Hyperbolic Quantum Time Evolution Formulation

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Abstract

We prove that the relativistic Schrödinger equation of a massive point particle in one dimension is equivalent to an alternative time evolution formulation where the values of the wave function depend on the wave function's past values on relativistic hyperbolas inside the past light cones.

Here we continue work from the previous article *Hyperbolic Quantum Time Evolution Kernel* [1]. In the previous article we found a conjecture that there exists a function $\Delta t \mapsto K(\Delta t)$ that works in a such way that if the time evolution of a wave function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $(t, x) \mapsto \psi(t, x)$ is defined according to the formula

$$\psi(t + \Delta t, x)$$
$$= K(\Delta t) \int_{-\infty}^{\infty} \psi\Big(t + \Delta t - \Delta t \cosh\Big(\frac{\xi}{c\Delta t}\Big),\ x + c\Delta t \sinh\Big(\frac{\xi}{c\Delta t}\Big)\Big) d\xi, \tag{1}$$

then a relation

$$i\hbar\partial_t \psi(t, x) \text{ `` = or $\approx$ '' } \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\,\psi(t, x)$$

will be true. We were unable to find a precise description or definition of the function $\Delta t \mapsto K(\Delta t)$, but we found a reason to believe that it most apparently can be written in the form

$$K(\Delta t) = \frac{\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\,e^{-\frac{i\Delta t mc^2}{\hbar}}}{f\big(\frac{\hbar}{\Delta t mc^2}\big)}, \tag{2}$$

where f is some function that has a Taylor series representation that looks like

$$f(z) = 1 - \frac{i}{8}z - \frac{9}{128}z^2 + \frac{75i}{1024}z^3 + \frac{3675}{32768}z^4 - \cdots.$$

We were unable to find a formula for an arbitrary coefficient of this Taylor series. Only the few first coefficients could be solved with a finite amount of effort.

Let's return to the task of finding a precise description or definition of the function $\Delta t \mapsto K(\Delta t)$. The key to making progress in this is to contemplate

on what we did with the parameter p earlier. We wrote equations that had Taylor series with respect to p on both sides of the equations. Then we demanded that the coefficients of the Taylor series should be the same on the both sides of the equations. The reason for this is that then the equations would be true with all values of p, which would imply that the time evolution equation would be true with all plane waves $e^{-\frac{it}{\hbar}\sqrt{(mc^2)^2+c^2p^2}}e^{\frac{i}{\hbar}px}$. We wanted this, because then the time evolution equation would also be true with arbitrary linear combinations of these plane waves. If those equations are true with all values of p, it means that in particular they are also true with the value $p = 0$. This means that in particular the time evolution equation is also true with the trivial plane wave $\psi(t,x) = e^{-\frac{itmc^2}{\hbar}}$. If we substitute this trivial plane wave into Equation (1), we get an equation

$$e^{-\frac{i(t+\Delta t)mc^2}{\hbar}} \;=\; K(\Delta t)\int\limits_{-\infty}^{\infty} e^{-\frac{imc^2}{\hbar}\left(t+\Delta t-\Delta t\cosh(\frac{\xi}{c\Delta t})\right)}d\xi.$$

Some things cancel, and we can solve $K(\Delta t)$ to be

$$K(\Delta t) \;=\; \frac{1}{\int\limits_{-\infty}^{\infty} e^{\frac{i\Delta tmc^2}{\hbar}\cosh(\frac{\xi}{c\Delta t})}d\xi}. \tag{3}$$

So it turned out that finding a precise description of $\Delta t \mapsto K(\Delta t)$ wasn't difficult after all. If we substitute $K(\Delta t)$ from Equation (3) into Equation (2), we can then solve the value of f to be

$$f\left(\frac{\hbar}{\Delta tmc^2}\right) \;=\; \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\int\limits_{-\infty}^{\infty} e^{\frac{i\Delta tmc^2}{\hbar}\left(\cosh(\frac{\xi}{c\Delta t})-1\right)}d\xi$$

$$=\; \sqrt{\frac{\Delta tmc^2}{2\pi i\hbar}}\int\limits_{-\infty}^{\infty} e^{\frac{i\Delta tmc^2}{\hbar}(\cosh(u)-1)}du.$$

We can express essentially the same equation without physical constants in the form

$$f(z) \;=\; \frac{1}{\sqrt{2\pi iz}}\int\limits_{-\infty}^{\infty} e^{\frac{i}{z}(\cosh(u)-1)}du. \tag{4}$$

So there are two different ways of defining the function $\Delta t \mapsto K(\Delta t)$. One is that we define it directly with Equation (3). Another one is that first we define a mathematical special function $z \mapsto f(z)$ without physical constants with Equation (4), and then we use this function to define $\Delta t \mapsto K(\Delta t)$ with Equation (2).

