Generalized random measures.Double stochastic processes and Generalized Chapman- Kolmogorov equations.Random Feinman measure.

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Abstract: In this paper generalized Chapman-Kolmogorov equation is derived.

1.Introduction

We remind that in probability theory, *random element* is a generalization of the concept of random variable to more complicated spaces than the simple real line. The concept was introduced by Maurice Fréchet (1948) who commented that the “development of probability theory and expansion of area of its applications have led to necessity to pass from schemes where (random) outcomes of experiments can be described by number or a finite set of numbers, to schemes where outcomes of experiments represent, for example, vectors, functions, processes, fields, series, transformations, and also sets or collections of sets.

Let $(\Omega, F, \mathbf{P})$ be a probability space, and (E, E) a measurable space.
A *random element* with values in E is a function $X: \ \Omega \to E$ which is (F, E) -measurable. That is, a function $X(\omega), \omega \in \Omega$ such that for any $B \in E$, the preimage of B lies in F.

Note if $(E,E) = (\mathbb{R},\mathfrak{B}(\mathbb{R}))$ where $\mathbb{R}$ are the real numbers, and $\mathfrak{B}(\mathbb{R})$ is its Borel σ-algebra, then the definition of random element is the classical definition of *random variable*.

Let $\Xi(\Sigma,\mathbb{R})$ be a set of the all $\mathbb{R}$- valued random variables defined on a probability space $\Sigma = (\Omega',\mathcal{F},\mathbf{P})$ A random element $X(\omega),\omega \in \Omega,\ X : \ \Omega \to \Xi(\Sigma,\mathbb{R})$ with values in $\Xi(\Sigma,\mathbb{R})$ we call the *double stochastic* random variable or simply *generalized random variable*.

Notation 1.1. A set of the all generalized random variables we will denote by $\widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$.

Definition 1.2. Double stochastic random process that is, a function $X(t,\omega) : [0,T] \times \Omega \to \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ with values in $\widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$.

Let E be a separable complete metric space and let Σ be its Borel σ-algebra.

Define $\widetilde{M} = \{\mu \mid \mu$ is measure on $(E,\Sigma)\}$ and the subset of locally finite measures by $M := \{\mu \in \widetilde{M} \mid \mu(\widetilde{B}) < \infty$ for all bounded measurable $\widetilde{B} \in \Sigma\}$

For all bounded measurable $\widetilde{B}$, define the mappings $I_{\widetilde{B}} : \mu \mapsto \mu(\widetilde{B})$ from $\widetilde{M}$ to $\mathbb{R}$. Let $\widetilde{\mathbf{M}}$ be the σ-algebra induced by the mappings $I_{\widetilde{B}}$ on $\widetilde{M}$ and $\mathbf{M}$ the σ-algebra induced by the mappings $I_{\widetilde{B}}$ on $\mathbf{M}$. Note that $\widetilde{\mathbf{M}} \restriction M = \mathbf{M}$.

Definition 1.3. A random measure is a random element from $(\Omega,A,\mathbf{P})$ to $(\widetilde{M},\widetilde{\mathbf{M}})$ that almost surely takes values in $(M,\mathbf{M})$.

Definition 1.4.Let E be a separable complete metric space and let Σ be its Borel σ-algebra. *Generalized* random measure it is a function $\mu : \Sigma \to \Xi(\Omega,\Sigma,\mathbb{R})$, that satisfies:

(1) If $E_1 \subset E_2$, then a.s.:$\mu(E_1) < \mu(E_2)$.

(2) If $E_n \in \mathcal{S},\ n = 1,2,\dots$ and $E_i \cap E_j = \varnothing\ (i \neq j)$, then a.s.:$\mu(\cup_n E_n) = \sum_n \mu(E_n)$.

If we further impose a condition a.s.:$\mu(\Omega) = 1$, then the generalized randommeasure so defined is called the *generalized random probability measure* and is usually denoted by $\widetilde{\mathbf{P}}$.

2.Generalized random variables and double stochastic processes.

2.1.(1).Canonical random variables and stochastic processes

Consider a set Ω whose elements are to be interpreted as the possible outcomes of a probabilistic experiment. On this set, we define a σ-algebra $\mathcal{S}$, which is a set whose elements are *some subsets* of Ω, such that the following properties are satisfied

(1) $\Omega \in \mathcal{S}$

(2) $\forall E \in \mathcal{S}, \exists E^C \equiv \Omega \backslash E \in \mathcal{S}$. That is, for every element E of $\mathcal{S}$, E^C which is defined as the complement of E in Ω is also in the set $\mathcal{S}$

(3) $\forall E_n \in \mathcal{S}, n = 1,2,\dots,\ \cup_n E_n \in \mathcal{S}$. That is, the finite union of elements in $\mathcal{S}$ also belongs to $\mathcal{S}$ With such an algebra defined, we call the tuple $(\Omega,\mathcal{S})$ a measurable space.

A measure is a function defined on a measurable space. It is a function $\mu : \mathcal{S} \to [0, \infty)$, that satisfies

(1) If $E_1 \subset E_2$, then $\mu(E_1) < \mu(E_2)$.

(2) If $E_n \in \mathcal{S},\ n = 1,2,\ldots$ and $E_i \cap E_j = \varnothing\ (i \neq j)$, then $\mu(\cup_n E_n) = \sum_n \mu(E_n)$

If we further impose a condition $\mu(\Omega) = 1$, then the measure so defined is called the probability measure and is usually denoted by $\mathbf{P}$.

Definition 2.1.1. The tuple $(\Omega, \mathcal{S}, \mathbf{P})$ is then called the probability space.

Suppose $(\Omega, \mathcal{S})$ and $(\Omega', \mathcal{S}')$ are two measurable spaces. The function $X : \Omega \to \Omega'$ is called a <u>*random element*</u>, if for every $A \in \mathcal{S}'$, $X^{-1}(A) \in \mathcal{S}$. If the first measurable space is equipped with a probability measure $\mathbf{P}$, then the random element induces a probability measure on the second space $(\Omega', \mathcal{S}')$ and given by $\mathcal{P} \equiv \mathbf{P} \circ X^{-1}$, called the distribution of X.

Definition 2.1.2. If the second measurable space $(\Omega', \mathcal{S}')$ is taken to be $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$, where $\mathfrak{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ the function $X : \Omega \to \Omega'$ is called a random variable. For this case, the distribution of X is completely determined by the *distribution function* defined as

$$F_X(x) = \mathbf{P}(X \leq x) \tag{2.1.1}$$

where $X \leq x$ is defined as the set $\{X \leq x\} = \{\omega \in \Omega \mid X(\omega) \leq x\}$.

