Part 3 of Guide to Hestenes's Geometric Algebra Treatment of Constant-Acceleration (Parabolic) Motion

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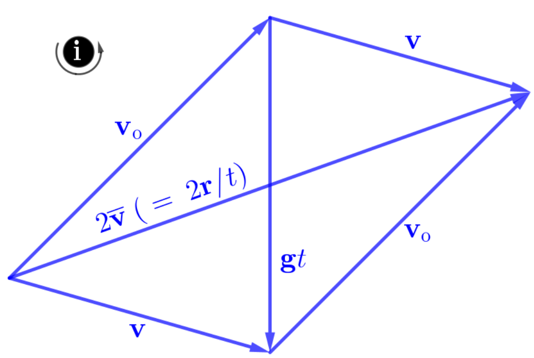
Abstract

As an aid to teachers and students who are learning to apply Geometric Algebra (GA) to high-school-level physics, we provide this third installment in our guide guide to Hestenes's treatment of constant-acceleration motion. A key element of this installment is our detailed discussion of how Hestenes uses "extended hodographs" as sources of geometric insights that can be expressed and transformed via GA to deduce relationships among factors that affect constant-acceleration motion.



The extended hodograph: a source of geometric insights that can be expressed and transformed via GA to deduce relationships among factors that affect constant-acceleration motion.

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1 Introduction

This document continues our presentation and explanation of Hestenes's "GA" treatment of constant-acceleration motion. Like the previous installments in this series ([1], [2]), the present one has been prepared in the spirit of Hestenes's observation that students will need "judicious guidance" to get through his book *New Foundations for Classical Mechanics*[3].[1] Therefore, this document is intended to be understandable by students and teachers who are still in the process of learning the basics of GA.

Our approach differs from what the reader may have experienced in mass-market textbooks, whose authors (because of length restrictions imposed by publishers) tend to present the most efficient derivations possible for their formulas. This approach can be intimidating for students, who are seldom aware

[1]"Though my book has been a continual best seller in the series for well over a decade, it is still unknown to most teachers of mechanics in the U.S. To be suitable for the series, I had to design it as a multipurpose book, including a general introduction to GA and material of interest to researchers, as well as problem sets for students. It is not what I would have written to be a mechanics textbook alone. Most students need judicious guidance by the instructor to get through it." [4]

that those formulas were almost never found via these efficient routes. Instead, someone in the past had an insight, then "followed her nose", thus arriving at a useful result that she (or others) later found a way to derive more efficiently. Indeed, that process is more or less the same way in which good problem-solvers (including students) often work. For that reason, our approach here has similar "exploratory" spirit.

In addition, our explanations and derivations are more detailed than is possible (again, because of length restrictions) in mass-market textbooks.

The examples that are usually presented when teaching constant-acceleration motion concern the trajectories of projectiles. That is the language that will be used here, but the analyses, equations, and solutions hold for any situations in which the acceleration is constant.

Please note that we don't use the terms "division of vectors" or "division of bivectors". Those terms (as well as equations that are written in terms of such divisions) can be ambiguous, and therefore a source of confusion to students. For that reason, we will use the multiplicative inverses that those "divisions" actually represent.

2 What We Will See in this Document

The previous installment [2] in this series ended with his Hestenes's Eq. 2.14 (*NFCM* p. 130). In the present document, we continue from that point, and end on p. 132, just before the beginning of the section entitled "Rectangular and polar coordinates".

After deriving his Eq. 2.14, Hestenes presents an "extended" hodograph, from which he derives

- A "conservation of energy" equation for constant-force motion.
- The relation between the two distinct trajectories that will hit a target at a given distance that is less than r_{max}, which is the the maximum range achievable along a given line of sight $\hat{\mathbf{r}}$.
- The launch vector for reaching a target at r_{max}.

3 Ideas that We Will Use

- Two multivectors (let's call them M and $\mathcal{M}$) are equal if and only if their respective parts of equal grade are equal. For example, if $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are vectors, then $\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}=\mathbf{c}\cdot\mathbf{d}+\mathbf{c}\wedge\mathbf{d}$ if and only if $\mathbf{a}\cdot\mathbf{b}=\mathbf{c}\cdot\mathbf{d}$ and $\mathbf{a}\wedge\mathbf{b}=\mathbf{c}\wedge\mathbf{d}$.

