

Residue classes and stopping time of the $3n+1$ problem

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Abstract

This paper presents an analysis of the stopping time of the $3n+1$ problem based on the residue class of n .

$3n + 1$ problem (or conjecture)

In the $3 \cdot n+1$ problem^[1] it is possible to define the function $s : N \rightarrow N$:

$$s(n) = \begin{cases} 3 \cdot n+1 & \text{if } n \equiv 1 \pmod{2} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \end{cases}$$

the sequence $s^k(n)$ for $k \in N$ obtained using the function $s(n)$ is as follows:

$$s^k(n) = \begin{cases} n & \text{for } k=0 \\ s(s^{k-1}(n)) & \text{for } k>0 \end{cases}$$

Stopping time

The $3 \cdot n+1$ conjecture is equivalent to the conjecture that for each $n \in N$, $n > 1$, there exists $k \in N$ such that $s^k(n) < n$. The least $k \in N$ such that $s^k(n) < n$ is called the stopping time of n ^[1].

If n is even $n \equiv 0 \pmod{2}$ then the stopping time $k=1$ and $s^1(n) = \frac{n}{2}$.

Let's analyze the case where n is odd.

Let m be the number of odd terms in the first k terms of the $3 \cdot n+1$ sequence, and d_i be the number of consecutive even terms immediately following the i -th odd term, then the next term $s^k(n)$ in the $3 \cdot n+1$ sequence is^[1]:

$$s^k(n) = \frac{3^m}{2^{k-m}} \cdot n + \sum_{i=1}^m \frac{3^{m-i}}{2^{d_i + \dots + d_m}}$$

note that for n odd $k-m = d_1 + \dots + d_m$.

Then

$$s^k(n) = \frac{3^m \cdot n + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{k-m}} = \frac{3^m \cdot n + r}{2^{k-m}}$$

where r depends on m and n with $r \geq 3^m - 2^m$.

It is possible to observe that if $n \equiv 1 \pmod{2^2}$ then $m=1$ and the stopping time is $k=3$

$$s^3(n) = \frac{3 \cdot n + 1}{2^2}$$

so we have $r=1=3^1-2^1$ and $d_1=2$.

If we now consider the numbers $n \equiv 3 \pmod{2^2}$ we have:

$$n = 3 + 4 \cdot a$$

$$s^1(n) = 3 \cdot n + 1 = 10 + 12 \cdot a$$

$$s^2(n) = \frac{3 \cdot n + 1}{2} = 5 + 6 \cdot a = 2 + 2 \cdot a + n \quad \text{odd number}$$

$$s^4(n) = \frac{3 \cdot \frac{3 \cdot n + 1}{2} + 1}{2} = 2 + a + 2n$$

if a odd then $n \equiv 7 \pmod{2^3}$, $d_1=1$, $d_2=1$ and $m>2$ since $s^2(n), s^4(n)$ odd and $s^4(n) > n$

if a even then $a=2 \cdot b$

$$s^5(n) = \frac{3 \cdot \frac{3 \cdot n + 1}{2} + 1}{4} = 1 + b + n$$

if b odd then $n \equiv 11 \pmod{2^4}$, $d_1=1$, $d_2=2$ and $m>2$ since $s^2(n), s^5(n)$ odd and $s^5(n) > n$

if b even $b=2 \cdot c$ then $n \equiv 3 \pmod{2^4}$

$$s^6(n) = \frac{3 \cdot \frac{3 \cdot n + 1}{2} + 1}{8} = c + \frac{(n+1)}{2} = \frac{3^2 \cdot n + 5}{2^4}$$

then $k=6$, $d_1=1$, $d_2=3$ and $m=2$ with $r=5=3+2^1=3^2-2^2$.

As seen for numbers $n \equiv 7 \pmod{2^3}$

$$s^4(n) = \frac{3 \cdot \frac{3 \cdot n + 1}{2} + 1}{2} = \frac{9 \cdot n + 5}{4} \quad \text{odd number and } d_1=1 \text{ and } d_2=1$$

$$s^6(n) = \frac{3 \cdot \frac{9 \cdot n + 5}{4} + 1}{2} = \frac{27 \cdot n + 19}{8}$$

if $n \equiv 7 \pmod{2^4}$ then $s^6(n)$ even and

$$s^7(n) = \frac{27 \cdot n + 19}{16}$$

if $n \equiv 23 \pmod{2^5}$ then $s^7(n)$ even and

$$s^8(n) = \frac{27 \cdot n + 19}{32} < n$$

then $k=8$, $d_1=1$, $d_2=1$, $d_3=3$ and $m=3$ from which

$$r = 19 = 3^3 - 2^3 = 3^2 + 3 \cdot 2^1 + 2^{(1+1)} = 3 \cdot (3+2^1) + 2^{(1+1)} = 3 \cdot 5 + 2^2 .$$

Note that the numbers $n \equiv 15 \pmod{2^4}$ and $n \equiv 7 \pmod{2^5}$ remain to be analyzed.