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It seems that we found a new mathematical special function, so it might be a good idea to try to recognize its obvious properties. Since division by zero is forbidden, we see that substituting $z = 0$ directly into Equation (4) is forbidden too. When using values $z \neq 0$, the integral converges if $\text{Im}(z) \leq 0$, and diverges if $\text{Im}(z) > 0$. So the definition of the function f should be equipped with the information that the initial domain is

$$\{z \in \mathbb{C} \mid \text{Im}(z) \leq 0\} \setminus \{0\}.$$

At this point many people will probably think that the fact that f appears to have a Taylor series representation around the point $z = 0$ supports the hypothesis that apparently f has an analytic continuation outside the initial domain. That might sound reasonable, but there is a problem that at this point we do not yet know the radius of convergence of the Taylor series, so consequently not much is known about the analytic continuation either. We will eventually learn the radius of convergence soon below.

Since the value of a complex square root can be interpreted in two different ways, we should clarify how $\sqrt{2\pi i z}$ should be interpreted here. Suppose z is an arbitrary complex number such that $\text{Im}(z) \leq 0$. Then it is possible to write z in the form $z = |z|e^{i\theta}$, where $-\pi \leq \theta \leq 0$. We can then decide that $\sqrt{2\pi i z}$ will be interpreted to be

$$\sqrt{2\pi i z} \;=\; \sqrt{2\pi |z|} e^{\frac{i\pi}{4} + \frac{i\theta}{2}}.$$

Two issues imply that we should chooce the square root this way. Firstly, in the physical application we are going to use z such that $\text{Re}(z) > 0$ and $\text{Im}(z) = 0$, and in this case we must choose $\sqrt{2\pi i z} = \sqrt{2\pi \text{Re}(z)} e^{\frac{i\pi}{4}}$, because we want this factor to cancel with the factor that comes from the Gaussian integral approximation. Secondly, it makes sense to demand that the mapping $z \mapsto \sqrt{2\pi i z}$ is continuous for all $\text{Im}(z) \leq 0$. These two demands determine the square root uniquely for all $\text{Im}(z) \leq 0$. There is no need to specify how the square root should be interpreted for $\text{Im}(z) > 0$.

By using the Taylor series of hyperbolic cosine and exponential function, and the Gaussian integral formulas, we can try to approximate the values of f with small $|z|$ as

$$f(z) \;=\; \frac{1}{\sqrt{2\pi i z}} \int\limits_{-\infty}^{\infty} e^{\frac{i}{z}\left(\frac{1}{2}u^2 + \sum\limits_{k=2}^{\infty} \frac{1}{(2k)!}u^{2k}\right)} du$$

$$=\; \frac{1}{\sqrt{2\pi i z}} \int\limits_{-\infty}^{\infty} e^{\frac{iu^2}{2z}} \left(1 \;+\; \frac{i}{z}\sum\limits_{k=2}^{\infty}\frac{1}{(2k)!}u^{2k} \;-\; \frac{1}{2z^2}\left(\sum\limits_{k=2}^{\infty}\frac{1}{(2k)!}u^{2k}\right)^2\right.$$

$$\left. -\; \frac{i}{6z^3}\left(\sum\limits_{k=2}^{\infty}\frac{1}{(2k)!}u^{2k}\right)^3 \;+\; \frac{1}{24z^4}\left(\sum\limits_{k=2}^{\infty}\frac{1}{(2k)!}u^{2k}\right)^4 \;+\; \cdots\right) du$$

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$$
= \frac{1}{\sqrt{2\pi i z}} \int\limits_{-\infty}^{\infty} e^{\frac{iu^2}{2z}} \Big(1
$$

$$
+ \frac{i}{z}\Big(\frac{1}{24}u^4 + \frac{1}{720}u^6 + \frac{1}{40320}u^8 + \frac{1}{3628800}u^{10} + \cdots\Big)
$$

$$
- \frac{1}{2z^2}\Big(\frac{1}{24}u^4 + \frac{1}{720}u^6 + \frac{1}{40320}u^8 + \cdots\Big)^2
$$

$$
- \frac{i}{6z^3}\Big(\frac{1}{24}u^4 + \frac{1}{720}u^6 + \cdots\Big)^3
$$

$$
+ \frac{1}{24z^4}\Big(\frac{1}{24}u^4 + \cdots\Big)^4 + \cdots\Big)du
$$

$$
= \frac{1}{\sqrt{2\pi i z}} \int\limits_{-\infty}^{\infty} e^{\frac{iu^2}{2z}} \Big(1
$$

$$
+ \frac{i}{z}\Big(\frac{1}{24}u^4 + \frac{1}{720}u^6 + \frac{1}{40320}u^8 + \frac{1}{3628800}u^{10} + \cdots\Big)
$$

$$
- \frac{1}{2z^2}\Big(\frac{1}{576}u^8 + \frac{1}{8640}u^{10} + \frac{29}{7257600}u^{12} + \cdots\Big)
$$

$$
- \frac{i}{6z^3}\Big(\frac{1}{13824}u^{12} + \frac{1}{138240}u^{14} + \cdots\Big)
$$