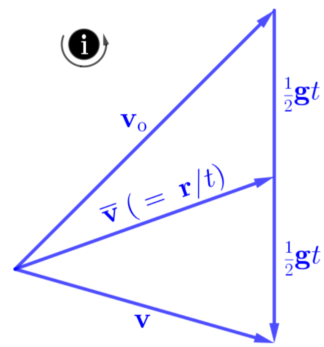

Figure 1: A hodograph.

- If two vectors $\mathbf{x}$ and $\mathbf{a}$ are perpendicular, then $\mathbf{xa} = \mathbf{x}\wedge\mathbf{a}$, because $\mathbf{x}\cdot\mathbf{a} = 0$. In such a case, we can solve the equation $\mathbf{x}\wedge\mathbf{a} = \mathbf{c}\wedge\mathbf{d}$ by rewriting it as $\mathbf{xa} = \mathbf{c}\wedge\mathbf{d}$, after which $\mathbf{x} = (\mathbf{c}\wedge\mathbf{d})\wedge\mathbf{a}^{-1}$.

4 Results from Analyses of the Extended Hodograph

4.1 Preliminary Observations, and the Extended Hodograph

"Playing around" like this is a good practice, and Hestenes' extension of the hodograph is an example of an often-productive technique in problem-solving: Asking "what can I add?" (For example, formulating a notation, or adding/extending a line in a diagram [5].)

At the beginning of the material that we are treating here, Hestenes seems to be "playing around" with the hodograph (Fig. 1) to see what might bubble up from the resulting analysis. First, he notes that although all of the previous results in the "Maximum Range" section had been derived by analyzing the upper triangle of the hodograph, the lower triangle contains additional information. Moreover, the information in both triangles can be represented in a symmetrical form by extending the hodograph, as shown in Fig. 2. Hestenes makes effective use of the interplay between gaining geometric insights from hodographs, and expressing/transforming those insights via GA.

4.2 Derivation of the Conservation of Energy Equation, and the Relation between Initial Velocity and the Velocity at r

From these two figures, we can deduce the equations

$$\mathbf{v} - \mathbf{v}_o = \mathbf{g}t \text{ (Hestenes's Eq. 2.15), and} \tag{1a}$$

$$\mathbf{v}_o + \mathbf{v} = \frac{2\mathbf{r}}{t} \quad \rightarrow \quad \frac{\mathbf{r}}{t}\,(=\overline{\mathbf{v}}) = \frac{\mathbf{v}_o + \mathbf{v}}{2} \text{ (Hestenes's Eq. 2.16).} \tag{1b}$$

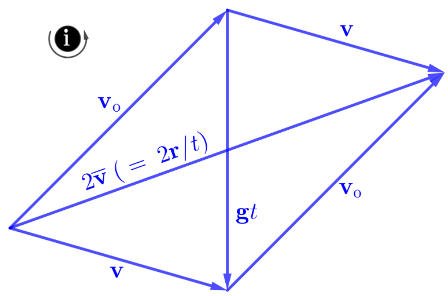

Figure 2: An extended hodograph. Hestenes makes effective use of the interplay between gaining geometric insights from hodographs, and expressing/transforming those insights via GA.

Hestenes further asserts that his Eqs. 2.15 and 2.16 can be derived algebraically from two equations that he presented earlier:

$$\mathbf{v} = \mathbf{v}_o + \mathbf{g}t \quad \text{(Hestenes's Eq. 2.3, p. 126 of } \textit{NFCM}\text{)}, \tag{2a}$$

$$\mathbf{r} = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2 \quad \text{(Hestenes's Eq. 2.4, p. 126 of } \textit{NFCM}\text{)}. \tag{2b}$$

Hestenes had derived both of these equations by integrating the equation of motion $\ddot{\mathbf{x}} = \mathbf{g}$, with the initial conditions $\mathbf{v}\,(t=0) = \mathbf{v}_o$ and $\mathbf{x}\,(t=0) = \mathbf{x}_o$.

Two points: (1) It is, of course, a good practice to investigate (as Hestenes has done) whether the inferences that we draw from diagrams are supported by mathematical analysis; and (2) Students should not accept assertions like Hestenes's (regarding the equivalence of his Eqs. 2.3/2.4 and 2.15/2.16) unquestioningly. Instead, students are encouraged to derive his Eq. 2.15 (1a) from his Eqs. 2.3 (2a) and 2.4 (8).