For numbers $n \equiv 11 \pmod{2^4}$

$$s^5(n) = \frac{3 \cdot \frac{3 \cdot n + 1}{2} + 1}{4} = \frac{9 \cdot n + 5}{8} \text{ odd number and } d_1=1 \text{ and } d_2=2$$

$$s^7(n) = \frac{3 \cdot \frac{9 \cdot n + 5}{8} + 1}{2} = \frac{27 \cdot n + 23}{16}$$

if $n \equiv 11 \pmod{2^5}$ then $s^7(n)$ even and

$$s^8(n) = \frac{3 \cdot \frac{9 \cdot n + 5}{8} + 1}{2} = \frac{27 \cdot n + 23}{32} < n$$

then $k=8$, $d_1=1$, $d_2=2$, $d_3=2$ and $m=3$ from which

$$r = 23 = 3^2 + 3 \cdot 2^1 + 2^{(1+2)} = 3 \cdot (3+2^1) + 2^{(1+2)} = 3 \cdot 5 + 2^3 .$$

For numbers $n \equiv 27 \pmod{2^5}$ then $s^7(n)$ odd and $d_3=1$

$$s^7(n) = \frac{3 \cdot \frac{9 \cdot n + 5}{8} + 1}{2} = \frac{27 \cdot n + 23}{16}$$

$$s^9(n) = \frac{3 \cdot \frac{27 \cdot n + 23}{16} + 1}{2} = \frac{81 \cdot n + 85}{32}$$

if $n \equiv 59 \pmod{2^6}$ then $s^9(n)$ even

$$s^{10}(n) = \frac{81 \cdot n + 85}{64}$$

if $n \equiv 59 \pmod{2^7}$ then $s^{10}(n)$ even

$$s^{11}(n) = \frac{81 \cdot n + 85}{128} < n$$

then $k=11$, $d_1=1$, $d_2=2$, $d_3=1$, $d_4=3$ and $m=4$ from which

$$r=85=3^3+3^2 \cdot 2^1+3 \cdot 2^{(1+2)}+2^{(1+2+1)}=3 \cdot (3^2+3 \cdot 2^1+2^{(1+2)})+2^{(1+2+1)}+2^{(1+2)}=3 \cdot 23+2^4 \text{ .}$$

Note that the numbers $n \equiv 27 \pmod{2^6}$ and $n \equiv 123 \pmod{2^7}$ remain to be analyzed.

For numbers $n \equiv 15 \pmod{2^4}$

$$s^6(n) = \frac{3 \cdot \frac{9 \cdot n + 5}{4} + 1}{2} = \frac{27 \cdot n + 19}{8} \text{ odd and } d_3=1$$

$$s^8(n) = \frac{3 \cdot \frac{27 \cdot n + 19}{8} + 1}{2} = \frac{81 \cdot n + 65}{16}$$

if $n \equiv 15 \pmod{2^5}$ then $s^8(n)$ even and

$$s^9(n) = \frac{81 \cdot n + 65}{32}$$

if $n \equiv 15 \pmod{2^6}$ then $s^9(n)$ even and

$$s^{10}(n) = \frac{81 \cdot n + 65}{64}$$

if $n \equiv 15 \pmod{2^7}$ then $s^{10}(n)$ even and

$$s^{11}(n) = \frac{81 \cdot n + 65}{128} < n$$

then $k=11$, $d_1=1$, $d_2=1$, $d_3=1$, $d_4=4$ and $m=4$ from which

$$r=65=3^4-2^4=3^3+3^2 \cdot 2^1+3 \cdot 2^{(1+1)}+2^{(1+1+1)}=3 \cdot (3^2+3 \cdot 2^1+2^{(1+1)})+2^{(1+1+1)}=3 \cdot 19+2^3 \text{ .}$$

For numbers $n \equiv 7 \pmod{2^5}$

$$s^7(n) = \frac{27 \cdot n + 19}{16} \text{ odd and } d_3=2$$

$$s^9(n) = \frac{3 \cdot \frac{27 \cdot n + 19}{16} + 1}{2} = \frac{81 \cdot n + 73}{32}$$

if $n \equiv 7 \pmod{2^6}$ then $s^9(n)$ even and

$$s^{10}(n) = \frac{81 \cdot n + 73}{64} < n$$

if $n \equiv 7 \pmod{2^7}$ then $s^{10}(n)$ even and $s^{11}(n) = \frac{81 \cdot n + 73}{128} < n$

then $k=11$, $d_1=1$, $d_2=1$, $d_3=2$, $d_4=3$ and $m=4$ from which

$$r=73=3^3+3^2 \cdot 2^1+3 \cdot 2^{(1+1)}+2^{(1+1+2)}=3 \cdot (3^2+3 \cdot 2^1+2^{(1+1)})+2^{(1+1+2)}=3 \cdot 19+2^4 \text{ .}$$

Note that the numbers $n \equiv 39 \pmod{2^6}$, $n \equiv 71 \pmod{2^7}$, $n \equiv 31 \pmod{2^5}$, $n \equiv 47 \pmod{2^6}$ and $n \equiv 79 \pmod{2^7}$ remain to be analyzed.