$$
+ \frac{1}{24z^4}\Big(\frac{1}{331776}u^{16} + \cdots\Big) + \cdots\Big)du
$$

$$
= \frac{1}{\sqrt{2\pi i z}} \int\limits_{-\infty}^{\infty} e^{\frac{iu^2}{2z}} \Big(1 + \frac{i}{24z}u^4 + \frac{i}{720z}u^6 + \Big(\frac{i}{40320z} - \frac{1}{1152z^2}\Big)u^8
$$

$$
+ \Big(\frac{i}{3628800z} - \frac{1}{17280z^2}\Big)u^{10}
$$

$$
+ \Big(O\Big(\frac{1}{z}\Big) - \frac{29}{14515200z^2} - \frac{i}{82944z^3}\Big)u^{12}
$$

$$
+ \Big(O\Big(\frac{1}{z^2}\Big) - \frac{i}{829440z^3}\Big)u^{14} + \Big(O\Big(\frac{1}{z^3}\Big) + \frac{1}{7962624z^4}\Big)u^{16}
$$

$$
+ \cdots\Big)du
$$

$$
= 1 + \frac{i}{24z}(-3z^2) + \frac{i}{720z}(-15iz^3) + \Big(\frac{i}{40320z} - \frac{1}{1152z^2}\Big)\cdot 105z^4
$$

$$
+ \Big(\frac{i}{3628800z} - \frac{1}{17280z^2}\Big)\cdot 945iz^5
$$

$$
+ \Big(O\Big(\frac{1}{z}\Big) - \frac{29}{14515200z^2} - \frac{i}{82944z^3}\Big)(-10395z^6)
$$

$$
+ \Big(O\Big(\frac{1}{z^2}\Big) - \frac{i}{829440z^3}\Big)(-135135iz^7)
$$

$$
+ \Big(O\Big(\frac{1}{z^3}\Big) + \frac{1}{7962624z^4}\Big)\cdot 2027025z^8 + \cdots
$$

$$= 1 - \frac{3i}{24}z + \Big(\frac{15}{720} - \frac{105}{1152}\Big)z^2 + \Big(\frac{105i}{40320} - \frac{945i}{17280} + \frac{10395i}{82944}\Big)z^3$$
$$+ \Big(-\frac{945}{3628800} + \frac{29 \cdot 10395}{14515200} - \frac{135135}{829440} + \frac{2027025}{7962624}\Big)z^4 + O(z^5)$$
$$= 1 - \frac{i}{8}z - \frac{9}{128}z^2 + \frac{75i}{1024}z^3 + \frac{3675}{32768}z^4 + O(z^5).$$

We have now derived the beginning of this Taylor series in two ways. First in the previous article [1], and now again here. Most people who have seen the previous calculation will probably agree that our new calculation here was relatively nicer. However, in some sense these calculations were the same. The only difference is that in the previous article the calculation of this series was hidden among the other calculations that were related to the terms proportional to $p, p^2, p^3, \ldots$, so it was more difficult to see where the series came from. Now we repeated the same calculation, but with the difference that we substituted $p = 0$, and also got rid of the physical constants with some changes of variables. Having gotten a clearer picture on where this series is coming from, it has now become reasonable to ask whether we could find some well-defined formula for an arbitrary coefficient of this series. An obvious challenge that may intimidate some people is that at one point in the calculation there was a need to raise a series to a power. Let's take a closer look at this challenge. Suppose a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots \in \mathbb{C}$ and a number $n \in \{2, 3, 4, \ldots\}$ have been fixed. The question is that is it possible to come up with a formula for new coefficients $\beta_{n,1}, \beta_{n,2}, \beta_{n,3}, \ldots \in \mathbb{C}$ such that

$$\Big(\sum_{k=0}^{\infty} \alpha_k X^k\Big)^n = \sum_{k=0}^{\infty} \beta_{n,k} X^k?$$

The answer is yes, because the formula

$$\beta_{n,k} = \sum_{\substack{k_1, k_2, \ldots, k_n = 0 \\ k_1 + k_2 + \cdots + k_n = k}}^{k} \alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_n}$$

is valid. This notation means that initially the indices $k_1, k_2, \ldots, k_n$ all independently traverse through the values $\{0, 1, 2, \ldots, k\}$, which is equivalent to the object $(k_1, k_2, \ldots, k_n)$ traversing through $(k+1)^n$ different values, but then out of those initial index values we reject all those that do not satisfy the condition $k_1 + k_2 + \cdots + k_n = k$. Some people might criticize this formula for being somehow implicit. For example, we cannot even easily see from it that what is the number of terms in this sum. Also, in this formula identical terms get added into the sum multiple times, so the formula probably isn't the most efficient possible. This criticism is reasonable, but the formula is a well-defined formula nonetheless. For example, the formula is sufficiently

clear that it is possible to write a computer program that computes the values of the coefficients $\beta_{n,k}$ by using it. Of course, there is a problem that if the number $(k+1)^n$ is so large that a computer cannot complete a loop over this number of iterations in a reasonable time, then a naive code will not be working anymore. Nevertheless, it is reasonable to write a naive computer implementation of this formula in the simplest possible way, then check that it works with small $(k+1)^n$, and then see how far we can go when we allow $(k+1)^n$ to increase.