Having obtained his Eqs. 2.15 (1a) and 2.16 (1b) from the extended hodograph, Hestenes now notes that by multiplying them, he can eliminate t. That's potentially useful, so let's do it:

$$\begin{aligned}(\mathbf{g}t)\left(\frac{2\mathbf{r}}{t}\right) &= (\mathbf{v}+\mathbf{v}_o)\,(\mathbf{v}+\mathbf{v}_o)\\ &= v^2 + \underbrace{\mathbf{v}\mathbf{v}_o - \mathbf{v}_o\mathbf{v}}_{=2\mathbf{v}\wedge\mathbf{v}_o} - v_o^2\\ \therefore 2\mathbf{g}\mathbf{r} &= v^2 - v_o^2 + 2\mathbf{v}\wedge\mathbf{v}_o. \end{aligned} \tag{3}$$

From the theorem for the equality of multivectors, the scalar parts of both sides of the preceding equation must be equal, and so must the bivector parts. Thus, we arrive at Hestenes's Eqs. 2.18 and 2.19:

$$2\mathbf{g}\cdot\mathbf{r} = v^2 - v_o^2 \quad \text{(Hestenes's Eq. 2.18, p. 130)} \tag{4a}$$

$$\mathbf{g} \wedge \mathbf{r} = \mathbf{v} \wedge \mathbf{v}_o \quad \text{(Hestenes's Eq. 2.19, p. 130)}. \tag{4b}$$

As Hestenes observes (p. 131), his Eq. 2.18 (4a) is an expression of the conservation of energy, because $\mathbf{g} \cdot \mathbf{r}$ is the change in gravitational potential energy. We also note that his Eq. 2.18 is the 2-dimensional version of the familiar one-dimensional equation "$v^2 - v_o^2 = 2as$", in which "a" is the acceleration, and s is the displacement. Thus, Eq. 2.18 is not surprising.

In contrast, Eq. 2.19 (4b) is intriguing, as we will see in the next section.

4.3 Relation Between the Launch Directions for Two Trajectories that Have the Same v_o, and Hit a Target at the Same $r < r_{max}$.

Hestenes finds an ingenious use for Eq. 2.19 (4b) via a geometric analysis that is inspired by the realization that for a given v_o, and for a target at any distance $r < r_{max}$ along a direction $\hat{\mathbf{r}}$, there are two distinct launch directions (i.e., two distinct $\hat{\mathbf{v}}_o$'s, with their resulting resulting trajectories) that will produce a "hit". Let's distinguish the two trajectories by calling the two launch vectors $\mathbf{v}_o$ and $\mathbf{v}_o'$, and calling the respective velocities upon arrival at the target $\mathbf{v}$ and $\mathbf{v}'$. Because of Eq. 2.18, $v_o' = v_o$. (That is, the speed of the projectile upon reaching the target is the same for both trajectories.) But what does Hestenes's Eq. 2.19 tell us about the relationship between $\hat{\mathbf{v}}_o$ and $\hat{\mathbf{v}}_o'$? To answer that question, let's follow Hestenes, and construct a hodograph for each launch vector (Fig. 3).

As explained in Fig. 3, the resulting hodographs are congruent parallelograms. Therefore, $gt = 2r/t'$, and also $2r/t = gt'$. Both of these equalities give

$$tt' = \frac{2r}{g} \quad \text{(Hestenes's Eq. 2.21, p. 131)}. \tag{5}$$

Hestenes's Eq. 2.7, p. 128 of *NFCM*:
$\mathbf{g} \wedge \mathbf{r} = t\mathbf{g} \wedge \mathbf{v}_o$.

Now, as Hestenes explains, this relation (i.e., his Eq. 2.21 (5)) can be used to determine either of the two trajectories if we are given the other. The problem is to find the direction $\hat{\mathbf{v}}_o'$ from the directions $\hat{\mathbf{r}}$ and $\hat{\mathbf{v}}_o$. This can be done by using his Eq. 2.7 to get the relations $t\mathbf{g} \wedge \mathbf{v}_o = \mathbf{g} \wedge \mathbf{r}$ and $t'\mathbf{g} \wedge \mathbf{v}_o' = \mathbf{g} \wedge \mathbf{r}$, from which

$$t'\mathbf{g} \wedge \mathbf{v}_o' = t\mathbf{g} \wedge \mathbf{v}_o \text{ , and}$$
$$\mathbf{g} \wedge \hat{\mathbf{v}}_o' = \frac{t}{t'}\mathbf{g} \wedge \hat{\mathbf{v}}_o. \tag{6}$$