By continuing with this procedure it is easy to verify the following results:

m	k	$n \pmod{2^{k-m}}$	d_m	E	r	G
1	3	1	2		1	a
2	6	3	3	a^1	$5=3+2^1$	b
3	8	23	3	b^1	$19=3^2+3 \cdot 2^1+2^{(1+1)}$	c
		11	2	b^2	$23=3^2+3 \cdot 2^1+2^{(1+2)}$	d
4	11	15	4	c^1	$65=3^3+3^2 \cdot 2^1+3 \cdot 2^{(1+1)}+2^{(1+1+1)}$	e
		7	3	c^2	$73=3^3+3^2 \cdot 2^1+3 \cdot 2^{(1+1)}+2^{(1+1+2)}$	f
		59	3	d^1	$85=3^3+3^2 \cdot 2^1+3 \cdot 2^{(1+2)}+2^{(1+2+1)}$	g
5	13	95	4	e^1	$211=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+1)}+3 \cdot 2^{(1+1+1)}+2^{(1+1+1+1)}$	h
		175	3	e^2	$227=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+1)}+3 \cdot 2^{(1+1+1)}+2^{(1+1+1+2)}$	i
		79	2	e^3	$259=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+1)}+3 \cdot 2^{(1+1+1)}+2^{(1+1+1+3)}$	j
		39	3	f^1	$251=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+1)}+3 \cdot 2^{(1+1+2)}+2^{(1+1+2+1)}$	k
		199	2	f^2	$283=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+1)}+3 \cdot 2^{(1+1+2)}+2^{(1+1+2+2)}$	l
		219	3	g^1	$287=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+2)}+3 \cdot 2^{(1+2+1)}+2^{(1+2+1+1)}$	m
		123	2	g^2	$319=3^4+3^3 \cdot 2^1+3^2 \cdot 2^{(1+2)}+3 \cdot 2^{(1+2+1)}+2^{(1+2+1+2)}$	n
6	16	575	5	h^1	$665=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+1)}+3 \cdot 2^{(1+1+1+1)}+2^{(1+1+1+1+1)}$	
		287	4	h^2	$697=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+1)}+3 \cdot 2^{(1+1+1+1)}+2^{(1+1+1+1+2)}$	
		735	3	h^3	$761=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+1)}+3 \cdot 2^{(1+1+1+1)}+2^{(1+1+1+1+3)}$	
		367	4	i^1	$745=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+1)}+3 \cdot 2^{(1+1+1+2)}+2^{(1+1+1+2+1)}$	
		815	3	i^2	$809=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+1)}+3 \cdot 2^{(1+1+1+2)}+2^{(1+1+1+2+2)}$	
		975	3	j^1	$905=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+1)}+3 \cdot 2^{(1+1+1+3)}+2^{(1+1+1+3+1)}$	
		999	4	k^1	$817=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+2)}+3 \cdot 2^{(1+1+2+1)}+2^{(1+1+2+1+1)}$	
		423	3	k^2	$881=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+2)}+3 \cdot 2^{(1+1+2+1)}+2^{(1+1+2+1+2)}$	
		583	3	l^1	$977=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+1)}+3^2 \cdot 2^{(1+1+2)}+3 \cdot 2^{(1+1+2+2)}+2^{(1+1+2+2+1)}$	
		923	4	m^1	$925=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+2)}+3^2 \cdot 2^{(1+2+1)}+3 \cdot 2^{(1+2+1+1)}+2^{(1+2+1+1+1)}$	
		347	3	m^2	$989=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+2)}+3^2 \cdot 2^{(1+2+1)}+3 \cdot 2^{(1+2+1+1)}+2^{(1+2+1+1+2)}$	
		507	3	n^1	$1085=3^5+3^4 \cdot 2^1+3^3 \cdot 2^{(1+2)}+3^2 \cdot 2^{(1+2+1)}+3 \cdot 2^{(1+2+1+2)}+2^{(1+2+1+2+1)}$	

The value in column E indicates the relationship with the value of r represented by the letter in column G.

$$s^k_m(n) = \frac{3^m \cdot n + r}{2^{k_m - m}} \text{ and } s^{k_{m+1}}(n') = \frac{3^{m+1} \cdot n' + r'}{2^{k_{m+1} - (m+1)}} \text{ with } r' = 3 \cdot r + 2^{(k_m - m - d_m + d'_m)} \text{ and } 1 \leq d'_m < d_m$$

example c^2 indicates $r=19$ e $d'_m=2$ then $r' = 3 \cdot 19 + 2^{(8-3-3+2)} = 73$.