Once we have learnt that it is possible to raise series to powers, we see that there is no major obstacle preventing us from solving a formula for the coefficients of the Taylor series. Let's do the work and see what comes out. If we first define coefficients $b_{n,k}$ for $n \in \{1,2,3,\ldots\}$ and $k \in \{0,1,2,\ldots\}$ with the formula

$$
b_{n,k} \;=\; \sum_{\substack{k_1,k_2,\ldots,k_n=0 \\ k_1+k_2+\cdots+k_n=k}}^{k} \frac{1}{(4+2k_1)!}\frac{1}{(4+2k_2)!}\cdots\frac{1}{(4+2k_n)!}, \tag{5}
$$

we can then use these coefficients to write the value of function f as

$$
f(z) = \frac{1}{\sqrt{2\pi i z}} \int_{-\infty}^{\infty} e^{\frac{iu^2}{2z}}\left(1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{z}\right)^n\left(\sum_{k=0}^{\infty} b_{n,k}u^{4n+2k}\right)\right)du
$$

$$
= \frac{1}{\sqrt{2\pi i z}} \int_{-\infty}^{\infty} e^{\frac{iu^2}{2z}}\Big(1
$$

$$
+ \sum_{j=1}^{\infty}\left(\left(\sum_{\ell=1}^{j}\frac{1}{(j-\ell+1)!}\left(\frac{i}{z}\right)^{j-\ell+1}b_{j-\ell+1,2\ell-2}\right)u^{4j}\right.
$$

$$
+ \left.\left(\sum_{\ell=1}^{j}\frac{1}{(j-\ell+1)!}\left(\frac{i}{z}\right)^{j-\ell+1}b_{j-\ell+1,2\ell-1}\right)u^{4j+2}\right)\Big)du
$$

$$
= 1 + \sum_{j=1}^{\infty}\left(\left(\sum_{\ell=1}^{j}\frac{1}{(j-\ell+1)!}\left(\frac{i}{z}\right)^{j-\ell+1}b_{j-\ell+1,2\ell-2}\right)\cdot(4j-1)!!\cdot(iz)^{2j}\right.
$$

$$
+ \left.\left(\sum_{\ell=1}^{j}\frac{1}{(j-\ell+1)!}\left(\frac{i}{z}\right)^{j-\ell+1}b_{j-\ell+1,2\ell-1}\right)\cdot(4j+1)!!\cdot(iz)^{2j+1}\right)
$$

$$
= 1 + \sum_{j=1}^{\infty}\left(\sum_{\ell=1}^{j}\frac{(4j-1)!!\cdot i^{3j-\ell+1}}{(j-\ell+1)!}b_{j-\ell+1,2\ell-2}z^{j+\ell-1}\right.
$$

$$
+ \left.\sum_{\ell=1}^{j}\frac{(4j+1)!!\cdot i^{3j-\ell+2}}{(j-\ell+1)!}b_{j-\ell+1,2\ell-1}z^{j+\ell}\right)
$$

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$$= 1 + \sum_{n=1}^{\infty} \left(\left(\sum_{\ell=1}^{n} \frac{(8n - 4\ell - 1)!! \cdot i^{2n+1}}{(2n - 2\ell + 1)!} b_{2n-2\ell+1,2\ell-2} \right) z^{2n-1} \right.$$

$$+ \left(\sum_{\ell=1}^{n} \frac{(8n - 4\ell + 3)!! \cdot i^{2n}}{(2n - 2\ell + 2)!} b_{2n-2\ell+2,2\ell-2} \right) z^{2n}$$

$$+ \left(\sum_{\ell=1}^{n} \frac{(8n - 4\ell + 1)!! \cdot i^{2n+2}}{(2n - 2\ell + 1)!} b_{2n-2\ell+1,2\ell-1} \right) z^{2n}$$

$$+ \left. \left(\sum_{\ell=1}^{n} \frac{(8n - 4\ell + 5)!! \cdot i^{2n+1}}{(2n - 2\ell + 2)!} b_{2n-2\ell+2,2\ell-1} \right) z^{2n+1} \right)$$

$$= 1 - 3ib_{1,0}z + \sum_{n=1}^{\infty} (-1)^n \Bigg($$

$$\sum_{\ell=1}^{n} \frac{(8n - 4\ell + 1)!!}{(2n - 2\ell + 1)!} \left(\frac{8n - 4\ell + 3}{2n - 2\ell + 2} b_{2n-2\ell+2,2\ell-2} - b_{2n-2\ell+1,2\ell-1} \right) z^{2n}$$

$$- i \Bigg(\sum_{\ell=1}^{n} \frac{(8n - 4\ell + 5)!!}{(2n - 2\ell + 2)!} \left(\frac{8n - 4\ell + 7}{2n - 2\ell + 3} b_{2n-2\ell+3,2\ell-2} \right.$$

$$- b_{2n-2\ell+2,2\ell-1} \Bigg) + (4n + 3)!! \cdot b_{1,2n} \Bigg) z^{2n+1} \Bigg).$$