The distribution function is continuously non-decreasing and gives the measure associated with $(x,\ x + dx) \in \mathfrak{B}$ as $d(F_X(x))$

Definition 2.1.3.A special case is when the distribution function can be written in the form

$$F(x) = \int_{-\infty}^{x} p(x)dx \tag{2.1.2}$$

In this case, the function $p(x)$ is called the probability density function of X, and the measure associated with dx is $d(F_X(x) = p(x)dx$.

With the knowledge of the distribution function, one can define the expectation value of any function $h(X)$ as

$$\mathbf{E}[h(X)] = \int h(x)d(F_X(x)) \tag{2.1.3}$$

With these definitions, we now define a stochastic process.

Definition 2.1.4. A stochastic process is an indexed set of random variables $X_t,\ t \in T$, where T is called the index set and is usually taken (in the continuous case) to be $T = [0, \infty)$. In a physical setting, the index is time and a stochastic process can be interpreted as, for each $t \in T$, X_t picks an event $E \in \mathcal{S}$ with probability $\mathbf{P}(E)$, and returns $X_t(E)$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and an n-dimensional time-dependent random variable $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, i.e. a $\mathbb{R}^n$-valued function that maps elements of the sample space to real numbers associated to the outcomes of a random experiment.

Assume now that the random variable is time-dependent: $X(t, \omega) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, then, given a sequence of timesteps $t_1, t_2, \ldots, t_N$ with $t_1 < t_2 < \cdots < t_N$, we can write

$$\{X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_N) = x_N\}, \tag{2.1.4}$$

where $x_1, x_2, \ldots, x_N \in \Omega$. The sequence defined in Eq.(2.1.4) is a stochastic process

if the joint probability density $p(x_1,t_1,;x_2,t_2,;\ldots x_N,t_N)$, fully describes the system.
Depending on how the joint probability density is defined, we can classify the stochastic processes. Here, we consider two cases.

1.Purely random process or separable stochastic process. If successive values of $X(t)$ are statistically independent, then we the joint probability density is written as

$$p(x_1,t_1,;x_2,t_2,;\ldots x_N,t_N) = \prod_{i=1}^{N} p(x_i,t_i). \tag{2.1.5}$$

The underlying idea is that the probability of an event x_i occurring at a time t_i does not depend on the past and in no way determines the future. In terms of conditional probabilities, we can write

$$p(x_N,t_N|x_1,t_1,;x_2,t_2,;\ldots x_{N-1},t_{N-1}) = p(x_N,t_N) \tag{2.1.6}$$

2.A second example of a stochastic process is the Markov process, whose joint probability density is written as

$$p(x_1,t_1,;x_2,t_2,;\ldots x_N,t_N) = \prod_{i=2}^{N} p(x_i,t_i|x_{i-1},t_{i-1})p(x_1,t_1), \tag{2.1.7}$$

or in terms of conditional probabilities as

$$p(x_N,t_N|x_1,t_1,;x_2,t_2,;\ldots x_{N-1},t_{N-1}) = p(x_N,t_N|x_{N-1},t_{N-1}), \tag{2.1.8}$$

i.e., a Markovian process is a process without memory, whose temporal evolution depends only on the present state, not on the past.

2.1.(2).Canonical space-time white noise.

Definition 2.1.5. A distribution valued Gaussian process with mean zero $\{\dot{W}(t,x) : t \in [0,T]\ ,\ x \in \mathbb{R}^d\}$is a space–time white noise if

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)\delta(x-y). \tag{2.1.9}$$

More precisely :

Definition 2.1.6.We denote by $\mathcal{D}((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ the space of the infinitely differentiable functions with compact support in $(0,T)\times\mathbb{R}^d$ and values in $\mathbb{R}^d$.

1.For any $\xi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the random variable $\dot{W}(\xi)$ is Gaussian variable with mean zero

$$\mathbb{E}(\dot{W}(\xi) = 0 \tag{2.1.10}$$

For any $\xi_1,\xi_2 \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the random variables $\dot{W}(\xi_1),\dot{W}(\xi_2)$ have covariance

- $$\mathbb{E}(\dot{W}(\xi_1)\dot{W}(\xi_2)) = \int_{[0,T]\times\mathbb{R}^d} \xi_1(t,x)\cdot\xi_2(t,x)dxdt. \tag{2.1.11}$$

It is not difficult to construct a space-time white noise. In fact, let $\{f_j : j \in \mathbb{N}\}$ be a complete orthonormal basis of $L^2([0,T]\times\mathbb{R}^d)$ and $\{Z_j(\omega) : j \in \mathbb{N}\}$ be a family of independent Gaussian random variables with mean zero and variance one. Then

$$\dot{W}(t,x) := \sum_{j=1}^{\infty} f_j(t,x)Z_j(\omega) \tag{2.1.12}$$

is a space–time white noise, where the action is defined as

$$(\dot{W},\xi)_{\mathcal{D}'} = \sum_{j=1}^{\infty}(\xi,f_j)Z_j(\omega). \tag{2.1.13}$$

Remark 2.1.1.It is well know that the last action can be extended to $\xi \in L^2([0,T]\times\mathbb{R}^d)$ using Itô isometry.

Definition 2.1.7.The cylindrical Wiener process $\{W_t : L^2(\mathbb{R}^d) \to L^2(\Omega) : t \in [0,T]\}$

associated to $\dot{W}$ is given by

$$W_t(\varphi) := (\dot{W}, \varphi \cdot \mathbf{1}_{[0,t]}). \quad (2.1.14)$$

It is clear that $W_t(\varphi)$ is a Brownian motion with variance $t\|\varphi\|^2$ for each $\varphi \in L^2(\mathbb{R}^d)$.

Definition 2.1.8. We say that a random field $\{S(t,x) : t \in [0,T],\ x \in \mathbb{R}^d\}$ is a spatially dependent semimartingale if for each $x \in \mathbb{R}^d$, $\{S(t,x) : t \in [0,T]\}$ is a $\mathbb{R}^d$–valued semimartingale in relation to the same filtration $\{\mathcal{F}_t : t \in [0,T]\}$. If $S(t,x)$ is a C^∞–function of x and continuous in t almost surely (i.e., for almost all ω), it is called a smooth semimartingale.

Definition 2.1.9. We say that a sequence of smooth semimartingales $(W_{n,t})$ is a *weak approximation* to the cylindrical Wiener W_t if for all $\varphi \in L^2(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x) W_{n,t}(x) dx = W_t(\varphi), \quad (2.1.15)$$

where the convergence is in $C([0,T], L^2(\Omega))$.