As seen $d_m = k - m - (d_1 + \dots + d_{m-1})$ and if for a certain value of m if we find a value of r , which we indicate as $r_m^i = 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}$ depends on the values of d_1, \dots, d_{m-1} , then from this value of r_m^i for $m+1$ we can obtain $r_{m+1}^j = 3 \cdot r_m^i + 2^{(d_1 + \dots + d_{m-1} + d_m^i)}$ with $1 \leq d_m^j < d_m^i$ and $x_{m+1}^j \cdot 3^{(m+1)} \equiv -r_{m+1}^j \pmod{2^{d_1 + \dots + d_{m-1} + d_m^j + d_{m+1}^j}}$ from which we can obtain x_{m+1}^j and for $n \equiv x_{m+1}^j \pmod{2^{d_1 + \dots + d_{m-1} + d_m^j + d_{m+1}^j}}$ we have $s^{m+1+d_1+\dots+d_{m-1}+d_m^j+d_{m+1}^j}(n) < n$.

If $n \equiv x \pmod{2^{d_1 + \dots + d_{m-1} + d_m}}$ with x odd, $m > 1$, $s^i(n) > n$ for $1 \leq i < m + d_1 + \dots + d_{m-1} + d_m$

and $s^{m+d_1+\dots+d_{m-1}+d_m}(n) = \frac{3^m \cdot n + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d_m}} < n$ then if $d_m > 1$ for $1 \leq d'_m < d_m$

$$s^{m+d_1+\dots+d_{m-1}+d'_m}(x) = \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d'_m}} \text{ is even}$$

$$\text{and } r_m = 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}$$

if $x = x' + b \cdot 2^{d_1 + \dots + d_{m-1} + d'_m}$ with $x' < 2^{d_1 + \dots + d_{m-1} + d'_m}$

$$s^{m+d_1+\dots+d_{m-1}+d'_m}(x) = \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d'_m}} = \frac{3^m \cdot x' + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d'_m}} + b \cdot 3^m$$

$s^{m+d_1+\dots+d_{m-1}+d'_m}(x)$ is even and for

$$y = x + (1+2 \cdot a) \cdot 2^{d_1 + \dots + d_{m-1} + d'_m} = x' + b \cdot 2^{d_1 + \dots + d_{m-1} + d'_m} + (1+2 \cdot a) \cdot 2^{d_1 + \dots + d_{m-1} + d'_m}$$

$$s^{m+d_1+\dots+d_{m-1}+d'_m}(y) = \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d'_m}} + (1+2 \cdot a) \cdot 3^m = s^{m+d_1+\dots+d_{m-1}+d'_m}(x) + (1+2 \cdot a) \cdot 3^m$$

$s^{m+d_1+\dots+d_{m-1}+d'_m}(y)$ is odd then

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+1}(y) = \frac{3^m \cdot y + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d'_m}} + 1 = \frac{3^{m+1} \cdot y + 3^m + 2^{d_1 + \dots + d_{m-1} + d'_m} + \sum_{i=2}^m 3^{m+1-i} \cdot 2^{d_1 + \dots + d_{i-1}}}{2^{d_1 + \dots + d_{m-1} + d'_m + 1}}$$

we obtain $r_{m+1} = 3 \cdot r_m + 2^{(d_1 + \dots + d'_m)} = 3^m + 2^{d_1 + \dots + d_{m-1} + d'_m} + \sum_{i=2}^m 3^{m+1-i} \cdot 2^{d_1 + \dots + d_{i-1}}$ with $1 \leq d'_m < d_m$.

As seen $s^{m+d_1+\dots+d_{m-1}+d_m}(x) = \frac{1}{2^{d_m-d'_m}} \cdot \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1+\dots+d_{i-1}}}{2^{d_1+\dots+d_{m-1}+d'_m}}$

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+1}(y) = \frac{3 \cdot \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1+\dots+d_{i-1}}}{2^{d_1+\dots+d_{m-1}+d'_m}} + (1+2 \cdot a) \cdot 3^{m+1} + 1}{2}$$

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+1}(y) = \frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + (1+2 \cdot a) \cdot 3^{m+1} + 1}{2}$$

with $y = x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m}$

if $s^{m+1+d_1+\dots+d_{m-1}+d'_m+1}(y)$ is odd

$$\frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + ((1+2 \cdot a)+2) \cdot 3^{m+1} + 1}{2} = s^{m+1+d_1+\dots+d_{m-1}+d'_m+1}(y) + 3^{m+1} \text{ is even}$$

for an appropriate value of a we can get

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+1}(x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + (1+2 \cdot a) \cdot 3^{m+1} + 1}{2}$$

if $\frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + (1+2 \cdot a) \cdot 3^{m+1} + 1}{4}$ is odd then

$$\frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + ((1+2 \cdot a)+4) \cdot 3^{m+1} + 1}{4} = \frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + (1+2 \cdot a) \cdot 3^{m+1} + 1}{4} + 3^{m+1}$$

is even and for an appropriate value of a we can get

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+2}(x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + (1+2 \cdot a) \cdot 3^{m+1} + 1}{4}$$

...

