

$$f(n) = \{\ddot{a}_{n=1}, \ldots, \ddot{a}_{1+r_f}, \ldots\}, \text{ with } n\text{-repetend difference } d_f = a_{1+r_f} - a_1 , \qquad \text{(T9-1)}$$

and

$$g(n) = \{\ddot{b}_{n=1}, \ldots, \ddot{b}_{1+r_g}, \ldots\}, \text{ with } n\text{-repetend difference } d_g = b_{1+r_g} - b_1 . \qquad \text{(T9-2)}$$

Then, for any constants A, B, C, E, we have

$$Af(n) + Bg(n) + Cn + E = \{(Aa_{n=1} + Bb_{n=1} + C + E)_{n=1}, (Aa_2 + Bb_2 + 2C + E)_2, \ldots\}, \qquad \text{(T9-3)}$$

by (R-2) and (R-7).

For any integral $k, r \geq 1$, let

$$D_{k,r} = \left(Aa_{k+r} + Bb_{k+r} + (k+r)C + E\right) - (Aa_k + Bb_k + kC + E)$$
$$= A(a_{k+r} - a_k) + B(b_{k+r} - b_k) + rC .$$

If $r = \text{lcm}(r_f, r_g)$, we then have

$$D_{k,r} = A\left(a_{k+\text{lcm}(r_f,r_g)} - a_k\right) + B\left(b_{k+\text{lcm}(r_f,r_g)} - b_k\right) + C\left(\text{lcm}(r_f, r_g)\right)$$
$$= A\left(\frac{\text{lcm}(r_f, r_g)}{r_f}\right)d_f + B\left(\frac{\text{lcm}(r_f, r_g)}{r_g}\right)d_g + C\left(\text{lcm}(r_f, r_g)\right), \qquad \text{(T9-4)}$$

by (T9-1), (T9-2), and **Proposition 4**,

is a constant, independent of k. Therefore, by (R-3), we have (T8-3) can be written

$$Af(n)+Bg(n)+Cn+E=\{(Aa_{n=1}+Bb_{n=1}+C+E\ddot{)}_{n=1},(Aa_{n=2}+Bb_{n=2}+2C+E)_{n=2},\ldots,$$
$$(Aa_{1+\mathrm{lcm}(r_f,r_g)}+Bb_{1+\mathrm{lcm}(r_f,r_g)}+C(\mathrm{lcm}(r_f,r_g)+1)+E\ddot{)}_{n=1+\mathrm{lcm}(r_f,r_g)},\ldots\},$$

i.e., $Af(n)+Bg(n)+Cn+E$ is an n-repetend function and has $\mathrm{lcm}(r_f,r_g)$ as an n-repetend length.

Now consider case 2 in which either of $f(n)$ or $g(n)$ has the first n-repetend starting at some value $n=n_f>1$ or $n=n_g>1$, respectively. By **Proposition 1**, we can choose the first n-repetends for both $f(n)$ and $g(n)$ to start at the same value of $n\geq\max(n_f,n_g)$, say n_0. Ignoring the terms $a_1,a_2,\ldots,a_{n_0-1}$ and $b_1,b_2,\ldots,b_{n_0-1}$ now allows using the same arguments as in case 1. This completes the proof. □

As a result of **Theorem 9**, if $\min(r_f)$ and $\min(r_g)$ are the minimum n-repetend lengths of $f(n)$ and $g(n)$, respectively, then **Proposition 7** dictates that the minimum n-repetend length of any linear combination of $f(n)$, $g(n)$, and n can only be 1 or a proper factor of $\mathrm{lcm}\big(\min(r_f),\min(r_g)\big)$. For example, let

$$f(n)=\{\ddot{1}_{n=1},1_2,\ddot{3}_3,\ldots\}\ \text{and}\ g(n)=\{\ddot{0}_{n=1},1_2,\ddot{0}_3,\ldots\}.$$

Then $\min(r_f)=\min(r_g)=2$ and $f(n)+g(n)=\{\ddot{1}_{n=1},2_2,\ddot{3}_3,\ldots\}=\{\ddot{1}_{n=1},\ddot{2}_2,\ldots\}$, so that the minimum n-repetend length of $f(n)+g(n)$ here is 1. Such a fortuitous combination for the right values of n cannot happen if $\min(r_f)$ and $\min(r_g)$ are not equal or one is not a multiple of the other, since otherwise the "corrective/reductionist" combinations can only occur at n-positions $\mathrm{lcm}\big(\min(r_f),\min(r_g)\big)$ from each other. In such situations, with or without "corrective/reductionist" combinations, the minimum n-repetend length of the linear combination of $f(n)$, $g(n)$, and n is $\mathrm{lcm}\big(\min(r_f),\min(r_g)\big)$.