This means that if we define a sequence $a_0, a_1, a_2, \ldots \in \mathbb{C}$ by setting $a_0 = 1$, $a_1 = -3ib_{1,0}$,

$$a_{2n} = (-1)^n \sum_{\ell=1}^{n} \frac{(8n - 4\ell + 1)!!}{(2n - 2\ell + 1)!} \left(\frac{8n - 4\ell + 3}{2n - 2\ell + 2} b_{2n-2\ell+2,2\ell-2} - b_{2n-2\ell+1,2\ell-1} \right) \tag{6}$$

for $n \in \{1, 2, 3, \ldots\}$, and

$$a_{2n+1} = (-1)^{n+1} i \Bigg(\sum_{\ell=1}^{n} \frac{(8n - 4\ell + 5)!!}{(2n - 2\ell + 2)!} \left(\frac{8n - 4\ell + 7}{2n - 2\ell + 3} b_{2n-2\ell+3,2\ell-2} \right.$$

$$- b_{2n-2\ell+2,2\ell-1} \Bigg) + (4n + 3)!! \cdot b_{1,2n} \Bigg) \tag{7}$$

for $n \in \{1, 2, 3, \ldots\}$, then we have a reason to hope that the equation

$$\frac{1}{\sqrt{2\pi i z}} \int_{-\infty}^{\infty} e^{\frac{i}{z}(\cosh(u) - 1)} du = \sum_{n=0}^{\infty} a_n z^n$$

is maybe true with some $z \in \mathbb{C}$.

If one writes a computer program that computes the values of $a_0, a_1, a_2, \ldots$

by using the equations (5), (6) and (7), the output turns out to be

$$a_0 = 1$$
$$a_1 = -\frac{i}{8}$$
$$a_2 = -\frac{9}{128}$$
$$a_3 = \frac{75i}{1024}$$
$$a_4 = \frac{3675}{32768}$$
$$a_5 = -\frac{59535i}{262144}$$
$$a_6 = -\frac{2401245}{4194304}$$
$$a_7 = \frac{57972915i}{33554432}$$
$$a_8 = \frac{13043905875}{2147483648}$$
$$a_9 = -\frac{418854310875i}{17179869184}$$
$$a_{10} = -\frac{30241281245175}{274877906944}$$
$$a_{11} = \frac{1212400457192925i}{2199023255552}$$
$$a_{12} = \frac{213786613951685775}{70368744177664}$$
$$a_{13} = -\frac{10278202593831046875i}{562949953421312}$$
$$a_{14} = -\frac{1070401384414690453125}{9007199254740992}$$
$$a_{15} = \frac{60013837619516978071875i}{72057594037927936}$$
$$\vdots$$

One interesting observation that we can make from here is that the sequence $a_0, a_1, a_2, \ldots$ is a divergent sequence. It doesn't converge to zero as one would usually hope with Taylor series. Most people who had initially only taken a glance at the first three coefficients were probably surprised by this fact. Now when we found out about this we should return to the first few coefficients and notice that yes indeed they satisfy the inequalities

$$|a_0| > |a_1| > |a_2| < |a_3| < |a_4| < \cdots.$$

It would be nice if we could somehow simplify the formulas that define the sequence $a_0, a_1, a_2, \ldots$. One idea that should always be tried is that we

look at the prime factorizations of the numerators and denominators, and try to recognize some repeating pattern. The prime factorizations are

$$a_0 = 1$$

$$a_1 = -\frac{i}{2^3}$$

$$a_2 = -\frac{3^2}{2^7}$$

$$a_3 = \frac{3 \cdot 5^2 i}{2^{10}}$$

$$a_4 = \frac{3 \cdot 5^2 \cdot 7^2}{2^{15}}$$

$$a_5 = -\frac{3^5 \cdot 5 \cdot 7^2 i}{2^{18}}$$

$$a_6 = -\frac{3^4 \cdot 5 \cdot 7^2 \cdot 11^2}{2^{22}}$$

$$a_7 = \frac{3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2 i}{2^{25}}$$

$$a_8 = \frac{3^6 \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13^2}{2^{31}}$$

$$a_9 = -\frac{3^4 \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 17^2 i}{2^{34}}$$

$$a_{10} = -\frac{3^4 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2}{2^{38}}$$

$$a_{11} = \frac{3^6 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 19^2 i}{2^{41}}$$

$$a_{12} = \frac{3^5 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23^2}{2^{46}}$$

$$a_{13} = -\frac{3^5 \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23^2 i}{2^{49}}$$

$$a_{14} = -\frac{3^{11} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23^2}{2^{53}}$$

$$a_{15} = \frac{3^{10} \cdot 5^5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29^2 i}{2^{56}}$$

$$\vdots$$

Most people, who stare at these prime factorizations, probably cannot immediately recognize a pattern. The key to succeeding in the recognition is to take a look at the prime factorizations of the numbers $n! \cdot a_n$. They turn