for an appropriate value of a we can get

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+(d_m-d'_m)+1}(x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{3 \cdot 2^{(d_m-d'_m)} \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) + (1+2 \cdot a) \cdot 3^{m+1} + 1}{2 \cdot 2^{d_m-d'_m}}$$

as seen $s^{m+d_1+\dots+d_{m-1}+d_m}(n) < n$ then $\frac{3^m}{2^{d_1+\dots+d_{m-1}+d_m}} < 1$

$$k_{m+1} = m+1+d_1+\dots+d_{m-1}+d'_m+d'_{m+1}$$

if $\frac{3^m}{2^{d_1+\dots+d_{m-1}+d_m}} < \frac{2}{3}$ then $\frac{3^{m+1}}{2 \cdot 2^{d_m-d'_m}} < 2^{d_1+\dots+d_{m-1}+d'_m}$, $d'_{m+1} = (d_m-d'_m)+1$, $k_{m+1} = k_m+2$

$$s^{k_{m+1}}(x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{3\cdot s^{m+d_1+\dots+d_{m-1}+d'_m}(x)}{2} + \frac{(1+2\cdot a)\cdot 3^{m+1}+1}{2\cdot 2^{d_m-d'_m}}$$

$$s^{k_{m+1}}(x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}) < x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}$$

else

$$\frac{3^{m+1}}{4} < 3^m < 2^{d_1+\dots+d_{m-1}+d'_m} \quad \text{and} \quad \frac{3^{m+1}}{4\cdot 2^{d_m-d'_m}} < \frac{3^m}{2^{d_m-d'_m}} < 2^{d_1+\dots+d_{m-1}+d'_m}$$

for an appropriate value of a we can get

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+(d_m-d'_m)+2}(x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{3\cdot 2^{(d_m-d'_m)}\cdot s^{m+d_1+\dots+d_{m-1}+d'_m}(x)+(1+2\cdot a)\cdot 3^{m+1}+1}{4\cdot 2^{d_m-d'_m}}$$

then $d'_{m+1}=(d_m-d'_m)+2$ and $k_{m+1}=k_m+3$ with

$$s^{k_{m+1}}(x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{3\cdot s^{m+d_1+\dots+d_{m-1}+d'_m}(x)}{4} + \frac{(1+2\cdot a)\cdot 3^{m+1}+1}{4\cdot 2^{d_m-d'_m}}$$

$$s^{k_{m+1}}(x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}) < x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m}$$

in both cases you can get b so

$$n \equiv ((x+2^{d_1+\dots+d_{m-1}+d'_m})(\text{mod } 2^{d_1+\dots+d_{m-1}+d'_{m+1}})+2\cdot b\cdot 2^{d_1+\dots+d_{m-1}+d'_m})(\text{mod } 2^{d_1+\dots+d_{m-1}+d_m+d'_m+d'_{m+1}})$$

$$n \equiv (x+(1+2\cdot a)\cdot 2^{d_1+\dots+d_{m-1}+d'_m})(\text{mod } 2^{d_1+\dots+d_{m-1}+d_m+d'_m+d'_{m+1}})$$

so

$$s^{k_{m+1}}(n) = \frac{3^{m+1}\cdot n + 3^m + 2^{d_1+\dots+d_{m-1}+d'_m} + \sum_{i=2}^m 3^{m+1-i}\cdot 2^{d_1+\dots+d_{i-1}}}{2^{d_1+\dots+d_{m-1}+d'_m+d'_{m+1}}} < n$$

$$b \text{ can be obtained using } r_{m+1} = 3\cdot r_m + 2^{(d_1+\dots+d'_m)} = 3^m + 2^{d_1+\dots+d_{m-1}+d'_m} + \sum_{i=2}^m 3^{m+1-i}\cdot 2^{d_1+\dots+d_{i-1}}$$

$$y\cdot 3^{(m+1)} \equiv -r_{m+1} (\text{mod } 2^{d_1+\dots+d_{m-1}+d'_m+d'_{m+1}}) .$$

Let's examine the relationship between k and m with $k-m=d_1+\dots+d_m$:

as seen if $\frac{3^m}{2^{d_1+\dots+d_{m-1}+d'_m}} \geq \frac{2}{3}$ then

$$d'_{m+1}=(d_m-d'_m)+2 \quad , \quad k_{m+1}=k_m+3$$

$$3^{m+1} \geq 2^{k_m+2-(m+1)} = 2^{k_{m+1}-(m+1)-1} \quad , \quad 2\cdot 3^{m+1} > 2^{k_{m+1}-(m+1)} \quad \text{and} \quad 2\cdot 3^{m+1} < 2^{k_{m+1}-(m+1)+1}$$

$$k_{m+1}-(m+1) = \lfloor \log_2(2\cdot 3^{m+1}) \rfloor = \lfloor 1+(m+1)\cdot \log_2(3) \rfloor$$

else if $\frac{3^m}{2^{d_1+\dots+d_{m-1}+d_m}} < \frac{2}{3}$

$$d'_{m+1} = (d_m - d'_m) + 1, \quad k_{m+1} = k_m + 2$$

if we assume that $k_{m+1} - (m+1) = 2 + [1 + (m-1) \cdot \log_2(3)] = 2 + [\log_2(2 \cdot 3^{m-1})]$