In addition, from Hestenes's Eq. 2.21 (5),

$$\frac{t}{t'} = \frac{gt^2}{2r}.$$

Substituting this expression for $\frac{t}{t'}$ in Eq. (6), we obtain

$$\mathbf{g} \wedge \hat{\mathbf{v}}_o' = \frac{gt^2}{2r}\mathbf{g} \wedge \hat{\mathbf{v}}_o. \tag{7}$$

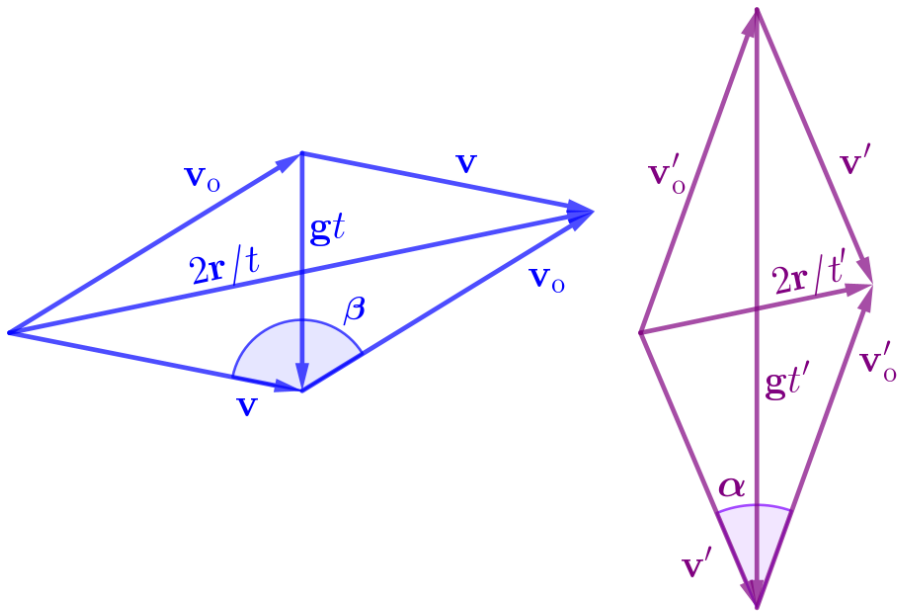

Figure 3: Hodographs for the two distinct trajectories, both with the same v_o, that reach the same target at the distance $r < r_{max}$. The two parallelograms are congruent because (1) $v'_o = v_o$; (2) $v' = v$ (as a consequence of Hestenes' Eq. 2.18 (4a)); and (3) the angles α and β must be supplementary as a consequence of Hestenes's Eq. 2.19 (4b), according to which $\mathbf{v}' \wedge \mathbf{v}'_o = \mathbf{v} \wedge \mathbf{v}_o = \mathbf{g} \wedge \mathbf{r}$. Specifically, the angles α and β must be supplementary because if they were equal, the two trajectories would be identical. Because the parallelograms are congruent, the corresponding lengths (actually, magnitudes of velocities) $\|\mathbf{g}t\|$ and $\|2\mathbf{r}/t'\|$ are equal, as are $\|\mathbf{g}t'\|$ and $\|2\mathbf{r}/t\|$.

Next, Hestenes transforms the right-hand side of Eq. (7) by using his Eq. 2.4, which was

$$\mathbf{r} = \mathbf{v}_o t + \frac{1}{2}\mathbf{g}t^2. \tag{8}$$

Taking the outer product of both sides with $\hat{\mathbf{v}}_o$, then rearranging,

$$\mathbf{g} \wedge \hat{\mathbf{v}}_o = \frac{2}{t^2}\mathbf{r} \wedge \hat{\mathbf{v}}_o.$$

Making this substitution for $\mathbf{g} \wedge \hat{\mathbf{v}}_o$ in Eq. (7), then simplifying,

$$\hat{\mathbf{g}} \wedge \hat{\mathbf{v}}_o' = \hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o \quad \text{(Hestenes's Eq. 2.22, p. 132), or}$$

$$\hat{\mathbf{v}}_o' \wedge (-\hat{\mathbf{g}}) = \hat{\mathbf{r}} \wedge \hat{\mathbf{v}}_o. \tag{9}$$

From Hestenes's Eq. 2.22 (9), we can infer a simple geometric relation between the two launch directions: the angle that $\hat{\mathbf{v}}_o'$ makes with the vertical ($-\mathbf{g}$) is equal to the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{v}}_o$. We used this relation in the previous installment in this series ([2], p. 14) to demonstrate that $\hat{\mathbf{v}}_o$ and $\hat{\mathbf{v}}_o'$ are symmetric with respect to the launch direction that gives the maximum range along $\hat{\mathbf{r}}$.