Definition 2.1.10.A weak approximation (W_n) to the cylindrical Wiener process W is

good if for each $n \in \mathbb{N}$, $W_{n,t}(x)$ is a Brownian motion with quadratic variation

$$\langle W_{n,t}(x)\rangle_t = C_n \cdot t, \quad (2.1.16)$$

Let $\rho : \mathbb{R}^d \to [0,\infty)$ be an infinitely differentiable symmetric function with compact support such that $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. We will consider the mollifiers $\rho_n(x) = n^d \rho(nx)$, with $n \in \mathbb{N}$.

Definition 2.1.11. The regularization by ρ of the space-time white noise $\dot{W}$, denoted by $\dot{W}_{\rho_n}$, are defined to be

$$\dot{W}_{\rho_n}(t,x) = \rho_n * \dot{W}(t,x). \quad (2.1.17)$$

Remark 2.3.2. Note that $\dot{W}_{\rho_n}$ is white in time and colored in space, in fact we have

that $\dot{W}_{\rho_n}(t,x)$ is a distribution valued Gaussian process with mean zero and covariance,

$$\mathbb{E}(\dot{W}_{\rho_n}(t,x)\dot{W}_{\rho_n}(s,y)) = \delta(t-s)h_n(x-y) \quad (2.1.18)$$

where $h_n : \mathbb{R}^d \to \mathbb{R}$ is given by

$$h_n(z) = \int_{\mathbb{R}^d} \rho_n(u)\rho_n(u+z)du. \quad (2.1.19)$$

In terms of the expansion (2.1.13) we have that

$$\dot{W}_{\rho_n}(t,x) := \sum_{j=1}^{\infty} \rho_n * f_j(t,x) Z_j. \quad (2.1.20)$$

The mollified cylindrical Wiener process $W_{\rho_n,t}(x)$ associated with the space-time white

noise $\dot{W}(t,x)$ is defined by

$$W_{\rho_n,t}(x) := W_t(\rho_n(x - \bullet)). \quad (2.1.21)$$

The distributional time derivative of $W_{\rho_n,t}(x)$ is $\dot{W}_{\rho_n}(t,x)$. We have that $W_{\rho_n}(x)$ is a Brownian motion with quadratic variation,

$$\langle W_{\rho_n}(x)\rangle_t = \|\rho\|^2 n^d \cdot t. \quad (2.1.22)$$

Proposition 2.1.1.$(W_{\rho_n}(x))$ is a good weak approximation to the cylindrical Wiener process W.

2.2.Classical Chapman-Kolmogorov equation.

From this point on, we consider Markov processes. Eq.(2.1.8) fully defines a Markov process, but it does not say anything about the probability density function p. The ChapmanKolmogorov Equation (CKE) states the property that the function p must satisfy to describe a Markov process. To derive the CKE, we proceed as follows. Consider two values x_1 and x_3 of the random variable $X(t) : \Omega \subset \mathbb{R} \to \mathbb{R}$, measured at times t_1 and t_3 with $t_1 < t_3$, then from Eq.(2.1.7) we obtain

$$p(x_1,t_1;x_3,t_3) = p(x_3,t_3|x_1,t_1)p(x_1,t_1). \tag{2.2.1}$$

Integrating over x_1, we define the marginal density

$$p(x_3,t_3) := \int_\Omega dx_1 p(x_1,t_1;x_3,t_3) = \int_\Omega dx_1 p(x_3,t_3|x_1,t_1)p(x_1,t_1). \tag{2.2.2}$$

Consider now an intermediate point x_2, then the joint probability is

$$p(x_1,t_1;x_2,t_2;x_3,t_3) = p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)p(x_1,t_1). \tag{2.2.3}$$

We integrate over x_2 and applying the definition in Eq.(2.1.8), we obtain

$$\int_\Omega dx_2 p(x_1,t_1;x_2,t_2;x_3,t_3) = \int_\Omega dx_2 p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)p(x_1,t_1) \tag{2.2.4}$$

$$p(x_1,t_1;x_3,t_3) = p(x_1,t_1)\int_\Omega dx_2 p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)p(x_1,t_1;x_3,t_3)p(x_1,t_1) \tag{2.2.5}$$

Thus

$$\frac{p(x_1,t_1;x_3,t_3)}{p(x_1,t_1)} = \int_\Omega dx_2 p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1). \tag{2.2.6}$$

The left-hand side of Eq.(2.2.6) is by definition a conditional probability, thus we obtain the Chapman-Kolmogorov equation

$$p(x_3,t_3|x_1,t_1) = \int_\Omega dx_2 p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1). \tag{2.2.7}$$

The conditional probability $p(x_3,t_3|x_1,t_1)$ defined in Eq.(2.2.7) satisfies the normalization condition

$$\int_\Omega dx_3\, p(x_3,t_3|x_1,t_1) = 1. \tag{2.2.8}$$

If $t_3 \to t_1$, then

$$p(x_3,t_3|x_1,t_1) = \delta(x_3 - x_1). \tag{2.2.9}$$

The CKE of the Brownian motion has the explicit form

$$p(x_3,t_3|x_1,t_1) = \frac{1}{\sqrt{4\pi D(t_3-t_1)}}\exp\left(-\frac{(x_3-x_1)^2}{4D(t_3-t_1)}\right) \tag{2.2.10}$$

2.3.1.The Path Integral for a Markov Stochastic Process

Here we consider one-dimensional Markovian processes describable through Langevin or Fokker-Planck equations

$$\dot{q} = f(q,t) + g(q,t)\eta(t), \tag{2.3.1}$$

where $f(q,t)$ and $g(q,t)$ are known (smooth) functions, and $\eta(t)$ is a Gaussian white noise with zero mean and δ-correlated.