$$2^{k_{m+1} - (m+1) - 2} \leq 2 \cdot 3^{m-1} < 2^{k_{m+1} - (m+1) - 1} \quad \text{and} \quad k_{m+1} = k_m + 2 = k_{m-1} + 4$$

$$\frac{3^{m-1}}{2^{k_{m-1} + 4 - (m+1) - 2}} = \frac{3^{m-1}}{2^{k_{m-1} - (m-1)}} \geq 1 \quad \text{this is impossible then it must be}$$

$$k_{m+1} - (m+1) = k_m - m + 1 = 1 + [1 + m \cdot \log_2(3)] = 1 + [\log_2(2 \cdot 3^m)]$$

$$2^{k_{m+1} - (m+1) - 1} \leq 2 \cdot 3^m < 2^{k_{m+1} - (m+1)}, \quad 4 \cdot 3^m \geq 2^{k_{m+1} - (m+1)}, \quad 2 \cdot 3^{m+1} \geq 4 \cdot 3^m \geq 2^{k_{m+1} - (m+1)} \quad \text{and}$$

$$2 \cdot 3^{m+1} < 2^{k_{m+1} - (m+1) + 1}$$

$$k_{m+1} - (m+1) = [1 + (m+1) \cdot \log_2(3)] .$$

We verified that $k_m - m = [1 + m \cdot \log_2(3)]$ with $k_m - m = d_1 + \dots + d_{m-1} + d_m$

then $2 \cdot 3^m \geq 2^{k_m - m}$

$$\frac{3^{m+1}}{2^{d_m - d'_m}} > \frac{2 \cdot 3^m}{2^{d_m - d'_m}} \geq 2^{d_1 + \dots + d_{m-1} + d_m - (d_m - d'_m)} \quad \text{with} \quad 1 \leq d'_m < d_m$$

for $1 \leq e \leq d_m - d'_m$

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+(d_m-d'_m)+1-e} (x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m}) = \frac{2^e \cdot 3 \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x)}{2} + 2^{e-1} \frac{(1+2 \cdot a) \cdot 3^{m+1} + 1}{2^{d_m - d'_m}}$$

$n \equiv x \pmod{2^{k_m - m}}$ has stopping time k_m then

$$s^{m+d_1+\dots+d_{m-1}+d'_m+(d_m-d'_m)-e}(x) = \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1+\dots+d_{i-1}}}{2^{d_1+\dots+d_{m-1}+d'_m+(d_m-d'_m)-e}} = 2^e \cdot s^{m+d_1+\dots+d_{m-1}+d_m}(x) > x$$

then since $2^{e-1} \geq 1$ for $1 \leq e \leq d_m - d'_m$

$$s^{m+1+d_1+\dots+d_{m-1}+d'_m+(d_m-d'_m)+1-e} (x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m}) > x + (1+2 \cdot a) \cdot 2^{d_1+\dots+d_{m-1}+d'_m} .$$

In summary it has been demonstrated that

For every $m > 1$, if $n \equiv x \pmod{2^{k_m - m}}$ has stopping time k_m ,

then for every d'_m , $1 \leq d'_m < d_m$ there exists the integer $b_{d'_m} \geq 0$ so for the numbers

$$n'_{d'_m} \equiv ((x + 2^{d_1 + \dots + d_{m-1} + d'_m}) \pmod{2^{d_1 + \dots + d_{m-1} + d'_m + 1}}) + 2 \cdot b_{d'_m} \cdot 2^{d_1 + \dots + d_{m-1} + d'_m} \pmod{2^{d_1 + \dots + d_{m-1} + d'_m + d'_{m+1}}}$$

$$s^{k_{m+1}}(n'_{d'_m}) < n'_{d'_m}$$

and if $\frac{3^m}{2^{d_1 + \dots + d_{m-1} + d_m}} < \frac{2}{3}$ then $d'_{m+1} = (d_m - d'_m) + 1$ else $d'_{m+1} = (d_m - d'_m) + 2$

the stopping time is $k_{m+1} = (m+1) + d_1 + \dots + d_{m-1} + d'_m + d'_{m+1} = (m+1) + [1 + (m+1) \cdot \log_2(3)]$

furthermore $s^{k_m}(n) = \frac{3^m \cdot n + r}{2^{k_m - m}}$ and $r = 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}$

then $n'_{d'_m} \equiv x'_{d'_m} \pmod{2^{k_{m+1} - (m+1)}}$

$$r'_{d'_m} = 3 \cdot r + 2^{(d_1 + \dots + d_{m-1} + d'_m)} = 3 \cdot r + 2^{(k_m - m - d_m + d'_m)} \quad \text{and} \quad s^{k_{m+1}}(n'_{d'_m}) = \frac{3^{m+1} \cdot n'_{d'_m} + r'_{d'_m}}{2^{k_{m+1} - (m+1)}}$$

$$x'_{d'_m} \cdot 3^{(m+1)} \equiv -r'_{d'_m} \pmod{2^{k_{m+1} - (m+1)}} \quad \text{from which we can obtain } x'_{d'_m}.$$