In **Theorem 9** we looked at linear combinations of repetend functions f and g. The next theorem looks at composition of repetend functions f and g.

Theorem 10. *Let $f(n)$ be an n-repetend function, with minimum n-repetend length r_f and corresponding minimum n-repetend difference d_f. Let $g(n)$ be an integral n-repetend function with values within the domain of f, minimum n-repetend length r_g, and corresponding minimum repetend n-difference d_g. Then,*

(*i*) *$f\big(g(n)\big)$ is an n-repetend function, with $r_{f(g)}=\mathrm{lcm}(r_f,r_g)$ as an n-repetend length*;

(*ii*) *if $d_g=0$ or d_g is not divisible by r_f, then $r_{f(g)}=\mathrm{lcm}(r_f,r_g)$ is the minimum n-repetend length of $f\big(g(n)\big)$; and*

(*iii*) *if $d_g=0$, then $d_{f(g)}=0$.*

Proof: Without loss of generality, we may assume $n, d_f, d_g \geq 0$, since otherwise we can always reverse directions or change the variable n. For convenience, we may also assume the n-repetends for $f(n)$ and $g(n)$ start at $n = 0$. Otherwise, we complicate notation by taking into account the least values of n that can start an n-repetend for each function, something that does not change the arguments otherwise.
The value $r_{f(g)} = \operatorname{lcm}(r_f, r_g)$ gives

$$f\left(g\left(n + r_{f(g)}\right)\right) - f\left(g(n)\right) = f\left(g(n) + \operatorname{lcm}(r_f, r_g)\frac{d_g}{r_g}\right) - f\left(g(n)\right),\ \forall n \geq 0,$$

by **Proposition 4**.

If $d_g = 0$, then $f\left(g\left(n + r_{f(g)}\right)\right) - f\left(g(n)\right) = 0$, so that $f\left(g(n)\right)$ is an n-repetend function with $d_{f(g)} = 0$.
Now consider $r_{f(g)} = \operatorname{lcm}(r_f, r_g)$ and $d_g > 0$.
Since $g(n)$ is an integral function, we have d_g is an integer, and $\frac{r_{f(g)}}{r_g} d_g = \frac{\operatorname{lcm}(r_f, r_g)}{r_g} d_g$ is a positive integer divisible by r_f, say $\frac{r_{f(g)}}{r_g} d_g = m r_f$, where $m \geq 1$.

Therefore, $f\left(g\left(n + r_{f(g)}\right)\right) - f\left(g(n)\right) = f\left(g(n) + m r_f\right) - f\left(g(n)\right),\ \forall n \geq 0$

$= f\left(g(n)\right) + m d_f - f\left(g(n)\right)$, by **Proposition 4**

$= m d_f$, a constant, independent of n.

Therefore, $f\left(g(n)\right)$ is an n-repetend function, by (R-3), with $r_{f(g)} = \operatorname{lcm}(r_f, r_g)$ as an n-repetend length.
Also, $r_{f(g)} = \operatorname{lcm}(r_f, r_g)$ is the smallest integer giving these results if d_g is not divisible by r_f, since an integer smaller than $\frac{r_{f(g)}}{r_g} d_g = \frac{\operatorname{lcm}(r_f, r_g)}{r_g} d_g$ cannot then be divisible by r_f without violating **Proposition 4** and r_f being the minimum n-repetend length of $f(n)$.
We have now shown all of (*i*), (*ii*), and (*iii*) hold. □

By **Theorem 10** , (R-9), and **Proposition 4**, we immediately have the following corollary:

Corollary 11. *Let $f(n)$ be an n-repetend function and let a and b be any positive*[17] *integers. Then, $f(an + b)$ is an n-repetend function with the same minimum n-repetend length as $f(n)$.* □

[17] There are many exceptions in which this corollary holds for either constant being negative.