As has been discussed in the section 2.2, $P(q,t|q',t')$ fulfills the Chapman-Kolmogorov equation $(t_1 < t_2 < t_3)$

$$P(q_3,t_3|q_1,t_1) = \int_{-\infty}^{\infty} dq_2 P(q_3,t_3|q_2,t_2)P(q_2,t_2|q_1,t_1). \qquad (2.3.2)$$

Such an equation allows, by making a partition of the time interval in N steps: $t_0 < t_1 <\ldots< t_f$, with $t_j = t_0 + j(t_f - t_0)/N$, to obtain a path-dependent representation of the propagator. With the given partition, we reiterate (2.3.2) N-times and get

$$P(q_f,t_f|q_0,t_0) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dq_1 dq_2\ldots dq_{N-1} P(q_f,t_f|q_{N-1},t_{N-1})\ldots \\ \ldots P(q_2,t_2|q_1,t_1)P(q_1,t_1|q_0,t_0). \qquad (2.3.3)$$

The last expression can be interpreted as an integration over all possible paths that the process could follow (corresponding to the different values of the sequence: propagator $P(q_{j+1},t_{j+1}|q_j,t_j)$ in order to find the more conventional representation of the integration over paths. Note that the probability that at a given time t, the process takes
a value between a and b is given by

$$P(a < q(t) < b) = \int_a^b dq P(q,t|q_0,t_0) \qquad (2.3.4)$$

Likewise, the probability that the process, starting at $q = q_0$ at $t = t_0$, has a value between a_1 and b_1 at t_1, between a_2 and b_2 at $t_2,\ldots,$ between a_{N-1} and b_{N-1} at t_{N-1} (with $a_j < b_j$ and $t_j < t_{j+1}$), and reaching point q_N at t_N, will be given by

$$P(q_f,t_f|q_0,t_0) = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{N-1}}^{b_{N-1}} dq_1 dq_2\ldots dq_{N-1} P(q_f,t_f|q_{N-1},t_{N-1}) \times\ldots \\ \ldots\times P(q_2,t_2|q_1,t_1)P(q_1,t_1|q_0,t_0). \qquad (2.3.5)$$

We define now the conditional probability $P(q_f,t_f|q_0,t_0;\delta_1,\delta_2)$ by

$$P(q_f,t_f|q_0,t_0;\delta_1,\delta_2) = \\ \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_{N-1}}^{b_{N-1}} dq_1 dq_2\ldots dq_{N-1}\theta\left(\sup_{0\le n\le N-1}|q_n|;\delta_1\right)\theta\left(\sum_{j=0}^{j=N} q_j^2(t_{j+1}-t_j);\delta_2\right)\times \\ \times P(q_f,t_f|q_{N-1},t_{N-1})\ldots P(q_2,t_2|q_1,t_1)P(q_1,t_1|q_0,t_0), \qquad (2.3.6)$$

where $\theta(x,\delta) = 1$if $|x| \le \delta$ and $\theta(x,\delta) = 0$ if $|x| > \delta.$ If we increase the number of time slices within the time partition where the intervals (a_j,b_j) are specified, and at the same time take the limit $b_j - a_j \to 0,$ the trajectory is defined with higher and higher precision. Clearly, a requisite is that the trajectories be continuous. This happens in particular for the Wiener process.With all this in mind (2.3.3) can be interpreted as an integration over all the paths that the process could follow corresponding to the different values of the sequence $q_0,q_1,q_2,\ldots..,q_N = q_f$ such that the inequalities

(2.3.7) hold

$$\sup_{0\le n\le N-1} |q_n| \le \delta_1 \text{ and } \sum_{j=0}^{j=N} q_j^2 (t_{j+1} - t_j) \le \delta_2. \tag{2.3.7}$$

For the Wiener process we have that

$$P(W_2, t_2 | W_1, t_1; \delta_1, \delta_2) = \frac{1}{\sqrt{2\pi D(t_2 - t_1)}} \exp\left[-\frac{(W_2 - W_1)^2}{2D(t_2 - t_1)} \right]. \tag{2.3.8}$$

By substituting (2.3.8) into (2.3.6) we get

$$\prod_{j=1}^{N} \frac{dW_j}{\sqrt{4\pi\varepsilon D}} \theta\left(\sup_{1\le j\le N} |W_j|; \delta_1 \right) \theta\left(\sum_{j=1}^{j=N} \varepsilon W_j^2; \delta_2 \right) \exp\left[-\frac{1}{4D\varepsilon} \sum_{j=1}^{N} (W_j - W_{j-1})^2 \right] \tag{2.3.9}$$

which is the desired probability of following a given path ander conditions (2.3.7). When $\varepsilon \to 0$ and $N \to \infty$, we can write the exponential in (2.3.9) in the continuous limit as

$$\exp\left[-\frac{1}{4D} \int_{t_0}^{t} d\tau \left(\frac{dW(\tau)}{d\tau} \right)^2 \right]. \tag{2.3.10}$$

If we integrate the expression in (2.3.5) over all the intermediate points (which is equivalent to a sum over all the possible paths), as all the integrands are Gaussian, and the convolution of two Gaussian is again a Gaussian, we recover the result of (2.3.8) for the probability density of the Wiener process. Hence, we have expressed the probability density as a Wiener path integral

$$P(W, t | W_0, t_0) = \int\limits_{\substack{W(t)=W \\ W(0)=W_0}} D[W(\tau)] \exp\left[-\frac{1}{4D} \int_{t_0}^{t} d\tau \left(\frac{dW(\tau)}{d\tau} \right)^2 \right]. \tag{2.3.11}$$

Thus we have expressed the conditional probability density as a Wiener path integral

$$P(W, t | W_0, t_0; \delta_1, \delta_2) = \int D[W(\tau)] \theta\left(\sup_{t_0 \le \tau \le t} W(\tau); \delta_1 \right) \theta\left(\int_{t_0}^{t} W^2(\tau) d\tau; \delta_2 \right) \exp\left[-\frac{1}{4D} \int_{t_0}^{t} d\tau \left(\frac{dW(\tau)}{d\tau} \right)^2 \right], \tag{2.3.12}$$

where the expression inside the integral represents the continuous version of the integral of (2.3.6), over all required values of the intermediate points $\{W_j\}$.
We revrite now RHS of (2.3.12) of the form

$$P(W, t | W_0, t_0; \delta_1, \delta_2) = \int D[W(\tau)] \times \exp\left[-\frac{1}{4D} \int_{t_0}^{t} \left(\frac{dW(\tau)}{d\tau} \right)^2 d\tau + \ln\theta\left(\sup_{t_0 \le \tau \le t} W(\tau); \delta_1 \right) + \ln\theta\left(\int_{t_0}^{t} W^2(\tau) d\tau; \delta_2 \right) \right]. \tag{2.3.13}$$

We introduce now generalized conditional probability density as a Colombeau-Wiener path integral

$$(P_\epsilon(W,t|W_0,t_0;\delta_1,\delta_2))_{\epsilon\in(0,1]} = \Bigg(\int D[W(\tau)] \times$$

$$\exp\left[-\frac{1}{4\epsilon}\int_{t_0}^{t}\left(\frac{dW(\tau)}{d\tau}\right)^2 d\tau + \ln\theta\left(\sup_{t_0\le\tau\le t} W(\tau);\delta_1\right) + \ln\theta\left(\int_{t_0}^{t} W^2(\tau)d\tau;\delta_2\right)\right]\Bigg)_{\epsilon\in(0,1]} . \qquad (2.3.14)$$