Application to cycle of the generalized function $3n + h$

The $3 \cdot n + h$ function^[2] $t_h: N \rightarrow N$ is defined for $h \in N$, $h \equiv 1 \pmod{2}$ by

$$t_h(n) = \begin{cases} 3 \cdot n + h & \text{if } n \equiv 1 \pmod{2} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \end{cases}$$

the sequence $t_h^k(n)$ for $k \in N$ obtained using the function $t_h(n)$ is as follows:

$$t_h^k(n) = \begin{cases} n & \text{for } k=0 \\ t_h(t_h^{k-1}(n)) & \text{for } k>0 \end{cases}$$

Let m be the number of odd terms in the first k terms of the $t_h^k(n)$ sequence, and d_i be the number of consecutive even terms immediately following the i -th odd term, then for n odd the next term $t_h^k(n)$ in the $3 \cdot n + h$ sequence is

$$t_h^k(n) = \frac{3^m \cdot n + h \cdot r}{2^{k-m}}$$

where $r = 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}$ and $k - m = d_1 + \dots + d_m$

r then identifies a specific trajectory up to $t_h^k(n)$.

For $h \geq 1$, $h \equiv 1 \pmod{6}$, in a primitive cycle^[2] , consisting only of integers relatively prime to h , we have:

$$y = \frac{3^m \cdot y + h \cdot r}{2^{k-m}} \quad \text{and} \quad h \cdot r = (2^{k-m} - 3^m) \cdot y$$

then y divides r .

As we have seen r must be $r = 3^{m-1} + \sum_{i=2}^m 3^{m-i} \cdot 2^{d_1 + \dots + d_{i-1}}$ fixed $m \geq 1$ if we take a value of r obtained for a given residue class that has stopping time $k = m + \lceil 1 + m \cdot \log_2(3) \rceil$ then for those divisors y of r for which $h \cdot r = (2^{k-m-d_m+a} - 3^m) \cdot y$ with $(2^{k-m-d_m+a} - 3^m) > 0$ the sequence $t_h^k(n)$ has the same trajectory as $s^k(n)$ up to the m -th odd term and therefore they have the same r , since r does not depend on d_m .

Some examples:

1)

$$m=1 , k=3 , d_m=2 , r=1$$

then $y=r=1$ and $h=(2^a-3^1)$ with $(2^a-3^1) > 0$

for $h=(2^a-3^1)$ with $a \geq 2$ the term $y=1$ belongs to a primitive cycle.

2)

$$m=2 , k=6 , d_m=3 , r=5$$

then $y=r=5$ and $h=(2^{6-2-3+a}-3^2)$ with $a \geq 3$

for $h=(2^{a+1}-3^2)$ with $a \geq 3$ the term $y=5$ belongs to a primitive cycle.

3)

$$m=3 , k=8 , d_m=3 , r=19$$

then $y=r=19$ and $h=(2^{8-3-3+a}-3^3)$ with $a \geq 3$

for $h=(2^{a+2}-3^3)$ with $a \geq 3$ the term $y=19$ belongs to a primitive cycle.

4)

$$m=4 , k=11 , d_m=4 , r=65=13 \cdot 5$$

then $y=13$ and $h \cdot 5 = (2^{11-4-4+a}-3^4)$ with $a > 3$ and $a \equiv 1 \pmod{4}$

for $h = \frac{(2^{a+3}-3^4)}{5}$ with $a > 3$ and $a \equiv 1 \pmod{4}$ the term $y=13$ belongs to a primitive cycle.

Below is the algorithm code to generate the residue classes for $m \geq 1$:

```
get_dim(m)={my(log2_3=log(3)/log(2),X1=matrix(2,floor(1+m*log2_3)-m), d=0,dim=1);X1[1,1]=0;
if(m>=3,for(x=3,m,y=floor(1+(x-1)*log2_3);
for(i=1,y-x,for(j=i,y-x,if(i==1,X1[1+(d+1)%2,j]=1);X1[1+(d+1)%2,j]=X1[1+(d+1)%2,j]+X1[d+1,i]));
d=(d+1)%2);for(i=1,floor(1+m*log2_3)-m,dim=dim+X1[d+1,i]));dim;}
{m_max=9;dim_max=get_dim(m_max);N=vector(dim_max);R=matrix(2,dim_max);Dm=matrix(2,dim_max);
nr=0;m=1;log2_3=log(3)/log(2);kmm=floor(1+m*log2_3);c=1;N[1]=1;R[1,1]=1;Dm[1,1]=2;
print1("m = ",m," - stopping time: ",kmm+m,"\nif n == ",N[1]);
print1(" (mod 2^",kmm,")\ntotal residue classes: ",c,"\n\n");
for(m=1,m_max-1, kmm=floor(1+m*log2_3);mp1=m+1;kmp1=floor(1+mp1*log2_3);
print1("m = ",mp1," - stopping time: ",kmp1+mp1,"\nif n == ");c1=0;
for(i=1,c,sdmm1=kmm-Dm[1+nr,i];for(j=1,Dm[1+nr,i]-1,rj=3*R[1+nr,i]+2^(sdmm1+j); c1=c1+1;
N[c1]=((2^kmp1-rj)*lift(Mod(1/3^mp1,2^kmp1)))%2^kmp1;R[1+(1+nr)%2,c1]=rj;
Dm[1+(1+nr)%2,c1]=kmp1-(sdmm1+j)); c=c1;N1=vector(c);N1=vecsort(N[1..c]);for(i1=1,c-1,
print1(N1[i1],", "));print1(N1[c]);nr=(1+nr)%2;print1(" (mod 2^",kmp1,")\ntotal residue classes: ",c,"\n\n"))}
```