EXAMPLES OF USEFULNESS

As the first example of the usefulness of our repetend notation and relations, we use them to prove one of Ramanujan's [1] problems:

if n is any positive integer, prove that

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor.$$

Proof: $$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \{\ddot{0}_{n=1}, 0_2, 1_3, \ddot{1}_4, 1_5, 2_6, 2_7, \ldots\} + \{\ddot{0}_{n=1}, 0_2, 0_3, 1_4, 1_5, 1_6, \ddot{1}_7, \ldots\}$$
$$+\{\ddot{0}_{n=1}, 1_2, 1_3, 1_4, 1_5, 1_6, \ddot{1}_7, \ldots\}$$
$$= \{\ddot{0}_{n=1}, 1_2, 2_3, 3_4, 3_5, 4_6, \ddot{4}_7, \ldots\}.$$

Similarly,

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor = \{\ddot{0}_{n=1}, 1_2, \ddot{1}_3, 2_4, 2_5, 3_6, 3_7, \ldots\} + \{\ddot{0}_{n=1}, 0_2, 1_3, 1_4, 1_5, 1_6, \ddot{1}_7, \ldots\}$$
$$= \{\ddot{0}_{n=1}, 1_2, 2_3, 3_4, 3_5, 4_6, \ddot{4}_7, \ldots\}. \ \square$$

The above proof does not actually require n to be an integer (although we would need to use a dummy integral index subscript) or even positive. A bit more involved problem, with similar proof, is showing that $\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor > \left\lfloor \frac{n+3}{7} \right\rfloor, \ \forall n \geq 3$.

Proof: $$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor = \{\ddot{1}_{n=3}, 1_4, 1_5, \ddot{2}_6, \ldots\} + \{\ddot{0}_3, 1_4, 1_5, 1_6, 1_7, \ddot{1}_8, \ldots\}$$
$$= \{\ddot{1}_{n=3}, 1_4, 1_5, 2_6, 2_7, 2_8, 3_9, 3_{10}, 3_{11}, 4_{12}, 4_{13}, 4_{14}, 5_{15}, 5_{16}, 5_{17}, \ddot{6}_{18}, \ldots\}$$
$$+\{\ddot{0}_3, 1_4, 1_5, 1_6, 1_7, 1_8, 2_9, 2_{10}, 2_{11}, 2_{12}, 2_{13}, 3_{14}, 3_{15}, 3_{16}, 3_{17}, \ddot{3}_{18}, \ldots\}$$
$$= \{\ddot{1}_{n=3}, 2_4, 2_5, 3_6, 3_7, 3_8, 5_9, 5_{10}, 5_{11}, 6_{12}, 6_{13}, 7_{14}, 8_{15}, 8_{16}, 8_{17}, \ddot{9}_{18}, \ldots\} \quad \text{(A)}$$
$$\geq \{\ddot{1}_{n=3}, 1_4, 1_5, \ddot{2}_6, 2_7, 2_8, 3_9, 3_{10}, 3_{11}, 4_{12}, 4_{13}, 4_{14}, 5_{15}, 5_{16}, 5_{17}, 6_{18}, \ldots\},$$ [18]

and $$\left\lfloor \frac{n+3}{7} \right\rfloor = \{\ddot{0}_{n=3}, 1_4, 1_5, 1_6, 1_7, 1_8, 1_9, \ddot{1}_{10}, \ldots\} \quad \text{(B)}$$
$$< \{\ddot{1}_{n=3}, 1_4, 1_5, \ddot{2}_6, 2_7, 2_8, 3_9, 3_{10}, \ldots\}, \ \forall n \geq 6.$$

Comparing (A) and (B) for $n = 3, 4, 5$ completes the proof. □

The method of proof used in this example shows a general technique for avoiding having to use the brute force technique of comparing repetends out to at least a least common multiple index, here of extending (A) and (B) out to $n = 3 + \text{lcm}(3,5,7) = 108$.

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[18] This is an example in which the right overdots can be moved to the left, as shown.

COMMENTS

Generalized repetends, as shown here, simplify calculations with floor functions. They can equally well be used with ceiling functions. But these are not the only situations in which they are useful. Surprisingly, they also help analyze the Sieve of Eratosthenes, since there are repeating patterns there as well [2]. This helps derive theorems concerning the distribution of primes.

REFERENCES

[1] S. Ramanujan, Question 723, J. Indian Math. Soc. 10 (1918), 357-358.

[2] I. Boxen, “Opening the Sieve of Eratosthenes, and Some Theorems on the Distribution of Primes”, viXra:2507.0134v2, 19Jul2025.