2.4.The Path Integral for a General Markov Process

We start by writing the discrete version of the Langevin equation given by (2.4.1) (in order to simplify the notation we adopt $g(q,t) = 1$ and $f(q,t)$ to be independent of t)

$$q_{j+1} - q_j \simeq \{\alpha f(q_{j+1}) + (1-\alpha)f(q_j)\}\varepsilon + [W_{j+1} - W_j], \qquad (2.4.1)$$

where $\varepsilon = t_{j+1} - t_j = (t_f - t_0)/N,$ and $W_j = W(t_j)$ is the Wiener process (just formally, $dW(t) \simeq \eta(t)dt$). The parameter $\alpha(0 \le \alpha \le 1)$ is arbitrary,the most usual choices being $\alpha = 0, 1/2, 1,$ corresponding to the prepoint, midpoint and endpoint discretization, respectively. According to the previous results, the probability that

$$\begin{gathered} W(t_0) = 0; W_1(t_1) < W_1 + dW_1;\ldots; \\ a.s.\, W_N(t_N) < W_N + dW_N, \\ \sum_{j=1}^{j=N} \varepsilon W_j^2 \le \delta_2 \end{gathered} \qquad (2.4.2)$$

is given by

$$P\Big(\{W_{j-1}\}_{j=1}^{N}\Big) = \prod_{j=1}^{N}\left(\frac{dW_j}{\sqrt{4\pi\varepsilon D}}\right)\exp\left[-\frac{1}{4D\varepsilon}\sum_{j=1}^{N}(W_j - W_{j-1})^2\right]. \qquad (2.4.3)$$

As our interest is to have the corresponding conditional probability in q-space, we need to transform the probability given in the last equation (2.4.3). As is well known, to do this

we need the Jacobian of the transformation connecting both sets of stochastic variables $\{W_j\} \to \{q_j\}$. To find it we write Eq.(2.4.1) as

$$W_j = q_j - q_{j-1} - \{\alpha f(q_j) + (1-\alpha)f(q_{j-1})\}\varepsilon + W_{j-1}. \qquad (2.4.4)$$

The Jacobian is given by

$$\mathbf{J} = \det\left(\frac{\partial W_j}{\partial q_k}\right) = \prod_{j=1}^{N}\left(1 - \varepsilon\alpha\frac{df(q_j)}{dq_j}\right). \qquad (2.4.5)$$

For $\varepsilon \to 0, N \to \infty,$ it can be approximated as

$$\mathbf{J} \approx \exp\left(-\varepsilon\alpha\sum_j \frac{df(q_j)}{dq_j}\right). \qquad (2.4.6)$$

Now, remembering that $P(\{q_j\}) = \mathbf{J}P(\{W_j\})$, and taking into account that the conditional probability $P(q,t|q_0,t_0;\delta_1,\delta_2)$ is given as a sum over all the possible paths such that the inequalities (2.4.2) hold, we get

$$P(q_f,t|q_0,t_0;\delta_1,\delta_2) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{4\pi\varepsilon D}}\right)^N dW_1 Wq_2\ldots dW_{N-1}\theta(|W_N|;\delta_1)\theta\left(\sum_{j=1}^{j=N}\varepsilon W_j^2;\delta_2\right)\times \times\delta(q_f-q_N)\exp\left[-\frac{1}{4D\varepsilon}\sum_{j=1}^{N}(W_j-W_{j-1})^2\right]. \tag{2.4.7}$$

Replacing (2.4.1) into (2.4.7) and going to the continuous limit $\varepsilon\to 0, N\to\infty$, the different terms in the exponentials, we get

$$\begin{gathered}
\lim_{\varepsilon\to 0,N\to\infty}\theta(|q_N|;\delta_1) = \theta(|q(t)|;\delta_1),\\
\lim_{\varepsilon\to 0,N\to\infty}\theta\left(\sum_{j=1}^{N}\varepsilon|q_j|;\delta_2\right) = \theta\left(\int_{t_0}^{t}q^2(\tau)d\tau;\delta_2\right),\\
\lim_{\varepsilon\to 0,N\to\infty}\sum_{j=1}^{N}\varepsilon f(q_j) = \int_{t_0}^{t}f(q(\tau))d\tau,\\
\lim_{\varepsilon\to 0,N\to\infty}\left(\varepsilon\alpha\sum_{j=1}^{N}\frac{df(q_j)}{dq_j}\right) = \alpha\int_{t_0}^{t}\frac{df(q(\tau))}{dq}d\tau,\\
\lim_{\varepsilon\to 0,N\to\infty}\left(\frac{\varepsilon}{2}\sum_{j=1}^{N}(\alpha f(q_{j+1})+(1-\alpha)f(q_j))^2\right) = \frac{1}{2}\int_{t_0}^{t}f^2(q(\tau))d\tau,\\
\lim_{\varepsilon\to 0,N\to\infty}\left(\frac{\varepsilon}{2}\sum_{j=1}^{N}\left(\frac{q_{j+1}-q_j}{\varepsilon}\right)^2\right) = \frac{1}{2}\int_{t_0}^{t}[\dot q(\tau)]^2d\tau,\\
\lim_{\varepsilon\to 0,N\to\infty}\left(\sum_{j=1}^{N}(q_{j+1}-q_j)(\alpha f(q_{j+1})+(1-\alpha)f(q_j))\right) = \int_{t_0}^{t}f(q(\tau))d\tau.
\end{gathered} \tag{2.4.8}$$

$$P_\epsilon(q_f,t|q_0,t_0;\delta_1,\delta_2) = \int D[q(\tau)]\exp\left[-\frac{1}{D}\int_{t_0}^{t}d\tau\mathcal{L}(\dot q(\tau),q(\tau),\tau,t)\right] \tag{2.4.9}$$

where

$$\mathcal{L}[\dot q(\tau),q(\tau),t,t_0;\delta_1,\delta_2] = (\dot q(\tau)-f(q(\tau),\tau))+D\frac{df(q(\tau),\tau)}{dq}+\ln\theta(|q(t)|;\delta_1) +\ln\theta\left(\int_{t_0}^{t}q^2(\tau)d\tau;\delta_2\right) \tag{2.4.10}$$

is the generalized Lagrangian.

2.5. Generalized random variables and generalized double stochastic processes

Definition 2.5.1.. Let $\Xi(\Sigma,\mathbb{R})$ be a set of the all $\mathbb{R}$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega, \mathcal{F}, \widetilde{\mathbf{P}}\right)$, see sect. 3, def.3.2.