PARI/GP code of the algorithm

```
get_dim(m)={my(dim_vec=floor(1+m*log(3)/log(2))-m,nr=0,dim=1,X1=matrix(2,dim_vec));X1[1,1]=0;
if(m>=3,kmxm1=4;for(x=3,m,for(i=1,kmxm1-x,for(j=i,kmxm1-x,if(i==1,X1[1+(nr+1)%2,j]=1);
X1[1+(nr+1)%2,j]=X1[1+(nr+1)%2,j]+X1[nr+1,i]));nr=(nr+1)%2;if(3^x<2^(kmxm1+1),kmxm1=kmxm1+1,
kmxm1=kmxm1+2));for(i=1,dim_vec,dim=dim+X1[nr+1,i]););dim;}
{m_max=9;dim_max=get_dim(m_max);N=vector(dim_max);R=matrix(2,dim_max);Dm=matrix(2,dim_max);
m=1;kmm=2;c=1;N[1]=1;R[1,1]=1;Dm[1,1]=2;
print1("m = ",m," - stopping time: ",kmm+m,"\nif n == ",N[1]);
print1(" (mod 2^",kmm,")\ntotal residue classes: ",c,"\n\n");
for(m=1,m_max-1,mp1=m+1;if(3^mp1<2^(kmm+1),kp1mmp1=kmm+1,kp1mmp1=kmm+2);c1=0;
print1("m = ",mp1," - stopping time: ",kp1mmp1+mp1,"\nif n == ");
for(i=1,c,sdmm1=kmm-Dm[1,i];for(j=1,Dm[1,i]-1,rj=3*R[1,i]+2^(sdmm1+j);if(c1,print1(", "));
c1=c1+1;N[c1]=((2^kp1mmp1-rj)*lift(Mod(1/3^mp1,2^kp1mmp1)))%2^kp1mmp1;print1(N[c1]);R[2,c1]=rj;
Dm[2,c1]=kp1mmp1-(sdmm1+j));c=c1;for(i2=1,c,R[1, i2]=R[2, i2];Dm[1, i2]=Dm[2, i2]);
print1(" (mod 2^",kp1mmp1,")\ntotal residue classes: ",c,"\n\n");kmm=kp1mmp1)}
```

PARI/GP code 2 of the algorithm

```
row(m)={if(m<1, [0], my(r=[1], d=[2], km=2); for(mi=1, m-1, my(temp1=[], temp2=[]),
c=if(3^(mi+1)<2^(km+1), 1, 2)); for(i=1, #d, temp1=concat(temp1, vector(d[i]-1, j, 3*r[i]+2^(km-d[i]+j))));
temp2=concat(temp2, vector(d[i]-1, j, d[i]-j+c)); km=km+c; r=temp1; d=temp2; );
vecsort(apply(x->((-x)*lift(Mod(1/3^m, 2^km)))%2^km, r));)
for(i=0,9,print1(row(i),"; "));)
```

PARI/GP code 3 of the algorithm

```
n_max=12;rows_max=16;
r_value(n)={if(n<1, [0], my(r=[1], d=[2], km=2); for(i=1, n-1, my(temp1=[], temp2=[]),
c=if(3^(i+1)<2^(km+1), 1, 2));
for(j=1, #d, temp1=concat(temp1, vector(d[j]-1, m, 3*r[j]+2^(km-d[j]+m))));
temp2=concat(temp2, vector(d[j]-1, m, d[j]-m+c)); km=km+c; r=temp1; d=temp2; ); r);}
triangle=matrix(rows_max,rows_max);index=vector(rows_max,i,1);log2_3=log(3)/log(2);
for(n=1, n_max, my(temp=r_value(n)); h=floor(1+n*log2_3); for(i=1, #temp,
my(r=temp[i],temp2=divisors(r)); for(i2=1, #temp2, for(j=1, rows_max, my(y=temp2[i2], k=3*j-1-j%2);
my(temp3=binary(k*(r/y)+3^n)); if(vecsum(temp3)=1 && #temp3>h,
triangle[j,index[j]]=y;if(index[j]<rows_max,index[j]++)))));)
for(i=1,rows_max,print1("h = ",3*i-1-i%2," - ",vecsort(select(x->x>0,triangle[i,])))
```

PARI/GP code of the algorithm for finding from r a term of the cycle of the generalized function $3n+h$

References

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