Generalized Energy in Special Relativity

Asutosh Kumar*
Department of Physics, KSS College, Lakhisarai 811311, India
P.G. Department of Physics, Munger University, Munger 811201, India and
Vaidic and Modern Physics Research Centre, Bhagal Bhim, Bhinmal, Jalore 343029, India

We obtain the generalized mass-energy equivalence using an elementary approach.

Key words: velocity-dependent mass; intrinsic or existential mass; existential energy; generalized mass-energy equivalence

I. INTRODUCTION

Within the framework of Special Relativity (SR) [1–5], the notion of velocity-dependent mass–historically termed *relativistic mass*–emerges from analyses of how a body's energy and inertia vary as its velocity nears the speed of light. This relativistic mass increases with velocity, signifying the enhanced resistance to acceleration experienced by an object as its speed approaches the speed of light. Conversely, the principle of mass-energy equivalence establishes that mass and energy are fundamentally interchangeable quantities. This equivalence entails that any variation in the mass of a physical system is accompanied by a corresponding release or absorption of energy. Such a concept proves instrumental in explaining a range of phenomena, including nuclear decay and particle-antiparticle annihilation.

For the velocity-dependent mass $m = m_0\gamma(u)$ in SR, where m_0 is the rest mass of a particle and $\gamma(u) = \left(1 - u^2/c^2\right)^{-1/2}$ is the Lorentz factor, the kinetic energy of a relativistic particle is given by

$$
\begin{aligned}
T(u) &= \int_0^u \vec{F}.\vec{dl} = \frac{\mathrm{d}(mu)}{\mathrm{d}t} dl \\
&= c^2 \int_{m_0}^{m} dm = (m - m_0)c^2 \qquad (1) \\
&= m_0c^2[\gamma(u) - 1]. \qquad (2)
\end{aligned}
$$

Consequently, total energy of the relativistic particle is

$$E = T + m_0c^2 = mc^2. \qquad (3)$$

In the non-relativistic (low-velocity) limit, ignoring the $\mathcal{O}(u^4/c^4)$ terms, T reduces to the classical expression of kinetic energy $T \to \frac{1}{2}m_0u^2$. Using $p = mu = m_0u\gamma(u)$, the relativistic energy $E = mc^2$ can be re-expressed as

$$E^2 = p^2c^2 + m_0^2c^4. \qquad (4)$$

It is remarked that Eqs.(3, 4) can also be arrived at using alternative approaches [6–11].

II. GENERALIZED ENERGY

There exist several velocity-dependent masses of the form $m = m_0A(u)$ [12, 13] for various space-time transformations between two inertial frames of reference (see Table I). One can see that not all these velocity-dependent masses satisfy the mass-energy equivalence Eq.(3) and the energy-momentum relation Eq.(4) identically. Moreover,

$$\frac{\mathrm{d}T}{\mathrm{d}u} = p\left(1 + \frac{A'(u)}{A(u)}u\right). \qquad (5)$$

In an elementary approach (using dynamical concepts) to obtain the energy of a particle having velocity $\vec{u}$ in SR, we can define $p := m_0uA(u)$ and kinetic energy $\widetilde{T} := m_0c^2B(u)$ [akin to Eq.(2)] such that energy is

$$
\begin{aligned}
E &= E_0 + \widetilde{T} = m_0c^2 + m_0c^2B(u) \\
&= m_0c^2[1 + B(u)] \qquad (6) \\
&= \frac{1 + B(u)}{A(u)}mc^2, \qquad (7)
\end{aligned}
$$

where m_0 is the intrinsic or existential mass (it does not depend on particle's velocity), $E_0 = m_0c^2$ is a constant (the existential energy), $A(u)$ and $B(u)$ are monotonic functions of u (see [14]), and we require that

$$\widetilde{T} \xrightarrow{u^2/c^2 \ll 1} \frac{1}{2}m_0u^2. \qquad (8)$$

Assuming the energy-momentum relation Eq.(4) holds true in general, one can obtain $B(u)$ in terms of $A(u)$ using Eq.(6) as follows:

$$[1 + B(u)]^2 = \left(1 + \frac{u^2}{c^2}A^2(u)\right) \equiv \Gamma^2(u). \qquad (9)$$

Thus, from Eqs.(6,7), we have

$$
\begin{aligned}
E &= m_0c^2[1 + B(u)] = m_0c^2\left(1 + \frac{u^2}{c^2}A^2(u)\right)^{1/2} \\
&= \Gamma(u)m_0c^2 = \frac{\Gamma(u)}{A(u)}mc^2. \qquad (10)
\end{aligned}
$$

Eq.(10) is the generalized mass-energy equivalence, a concept discussed earlier in Refs. [15–18].