Definition 2.5.2. Let $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega, \mathcal{F}, \widetilde{\mathbf{P}}\right)$, i.e.,
$q_{1,2}(\omega,\omega') : \Omega \to \boldsymbol{\Omega}' = \boldsymbol{\Xi}(\boldsymbol{\Sigma},\mathbb{R})$. Assume that a.s.: $-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$.
Let $\widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ be a set of the all $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega, \mathcal{F}, \widetilde{\mathbf{P}}\right)$. Closed random interval
$[q_1(\omega,\omega'), q_2(\omega,\omega')]$
that is a subset $[q_1(\omega,\omega'), q_2(\omega,\omega')] \subset \widetilde{\Xi}(\Omega',\Sigma,\mathbb{R})$ such that a.s.: $-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$ and

$$\forall \omega \{q(\omega,\omega') \in [q_1(\omega,\omega'), q_2(\omega,\omega')] \Leftrightarrow \left\{q(\omega,\omega') \in \widetilde{\Xi}(\Omega',\Sigma,\mathbb{R}) \wedge a.s.: (q_1(\omega,\omega') < q(\omega,\omega') < q_2(\omega,\omega'))\right\}\}. \tag{2.5.1}$$

Notation 2.5.1. Assume that a.s.: $q_1(\omega,\omega') < q_2(\omega,\omega')$. Then we will write:

$$q_1(\omega,\omega') \prec q_2(\omega,\omega').$$

Assume that a.s. $q_1(\omega) \leq q_2(\omega)$. Then we will write:

$$q_1(\omega,\omega') \preccurlyeq q_2(\omega,\omega').$$

Definition 2.5.3. Let E be a separable complete metric space and let Σ be its Borel σ-algebra. <u>*Generalized*</u> random measure it is a function $\mu : \Sigma \to \Xi(\Sigma,\mathbb{R})$, that satisfies:

(1) If $E_1 \subset E_2$, then a.s.: $\mu(E_1) < \mu(E_2)$.

(2) If $E_n \in \mathcal{S}$, $n = 1,2,\ldots$ and $E_i \cap E_j = \varnothing$ $(i \neq j)$, then a.s.: $\mu(\cup_n E_n) = \sum_n \mu(E_n)$.

If we further impose a condition $\mu(\Omega) = 1$, then the generalized random measure so defined is called the generalized random probability measure and is usually denoted by $\widetilde{\mathbf{P}}$.

Definition 2.5.4. The tuple $(\Omega, \mathcal{S}, \widetilde{\mathbf{P}})$ is then called the *generalized probability space*. Suppose $(\Omega, \mathcal{S})$ and $(\Omega', \mathcal{S}')$ are two measurable spaces. The function $X : \Omega \to \Omega'$ is called a <u>*generalized*</u> <u>*random element*</u>, if for every $A \in \mathcal{S}'$, $X^{-1}(A) \in \mathcal{S}$. If the first measurable space is equipped with a generalized random probability measure $\widetilde{\mathbf{P}}$, then the generalized random element $X : \Omega \to \Omega'$ induces a generalized random probability measure on the second space $(\Omega', \mathcal{S}')$ and given by $\mathcal{P} \equiv \widetilde{\mathbf{P}} \circ X^{-1}$, called the generalized random distribution of X.

Definition 2.5.5. If the second measurable space $(\Omega', \mathcal{S}')$ is taken to be
$(\Xi(\Sigma,\mathbb{R}), \mathfrak{B}_{\Xi(\Sigma,\mathbb{R})})$,
where $\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})}$ is the Borel σ-algebra on $\Xi(\Sigma,\mathbb{R})$ the function $X(\omega) : \Omega \to \Omega'$ is called a *generalized random variable*. For this case, the generalized random distribution of X is completely determined by the *random distribution function* defined as

$$F_{X(\omega)}(x(\omega,\omega')) = \widetilde{\mathbf{P}}(X(\omega) \preccurlyeq x(\omega,\omega')) \tag{2.5.2}$$

where $X \preccurlyeq x$ is defined as the set $\{X(\omega) \preccurlyeq x(\omega,\omega')\} = \{\omega \in \Omega \mid X(\omega) \preccurlyeq x(\omega,\omega')\}$.
The distribution function is continuously non-decreasing and gives the generalized random measure associated with $(x(\omega,\omega'), x(\omega') + d[x(\omega,\omega')]) \in \mathfrak{B}$ as
$d[F_{X(\omega)}(x(\omega,\omega')))]$.

Definition 2.5.6. A special case is when the distribution function can be written in the form

$$F(x(\omega,\omega')) = \int_{-\infty(\omega,\omega')}^{x(\omega,\omega')} p(x(\omega,\omega'))d[x(\omega,\omega')] \quad (2.5.3)$$

In this case, the function $p(x(\omega,\omega')) : \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R}) \to \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ is called the *random probability density function* of generalized random variable $X(\omega,\omega')$, and the generalized random measure associated with $d[x(\omega,\omega')]$ is $d[F_{X(\omega)}(x(\omega,\omega'))] = p(x(\omega,\omega'))d[x(\omega,\omega')]$.

Definition 2.5.7.Gaussian random distribution is a type of continuous random probability distribution for a real-valued generalized random variable. The general form of its random probability density function is

$$p(x(\omega,\omega')) = \frac{1}{\sqrt{2\pi(\omega,\omega')\sigma^2}} \exp\left[-\frac{(x(\omega,\omega')-\mu)^2}{2\sigma^2},\right] \quad (2.5.3)$$

where a.s.: $\pi(\omega,\omega') = \pi$

With the knowledge of the distribution function, one can define the expectation value of any function $h(X)$ as

$$\mathbf{E}[h(X(\omega,\omega'))] = \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} h(x(\omega,\omega'))d[F_{X(\omega)}(x(\omega,\omega'))]). \quad (2.5.4)$$

Definition 2.5.8. A double stochastic process is an indexed set of generalized random variables $X_t(\omega)$, $t \in T$, where T is called the index set and is usually taken (in the continuous case) to be $T = [0,\infty)$. In a physical setting, the index is time and a stochastic process can be interpreted as, for each $t \in T$, $X_t(\omega)$ picks an event $E \in \mathcal{S}$ with random probability $\widetilde{\mathbf{P}}(E)$, and returns $X_t(E)$.

Consider generalized probability space $(\Omega,\mathcal{F},\widetilde{\mathbf{P}})$ and an 1-dimensional time-dependent generalized random variable $X_t(\omega) : \Omega \subset \mathbb{R} \to \Xi(\Sigma,\mathbb{R})$, i.e. a $\Xi(\Sigma,\mathbb{R})$-valued function that maps elements of the sample space to $\Xi(\Sigma,\mathbb{R})$.