out to be

$$1! \cdot a_1 = -\frac{i}{2^3}$$

$$2! \cdot a_2 = -\frac{3^2}{2^6}$$

$$3! \cdot a_3 = \frac{3^2 \cdot 5^2 i}{2^9}$$

$$4! \cdot a_4 = \frac{3^2 \cdot 5^2 \cdot 7^2}{2^{12}}$$

$$5! \cdot a_5 = -\frac{3^6 \cdot 5^2 \cdot 7^2 i}{2^{15}} = -\frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 i}{2^{15}}$$

$$6! \cdot a_6 = -\frac{3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2}{2^{18}} = -\frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2}{2^{18}}$$

$$7! \cdot a_7 = \frac{3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 i}{2^{21}} = \frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2 \cdot 13^2 i}{2^{21}}$$

$$8! \cdot a_8 = \frac{3^8 \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13^2}{2^{24}} = \frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2 \cdot 13^2 \cdot 15^2}{2^{24}}$$

$$9! \cdot a_9 = -\frac{3^8 \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 i}{2^{27}}$$

$$= -\frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2 \cdot 13^2 \cdot 15^2 \cdot 17^2 i}{2^{27}}$$

$$10! \cdot a_{10} = -\frac{3^8 \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2}{2^{30}}$$

$$= -\frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2 \cdot 13^2 \cdot 15^2 \cdot 17^2 \cdot 19^2}{2^{30}}$$

$$\vdots$$

From here we see that most apparently the formula

$$a_n = \frac{(-i)^n ((2n-1)!!)^2}{n! \cdot 2^{3n}} \tag{8}$$

is true. We encountered a new mathematical problem: How do we prove that the equations (5), (6) and (7) define the same sequence as Equation (8) does? I'll omit the attempts to solve this problem now, because there are other novel findings that should be gone through.

According to Stirling's approximation [2]

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + o(1)\right)$$

in the limit $n \to \infty$ the absolute values of the coeffients defined by Equation

(8) are approximately

$$|a_n| = \frac{((2n-1)!!)^2}{n! \cdot 2^{3n}} = \frac{((2n-1)!)^2}{n! \cdot ((n-1)!)^2 \cdot 2^{5n-2}}$$

$$= \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n \left(\frac{1}{\sqrt{2\pi(n-1)}}\left(\frac{e}{n-1}\right)^{n-1}\right)^2$$

$$\left(\sqrt{2\pi(2n-1)}\left(\frac{2n-1}{e}\right)^{2n-1}\right)^2 \frac{1}{2^{5n-2}}(1+o(1)) = \cdots$$

It might be a good idea to prove

$$(n-1)^{-2n+2} = n^{-2n+2}(e^2 + o(1))$$

and

$$(2n-1)^{4n-2} = (2n)^{4n-2}(e^{-2} + o(1))$$

as separate exercises with L'Hopital's rule, and then complete the simplification of $|a_n|$ as

$$\cdots = \frac{1}{\sqrt{2\pi}}\frac{2n-1}{\sqrt{n(n-1)}}n^{-n}n^{-2n+2}(2n)^{4n-2}e^n e^{2n-2}e^{-4n+2}\frac{1}{2^{5n-2}}$$

$$\left(e^2 + o(1)\right)\left(e^{-2} + o(1)\right)\left(1 + o(1)\right) = \sqrt{\frac{2}{\pi n}}\left(\frac{n}{2e}\right)^n (1 + o(1)).$$

This approximation of the coefficients allows us to solve the radius of convergence [3] of the Taylor series. The answer is

$$r = \liminf_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty}\left(\sqrt{\frac{2}{\pi n}}\left(\frac{n}{2e}\right)^n (1+o(1))\right)^{-\frac{1}{n}}$$

$$= \lim_{n\to\infty}2e\left(\frac{\pi}{2}\right)^{\frac{1}{2n}}n^{\frac{1}{2n}}\frac{1}{n}(1+o(1)) = 2e \cdot 1 \cdot 1 \cdot 0 \cdot 1 = 0.$$

The radius of convergence is zero, and the Taylor series that we've been studying is useless! This means that at this point we do not yet know anything about the possible analytic continuation of the function $z \mapsto f(z)$. Earlier [1] we made an obvious observation that the Taylor series appeared to be more useful with large Δt, and less useful with small Δt. The claim maybe was roughly in the right direction in some sense, but now we see that the claim wasn't really correct. The truth is that it makes no difference how large Δt is; the series will never converge.