* asutoshk.phys@gmail.com

Transformation	$m'a' = ma$	$\frac{\mathrm{d}(m'u')}{\mathrm{d}t'} = \frac{\mathrm{d}(mu)}{\mathrm{d}t}$
Galilean $x' = x - vt$ $y' = y, \quad z' = z$ $t' = t$	$m'_k = m_k$ $(k = 1, 2, 3)$	complementary: $m_k u_k = m_{k0} u_{k0}$ particular: $m_k = m_{k0} \left(1 + \frac{u_{k0}}{u_k}\right)$ $(k = 1, 2, 3)$
Lorentz $x' = \gamma(v)(x - vt)$ $y' = y, \quad z' = z$ $t' = \gamma(v)(t - \frac{v}{C^2} x)$	$m_1 = m_{10} \gamma^3(u_1)$	complementary: $m_1 u_1 = m_{10} u_{10}$ particular: $m_1 = m_{10} \gamma(u_1)$
Voigt $x' = x - vt$ $y' = y/\gamma(v), \quad z' = z/\gamma(v)$ $t' = t - \frac{v}{C^2} x$	$m_1 = m_{10} \gamma^3(u_1)$	complementary: $m_1 u_1 = m_{10} u_{10}$ particular: $m_1 = m_{10} \gamma(u_1)$
Selleri $x' = \gamma(v)(x - vt)$ $y' = y, \quad z' = z$ $t' = t/\gamma(v)$	$m'_1 = m_1 \gamma^3(v)$ $m'_k = m_k \gamma^2(v)$ $(k = 2, 3)$	$m'_1 + u'_1 \frac{\mathrm{d}m'_1}{\mathrm{d}u'_1} = \left(m_1 + u_1 \frac{\mathrm{d}m_1}{\mathrm{d}u_1}\right) \gamma^3(v)$ $m'_k + u'_k \frac{\mathrm{d}m'_k}{\mathrm{d}u'_k} = \left(m_k + u_k \frac{\mathrm{d}m_k}{\mathrm{d}u_k}\right) \gamma^2(v)$ $(k = 2, 3)$
Spacetime-symmetric $x'_k = \gamma(v_k)(x_k - v_k t_k)$ $t'_k = \gamma(v_k)(t_k - \frac{v_k}{C^2} x_k)$	$m_k = m_{k0} \gamma^3(u_k)$ $(k = 1, 2, 3)$	complementary: $m_k u_k = m_{k0} u_{k0}$ particular: $m_k = m_{k0} \gamma(u_k)$ $(k = 1, 2, 3)$

TABLE I. Velocity-dependent masses obtained using invariance of Newton's second law [13] for various space-time transformations. Here $\gamma(\xi) = \left(1 - \xi^2/C^2\right)^{-1/2}$ with $C \geq c$ and c is the speed of light in vacuum.

III. CONCLUSION

In summary, it is possible to obtain the mass-energy equivalence using elementary approaches rather than quantum, electrodynamic or four-vector considerations. Here, we showed that the Einstein's mass-energy equivalence is a particular case of a more general mass-energy relation.

[1] M. Saha and S. Bose, *The Principle of Relativity: Original papers by A. Einstein and H. Minkowski* (Dover, New York, 1920).

[2] R. Resnick, *Introduction to Special Relativity* (John Wiley & Sons, Singapore, 1998).

[3] D. Bohm, *The special theory of relativity* (Routledge, 2015).

[4] W. G. V. Rosser, *Introductory special relativity* (CRC Press, 2017).

[5] A. Beiser, *Perspectives of modern Physics* (Singapore: McGraw-Hill Book Company, 1987).

[6] A. Einstein, *An Elementary Derivation of the Equivalence of Mass and Energy*, Bull. Amer. Math. Soc. **41(4)**, 223-230 (1935).

[7] D. J. Steck and F. Rioux, *An elementary development of mass-energy equivalence*, Am. J. Phys. **51(5)**, 461-462 (1983).

[8] F. Rohrlich, *An elementary derivation of $E = mc^2$*, Am. J. Phys. **58**, 348-349 (1990).

[9] L. Peliti, *Elementary derivation of the expressions of momentum and energy in special relativity*, arXiv:1504.07287.

[10] Y. Dai and L. Dai, *Deriving mass-energy equivalence and mass-velocity relation without light*, Am. J. Phys. **86**, 284-289 (2018).

[11] A. V. Nenashev, S. D. Baranovskii, and F. Gebhard, *A pedestrian approach to Einstein's formula $E = mc^2$ with an application to photon dynamics*, arXiv:2308.02612.

[12] A. Sharma, *The Various Equations of Variation of Mass with Velocity*, Ind. J. Sci. Tech. 10(21), 1-10 (2017).

[13] A. Kumar, *Relativity of mass: Mass can decrease with velocity*, SSRN (2025).

[14] The isotropy of space allows for the velocity-dependent functions $A(u)$ and $B(u)$ to depend solely on the magnitude u of the velocity [9].

[15] A. Sharma, *The Origin of the Generalized Mass-Energy Equation $\Delta E = Ac^2 \Delta M$ and Its Applications in General Physics and Cosmology*, Phys. Essays **17**, 195-222 (2004).

[16] M. S. El Naschie, *Why E Is Not Equal to mc^2*, J. Mod. Phys. **5**, 743-750 (2014).

[17] T. S. Santhanam, *On a Generalization of Einsteins $E = mc^2$*, J. Applied Maths. Phys. **6**, 1012-1016 (2018).

[18] M. S. El Naschie, *From Nikolay Umov $E = kmc^2$ via Albert Einsteins $E = \gamma mc^2$ to the Dark Energy Density of the Cosmos $E = (21/22)mc^2$*, World J. Mech. **8**, 73-81 (2018).