Assume now that the random variable is time-dependent: $X(t,\omega) : \Omega \subset \mathbb{R} \to \Omega' = \Xi(\Sigma,\mathbb{R})$, then, given a sequence of timesteps $t_1,t_2,\ldots,t_N$ with $t_1 < t_2 <\ldots< t_N$, we can write

$$\{X(t_1,\omega) = x_1(\omega,\omega'),X(t_2,\omega) = x_2(\omega,\omega'),\ldots,X(t_N,\omega) = x_N(\omega,\omega')\}, \quad (2.5.5)$$

where $x_1(\omega,\omega'),x_2(\omega,\omega'),\ldots,x_N(\omega,\omega') \in \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$.

The sequence defined in Eq.(2.5.5) is a *double stochastic process* if the joint random probability density$p_\omega(x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_N,t_N(\omega,\omega'))$,fully describes the system.

Depending on how the joint random probability density is defined, we can classify the double stochastic processes. Here, we consider two cases.

1.Purely double stochastic process or separable stochastic process. If successive values of $X(t)$ are statistically independent, then we the joint random probability density is written as

$$a.s.: \ p(x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_N(\omega,\omega'),t_N) = \prod_{i=1}^{N} p(x_i(\omega,\omega'),t_i). \quad (2.5.6)$$

The underlying idea is that the random probability of an event $x_i(\omega,\omega')$ occurring at a time t_i does not depend on the past and in no way determines the future. In terms of conditional random probabilities, we can write

$$a.s.: p(x_N(\omega,\omega'),t_N|x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_{N-1}(\omega,\omega'),t_{N-1}) \\ = p(x_N(\omega,\omega'),t_N). \tag{2.5.7}$$

2.A second example of a stochastic process is the Markov process, whose joint probability density is written as

$$p(x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_N(\omega,\omega'),t_N) = \\ = \prod_{i=2}^{N} p(x_i(\omega,\omega'),t_i|x_{i-1}(\omega,\omega'),t_{i-1})p(x_1(\omega,\omega'),t_1), \tag{2.5.8}$$

or in terms of conditional probabilities as

$$p(x_N(\omega,\omega'),t_N|x_1(\omega,\omega'),t_1,;x_2(\omega,\omega'),t_2,;\ldots x_{N-1}(\omega,\omega'),t_{N-1}) = \\ p(x_N(\omega,\omega'),t_N|x_{N-1}(\omega,\omega'),t_{N-1}), \tag{2.5.9}$$

i.e., double stochastic Markovian process is a process without memory, whose temporal evolution depends only on the present state, not on the past.

2.6.Double stochastic space-time white noise.

Definition 2.6.1. A distribution valued double stochastic Gaussian process with almost surely mean zero $\{\dot{W}(t,x;\omega,\omega') : t \in [0,T], x \in \mathbb{R}^d\}$ is a *double stochastic space-time white noise* if

$$\mathbb{E}(\dot{W}(t,x;\omega,\omega')\dot{W}(s,y;\omega,\omega')) = \delta(t-s)\delta(x-y). \tag{2.6.9}$$

More precisely :

Definition 2.6.2. We denote by $\mathcal{D}((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ the space of the infinitely differentiable functions with compact support in $(0,T)\times\mathbb{R}^d$ and values in $\mathbb{R}^d$.

1.For any $\xi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the random variable $\dot{W}(\xi)$ is double stochastic Gaussian variable with almost surely mean zero

$$\mathbb{E}(\dot{W}(\xi) = 0 \tag{2.6.10}$$

For any $\xi_1(t,x),\xi_2(t,x) \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ the double stochastic random variables $\dot{W}(\xi_1),\dot{W}(\xi_2)$ almost surely have covariance

- $$\mathbb{E}(\dot{W}(\xi_1)\dot{W}(\xi_2)) = \int_{[0,T]\times\mathbb{R}^d}\xi_1(t,x)\cdot\xi_2(t,x)dxdt. \tag{2.6.11}$$

It is not difficult to construct a space-time white noise. In fact, let $\{f_j : j \in \mathbb{N}\}$ be a complete orthonormal basis of $L^2([0,T]\times\mathbb{R}^d)$ and $\{Z_j(\omega,\omega') : j \in \mathbb{N}\}$ be a family of independent Gaussian generalized random variables with mean zero and variance one. Then

$$\dot{W}(t,x;\omega,\omega') := \sum_{j=1}^{\infty} f_j(t,x)Z_j(\omega,\omega') \tag{2.6.12}$$

is a space-time white noise, where the action is defined as

$$(\dot{W},\xi)_{\mathcal{D}'} = \sum_{j=1}^{\infty}(\xi,f_j)Z_j(\omega,\omega'). \tag{2.6.13}$$

Remark 2.6.1. It is well know that the last action can be extended to $\xi \in L^2([0,T]\times\mathbb{R}^d)$ using Itô isometry.

Let $\rho : \mathbb{R}^d \to [0,\infty)$ be an infinitely differentiable symmetric function with compact support such that $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. We will consider the mollifiers $\rho_n(x) = n^d \rho(nx)$, with $n \in \mathbb{N}$.

Definition 2.6.3. The regularization by ρ of the space-time white noise $\dot{W}(t,x;\omega,\omega')$, denoted by $\dot{W}_{\rho_n}$, are defined to be

$$\dot{W}_{\rho_n}(t,x;\omega,\omega') := \rho_n * \dot{W}(t,x;\omega,\omega'). \tag{2.6.14}$$

Remark 2.6.2. Note that $\dot{W}_{\rho_n}$ is white in time and colored in space, in fact we have that $\dot{W}_{\rho_n}(t,x)$ is a distribution valued Gaussian process with mean zero and covariance,

$$\mathbb{E}(\dot{W}_{\rho_n}(t,x;\omega,\omega')\dot{W}_{\rho_n}(s,y;\omega,\omega')) = \delta(t-s)h_n(x-y) \tag{2.6.15}$$

where $h_n : \mathbb{R}^d \to \mathbb{R}$ is given by

$$h_n(z) = \int_{\mathbb{R}^d} \rho_n(u)\rho_n(u+z)du. \tag{2.6.16}$$

In terms of the expansion (2.6.13) we have that

$$\dot{W}_{\rho_n}(t,x;\omega,\omega') := \sum_{j=1}^{\infty} \rho_n * f_j(t,x)Z_j(\omega,\omega'). \tag{2.6.17}$$

The mollified cylindrical Wiener process $W_{\rho_n,t}(x)$ associated with the space-time white noise $\dot{W}(t,x)$ is defined by

$$W_{\rho_n,t}(x;\omega,\omega') := W_t(\rho_n(x-\bullet)). \tag{2.6.18}$$

The distributional time derivative of $W_{\rho_n,t}(x;\omega,\omega')$ is $\dot{W}_{\rho_n}(t,x;\omega,\omega')$. We have that $W_{\rho_n}(x;\omega,\omega')$ is a double stochastic Brownian motion with quadratic variation,

$$\langle W_{\rho_n}(x;\omega,\omega')\rangle_t = \|\rho\|^2 n^d \bullet t. \tag{2.6.19}$$

Proposition 2.6.1.$(W_{\rho_n}(x))$ is a good weak approximation to the cylindrical double stochastic Wiener process W.