The seriously interesting new question is this: If we define the function $\Delta t \mapsto K(\Delta t)$ with the equations (2), (3) and (4), and then define the time evolution of the wave function ψ with Equation (1), will these definitions imply that the wave function ψ then satisfies the relativistic Schrödinger equation? We should clarify what we assume about the past values of the

wave function ψ in this question. Of course we are not going to pose the question in the usual way by first assuming that the initial values $\psi(t, x)$ would only be known at some time t, and that then new values $\psi(t + \Delta t, x)$ would be generated for some future time $t + \Delta t$, because this wouldn't be compatible with our use of hyperbolas. What we'll have to do is that we'll fix some time t, and then we'll assume that the initial values $\psi(t', x')$ are known for all past times t' such that $t' \leq t$. However, we are not going to pose the question in a such way that $\psi(t', x')$ would be some arbitrary function for all the past time values $t' \leq t$, because we don't have a reason to believe that the hypothesis would be true with such initial condition. We will assume that the values $\psi(t', x')$ for the past times $t' \leq t$ will be such that $\psi(t', x')$ already satisfies the relativistic Schrödinger equation for all those past times. Then the question that we'll be interested in is that when new values $\psi(t + \Delta t, x)$ are generated for times $t + \Delta t$, will the time evolution be of such kind that ψ maintains the form that the relativistic Schrödinger equation remains satisfied for the new time values $t + \Delta t$. So we fix some t, and we assume that

$$i\hbar \partial_t \psi(t', x') = \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\,\psi(t', x')$$

is true for all $t' \leq t$ and $x' \in \mathbb{R}$. This is equivalent to stating that these initial values of ψ can be written in the form

$$\psi(t', x') = \frac{1}{2\pi\hbar} \int\limits_{-\infty}^{\infty} e^{-\frac{i(t'-t)}{\hbar}\sqrt{(mc^2)^2+c^2p^2}} e^{\frac{i}{\hbar}px'} \hat{\psi}(t, p)dp,$$

where $\hat{\psi}(t, p)$ is the Fourier transform of $\psi(t, x)$ defined using the convention

$$\hat{\psi}(t, p) = \int\limits_{-\infty}^{\infty} e^{-\frac{i}{\hbar}px} \psi(t, x)dx.$$

We define $\psi(t + \Delta t, x)$ using Equation (1), and then we are interested to know whether the relation

$$\psi(t + \Delta t, x) = \frac{1}{2\pi\hbar} \int\limits_{-\infty}^{\infty} e^{-\frac{i\Delta t}{\hbar}\sqrt{(mc^2)^2+c^2p^2}} e^{\frac{i}{\hbar}px} \hat{\psi}(t, p)dp$$

is true or not. If the answer is yes, it will mean that equation

$$i\hbar \partial_t \psi(t + \Delta t, x) = \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\,\psi(t + \Delta t, x)$$

is true for all $\Delta t > 0$ and $x \in \mathbb{R}$. Then we can say that the wave function ψ remains in a such state that the relativistic Schrödinger equation remains satisfied.

So the values $\psi(t + \Delta t, x)$ are supposed to come from the formula

$$\psi(t + \Delta t, x)$$

$$= K(\Delta t) \int_{-\infty}^{\infty} \psi\left(t + \Delta t - \Delta t \cosh\left(\frac{\xi}{c\Delta t}\right),\ x + c\Delta t \sinh\left(\frac{\xi}{c\Delta t}\right)\right) d\xi$$

$$= \frac{1}{\int_{-\infty}^{\infty} e^{\frac{i\Delta t mc^2}{\hbar} \cosh\left(\frac{\xi}{c\Delta t}\right)} d\xi} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{i\Delta t}{\hbar}\left(\cosh\left(\frac{\xi}{c\Delta t}\right)-1\right)\sqrt{(mc^2)^2+c^2p^2}}\right.$$

$$\left. e^{\frac{i}{\hbar}p\left(x+c\Delta t \sinh\left(\frac{\xi}{c\Delta t}\right)\right)} \hat{\psi}(t,p) dp\right) d\xi$$

$$= \frac{1}{\int_{-\infty}^{\infty} e^{\frac{i\Delta t mc^2}{\hbar} \cosh\left(\frac{\xi}{c\Delta t}\right)} d\xi} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{\frac{i\Delta t}{\hbar} \cosh\left(\frac{\xi}{c\Delta t}\right)\sqrt{(mc^2)^2+c^2p^2}}\right.$$

$$\left. e^{\frac{i}{\hbar}pc\Delta t \sinh\left(\frac{\xi}{c\Delta t}\right)} d\xi\right) e^{-\frac{i\Delta t}{\hbar}\sqrt{(mc^2)^2+c^2p^2}} e^{\frac{i}{\hbar}px} \hat{\psi}(t,p) dp.$$