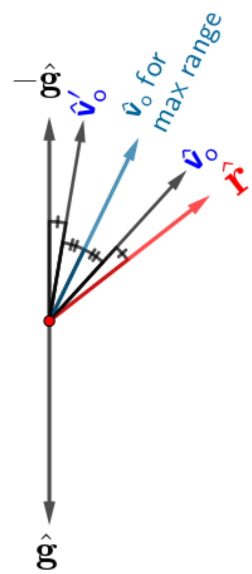

Figure 4: Hestenes demonstrates (*NFCM*, p. 132) that the angle between $\hat{\mathbf{v}}_o$ and $\hat{\mathbf{r}}$ is equal to the angle between $\hat{\mathbf{v}}_o'$ and $-\hat{\mathbf{g}}$. Because $\hat{\mathbf{v}}_0$ for r_{max} bisects the angle between $\hat{\mathbf{r}}$ and $-\hat{\mathbf{g}}$, $\hat{\mathbf{v}}_o$ and $\hat{\mathbf{v}}_o'$ are symmetric to each other with respect to the $\hat{\mathbf{v}}_o$ for r_{max}.

4.4 Derivation of the Velocity Vector of a Projectile Upon Reaching r_{max}

Hestenes's Eq. 2.22 (9) has proved useful. Next, he proceeds to derive others, again from an analysis of the extended hodographs in Fig. 3. First, he notes that as r approaches r_{max}, the two hodographs must converge to a single parallelogram—specifically, the parallelogram that is the extended hodograph for the $\mathbf{v}_o$ that gives r_{max}. Via a geometric "limiting" argument, Hestenes demonstrates that that parallelogram is actually a rectangle; therefore, $(\mathbf{v}\ at\ r_{max}) \perp (\mathbf{v}_o\ for\ r_{max})$. (We present an algebraic proof of the perpendicularity in the Appendix.)

Because $(\mathbf{v}\ at\ r_{max}) \perp (\mathbf{v}_o\ for\ r_{max})$, $(\mathbf{v}\ at\ r_{max}) \wedge (\mathbf{v}_o\ for\ r_{max}) = (\mathbf{v}\ at\ r_{max})(\mathbf{v}_o\ for\ r_{max})$. Hestenes uses this result to solve his Eq. 2.19 (4b) for $(\mathbf{v}\ at\ r_{max})$:

$$(\mathbf{v}\ at\ r_{max}) \wedge (\mathbf{v}_o\ for\ r_{max}) = \mathbf{g} \wedge \mathbf{r} \quad \text{(Hestenes's Eq. 2.19, p. 130)}$$
$$\therefore (\mathbf{v}\ at\ r_{max})(\mathbf{v}_o\ for\ r_{max}) = \mathbf{g} \wedge \mathbf{r}, \text{ and}$$
$$(\mathbf{v}\ at\ r_{max}) = [\mathbf{g} \wedge \mathbf{r}](\mathbf{v}_o\ for\ r_{max})^{-1}.$$

Hestenes writes this result as

$$(\mathbf{v}\ at\ r_{max}) = \frac{(\mathbf{g} \wedge \mathbf{r}) \cdot (\mathbf{v}_o\ for\ r_{max})}{v_o^2}. \tag{10}$$

This form takes advantage of the fact that because $\mathbf{g}$, $\mathbf{r}$, and $(\mathbf{v}_o\ for\ r_{max})$ are coplanar, $(\mathbf{g} \wedge \mathbf{r})(\mathbf{v}_o\ for\ r_{max}) = (\mathbf{g} \wedge \mathbf{r}) \cdot (\mathbf{v}_o\ for\ r_{max})$.