Here we first substituted the formula for the past values $\psi(t', x')$ into the time evolution equation, and then we changed the order of the integrals. At this point we encounter the problem that it's not obvious what should be done with the new inner integral. One of the properties of the inner integral is that if one substitutes $p = 0$ into it, the integral becomes the same integral as whose reciprocal is in front of the expression. If we assume that our hypothesis is true, and that it will somehow be possible to prove it, it is possible to guess how the proof will work. The guess is that the inner integral is a such kind of integral that it only looks like that its value would depend on the parameter p, but for some technical reason the value of the integral actually does not depend on p. Suppose we had learnt from somewhere that the value of the inner integral did not depend on p. We could then substitute $p = 0$ into the inner integral without changing its value. The inner integral would then be a constant with respect to the outer integral over the parameter p. We could take this constant outside the outer integral, and the constant would then cancel with the reciprocal of the same constant that's in front of the expression. After the cancellation what remains is precisely what we want for us to then be able to conclude that the wave function ψ is satisfying the relativistic Schrödinger equation for the future times $t + \Delta t$. So the proof of the hypothesis is almost already complete. Only thing that remains to be done is that we should somehow prove that the value of the inner integral does not depend on p. This can be accomplished with a change of integration variable by defining a new variable as

$$\mu = cp \cosh\left(\frac{\xi}{c\Delta t}\right) + \sqrt{(mc^2)^2 + c^2p^2} \sinh\left(\frac{\xi}{c\Delta t}\right).$$

This relation between the integration variables ξ and μ can also be expressed as

$$\xi \;=\; c\Delta t \ln\Big(\frac{\sqrt{\mu^2 + (mc^2)^2} + \mu}{\sqrt{(mc^2)^2 + c^2p^2} + cp} \Big).$$

The formula

$$-\xi \;=\; c\Delta t \ln\Big(\frac{\sqrt{\mu^2 + (mc^2)^2} - \mu}{\sqrt{(mc^2)^2 + c^2p^2} - cp} \Big)$$

turns out to be useful. The values of the hyperbolic cosine and sine can be expressed as functions of μ as

$$\cosh\Big(\frac{\xi}{c\Delta t}\Big) \;=\; \frac{\sqrt{(mc^2)^2 + c^2p^2}\sqrt{\mu^2 + (mc^2)^2} \;-\; cp\mu}{(mc^2)^2}$$

and

$$\sinh\Big(\frac{\xi}{c\Delta t}\Big) \;=\; \frac{\sqrt{(mc^2)^2 + c^2p^2}\,\mu \;-\; cp\sqrt{\mu^2 + (mc^2)^2}}{(mc^2)^2}.$$

Then the expression in the exponent of the integrand turns out to be

$$\frac{i\Delta t}{\hbar} \cosh\Big(\frac{\xi}{c\Delta t}\Big)\sqrt{(mc^2)^2 + c^2p^2} \;+\; \frac{i}{\hbar}pc\Delta t \sinh\Big(\frac{\xi}{c\Delta t}\Big)$$
$$= \frac{i\Delta t}{\hbar}\sqrt{\mu^2 + (mc^2)^2}.$$

We see that this is an interesting change of variable, because it makes the parameter p vanish from the integrand. The Jacobian related to the change of variable is

$$\frac{d\xi}{d\mu} \;=\; \frac{c\Delta t}{\sqrt{\mu^2 + (mc^2)^2}},$$

and there is no parameter p here either. The application of change of variable gives us an equation

$$\int_{-\infty}^{\infty} e^{\frac{i\Delta t}{\hbar}\cosh\big(\frac{\xi}{c\Delta t}\big)\sqrt{(mc^2)^2+c^2p^2}} e^{\frac{i}{\hbar}pc\Delta t \sinh\big(\frac{\xi}{c\Delta t}\big)}\, d\xi$$

$$= \int_{-\infty}^{\infty} e^{\frac{i\Delta t}{\hbar}\sqrt{\mu^2+(mc^2)^2}} \frac{c\Delta t}{\sqrt{\mu^2 + (mc^2)^2}}\, d\mu,$$

and from here we see that the value of the inner integral does not depend on the parameter p.

Since the proof was successful, we now know that the earlier idea that there would maybe need to be an approximation sign "$\approx$" in the relativistic Schrödinger equation was unnecessary. If the time evolution of a wave function is defined with the equations (1), (2), (3) and (4), then at least

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with the right initial conditions, the relativistic Schrödinger equation will be satisfied precisely with an equality sign "=".

We could now decide that if the time evolution of a wave function is defined with the equations (1), (2), (3) and (4), we say that we are using a *hyperbolic time evolution formulation.* The hyperbolic time evolution formulation does not rely on a use of divergent series. We used divergent series in some calculations in a symbolic way only for the purpose of discovering the path to this formulation.

It can be deduced from the nonlocal nature of the pseudo-differential operator in the relativistic Schrödinger equation that the hyperbolic time evolution formulation cannot produce relativistic Schrödinger equation satisfying wave functions out from arbitrary initial conditions. One question that maybe interests some is that what happens then if the wave function does not satisfy the relativistic Schrödinger equation in the past. What will the future values of the wave function then be like according to the hyperbolic time evolution formulation? We are not going to be able to answer this question in this article, but it would be a reasonable speculation that maybe in such situation the wave function somehow converges into a state where the relativistic Schrödinger equation eventually becomes satisfied?

References

[1] *Hyperbolic Quantum Time Evolution Kernel*, J. S. Puuronen, 2024, viXra:2407.0064

[2] https://en.wikipedia.org/wiki/Stirling%27s_approximation

[3] https://en.wikipedia.org/wiki/Radius_of_convergence