A Proof that for the Trajectory that Achieves r_{max}, $(\mathbf{v}_o\ for\ r_{max}) \perp (\mathbf{v}\ at\ r_{max})$

To prove the perpendicularity, we will demonstrate that

$$(\mathbf{v}_o\ for\ r_{max}) \cdot (\mathbf{v}\ at\ r_{max}) = 0.$$

To find $\mathbf{v}\ at\ r_{max}$, we can use the equation

$$\mathbf{v}\ at\ r_{max} = (\mathbf{v}_o\ for\ r_{max}) + \mathbf{g}(t\ at\ r_{max}).$$

To find t at r_{max}, we use Hestenes's EQ. 2.14 (*NFCM*, p. 130):

$$(t\ at\ r_{max})^2 = \left[\frac{2(v_o\ for\ r_{max})^2}{g^2}\right]\left[\frac{1}{1-\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}}\right].$$

Thus,

$$(\mathbf{v}_o\ for\ r_{max}) \cdot (\mathbf{v}\ at\ r_{max}) = (\mathbf{v}_o\ for\ r_{max}) \cdot \left\{(\mathbf{v}_o\ for\ r_{max}) + \mathbf{g}\sqrt{\left[\frac{2(v_o\ for\ r_{max})^2}{g^2}\right]\left[\frac{1}{1-\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}}\right]}\right\}.$$

Expanding, rearranging, and simplifying,

$$(\mathbf{v}_o\ for\ r_{max}) \cdot (\mathbf{v}\ at\ r_{max}) = (v_o\ for\ r_{max})^2 + \{(\mathbf{v}_o\ for\ r_{max}) \cdot \mathbf{g}\}\left[\frac{v_o\ for\ r_{max}}{g}\right]\sqrt{\frac{2}{1-\hat{\mathbf{g}}\cdot\hat{\mathbf{r}}}}.$$

Hestenes's Eq. 2.11:
$\hat{\mathbf{v}}_o \; for \; r_{max} = \frac{\hat{\mathbf{r}} - \hat{\mathbf{g}}}{\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\|}$.

Next, we use Hestenes's Eq. 2.11 (*NFCM*, p. 129) to express ($\mathbf{v}_o \; for \; r_{max}$) in terms of $\hat{\mathbf{r}}$ and $\hat{\mathbf{g}}$:

$$\begin{aligned}(\mathbf{v}_o \; for \; r_{max}) \cdot (\mathbf{v} \; at \; r_{max}) &= (v_o \; for \; r_{max})^2 \\ &\quad + \left\{ (v_o \; for \; r_{max}) \left[\frac{\hat{\mathbf{r}} - \hat{\mathbf{g}}}{\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\|} \right] \cdot \mathbf{g} \right\} \left[\frac{v_o \; for \; r_{max}}{g} \right] \sqrt{\frac{2}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}}} \\ &= (v_o \; for \; r_{max})^2 + (v_o \; for \; r_{max})^2 \left\{ \left[\frac{\hat{\mathbf{r}} - \hat{\mathbf{g}}}{\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\|} \right] \cdot \hat{\mathbf{g}} \right\} \sqrt{\frac{2}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}}} \; .\end{aligned}$$

Using $\|\hat{\mathbf{r}} - \hat{\mathbf{g}}\| = \sqrt{(\hat{\mathbf{r}} - \hat{\mathbf{g}})^2} = \sqrt{2\,(1 - \hat{\mathbf{r}} \cdot \hat{\mathbf{g}})}$,

$$\begin{aligned}(\mathbf{v}_o \; for \; r_{max}) \cdot (\mathbf{v} \; at \; r_{max}) &= (v_o \; for \; r_{max})^2 + (v_o \; for \; r_{max})^2 \left\{ \left[\frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{g}} - \hat{\mathbf{g}}}{\sqrt{2\,(1 - \hat{\mathbf{r}} \cdot \hat{\mathbf{g}})}} \right] \cdot \hat{\mathbf{g}} \right\} \sqrt{\frac{2}{1 - \hat{\mathbf{g}} \cdot \hat{\mathbf{r}}}} \\ &= (v_o \; for \; r_{max})^2 + (v_o \; for \; r_{max})^2 \left\{ \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{g}} - 1}{1 - \hat{\mathbf{r}} \cdot \hat{\mathbf{g}}} \right\} \\ &= 0.\end{aligned}$$

Therefore, $(\mathbf{v}_o \; for \; r_{max}) \perp (\mathbf{v} \; at \; r_{max})$.

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