3.1.Integration over random interval.

Definition 3.1.1. Let $q_1(\omega)$ and $q_2(\omega)$ are $\mathbb{R}$- valued random variables defined on a probability space $\Sigma = (\mathbf{\Omega},\mathcal{F},\mathbf{P})$,i.e., $q_{1,2}(\omega) : \mathbf{\Omega} \to \mathbb{R}$.Assume that a.s.$-\infty < q_1(\omega) < q_2(\omega) < \infty$.

Let $\Xi(\Sigma,\mathbb{R})$ be a set of the all $\mathbb{R}$- valued random variables defined on a probability space

$\Sigma = (\mathbf{\Omega},\mathcal{F},\mathbf{P})$.Closed random interval $[q_1(\omega),q_2(\omega)]$ that is a subset $[q_1(\omega),q_2(\omega)] \subset \Xi(\Sigma,\mathbb{R})$ such that $-\infty < q_1(\omega) < q_2(\omega) < \infty$ and

$$\forall q(\omega)\{q(\omega) \in [q_1(\omega),q_2(\omega)] \Leftrightarrow \{q(\omega) \in \Xi(\Sigma,\mathbb{R}) | a.s.(q_1(\omega) < q(\omega) < q_2(\omega))\}\} \tag{3.1.1}$$

Definition 3.1.2. The lengths $l([q_1(\omega),q_2(\omega)])$ of the random interval $[q_1(\omega),q_2(\omega)]$ is defined by

$$l([q_1(\omega),q_2(\omega)]) = \mathbf{ess}\sup q_3(\omega), \tag{3.1.2}$$

where $q_3(\omega) = q_2(\omega) - q_1(\omega)$.

Notation 3.1.2. Assume that a.s.$q_1(\omega) < q_2(\omega)$.Then we will write: $q_1(\omega) \prec q_2(\omega)$.
Assume that a.s.$q_1(\omega) \leq q_2(\omega)$.Then we will write: $q_1(\omega) \preccurlyeq q_2(\omega)$.

Definition 3.1.3. A partition P of a closed random interval $[q_1(\omega),q_2(\omega)]$ is a finite system of random variables $x_0(\omega),\ldots,x_n(\omega)$ such that
$q_1(\omega) = x_0(\omega) \prec x_1(\omega) \prec \ldots x_{n-1}(\omega) \prec x_n(\omega) = q_2(\omega)$.
The closed random intervals $[x_{i-1}(\omega),x_i(\omega)]$, $(i = 1,\ldots,n)$ are called the intervals of the partition P.

Definition 3.1.4. The largest of the lengths of the intervals of the partition P, denoted $\lambda(P)$, is called the *mesh* of the partition.

Definition 3.1.5. We speak of a partition with distinguished points (P,ξ) on the closed random interval $[q_1(\omega),q_2(\omega)]$ if we have a partition P of $[q_1(\omega),q_2(\omega)]$ and a point $\xi_1(\omega) \in [x_{i-1}(\omega),x_i(\omega)]$ has been chosen in each of the intervals of the partition $[x_{i-1}(\omega),x_i(\omega)]$, $(i = 1,\ldots,n)$.

We denote the set of points $(\xi_1(\omega),\ldots,\xi_n(\omega))$ by the single letter $\xi(\omega)$.

Definition 3.1.6. In the set $\widetilde{P}$ of partitions with distinguished points on a given random interval $[q_1(\omega),q_2(\omega)]$, we consider the following base $\mathbf{B} = \{B_d\}$. The element $B_d, d > 0$, of the base B consists of all partitions with distinguished points (P,ξ) on $[q_1(\omega),q_2(\omega)]$ for which $\lambda(P) < d$.

Proposition 3.1.1. Let us verify that $\{B_d\}, d > 0$ is actually a base in $\widetilde{P}$.

Proof. First $B_d \neq \varnothing$. In fact, for any number $d > 0$, it is obvious that there exists a partition P of $[q_1(\omega),q_2(\omega)]$ with mesh $\lambda(P) < d$ (for example, a partition into n congruent closed random intervals). But then there also exists a partition $(P,\xi(\omega))$ with distinguished points for which $\lambda(P) < d$.

Second, if $d_1 > 0, d_2 > 0$, and $d = \min\{d_1,d_2\}$, it is obvious that $B_{d_1} \cap B_{d_2} = B_d \in \mathbf{B}$. Hence $\mathbf{B} = \{B_d\}$ is indeed a base in $\widetilde{P}$.

Definition 3.1.7. (**Random Riemann Sums**) (i) If a function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega),q_2(\omega)]$ and $(P,\xi(\omega))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega)] = \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega), \tag{3.1.3}$$

where $\Delta x_i(\omega) = x_i(\omega) - x_{i-1}(\omega)$, is the random Riemann sum of the function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega))$ with distinguished points on $[q_1(\omega),q_2(\omega)]$. Thus, when the function f is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega)]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\widetilde{P}$ of all partitions $p = (P,\xi(\omega))$ with distinguished points on the closed interval $[q_1(\omega),q_2(\omega)]$.

Since there is a base $\mathbf{B}$ in $\widetilde{P}$, one can ask about the limit of the function $\Phi(p(\omega))$ over that base.

(ii) If a function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega),q_2(\omega)]$ and $(P,\xi(\omega))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega)] = \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega), \tag{3.1.3$'$}$$

where $\Delta x_i(\omega) = x_i(\omega) - x_{i-1}(\omega)$, is the random Riemann sum of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega))$ with distinguished points on $[q_1(\omega),q_2(\omega)]$. Thus, when the function $f : \Omega \times [q_1(\omega),q_2(\omega)] \to \Xi(\Sigma,\mathbb{R})$ is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega)]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\widetilde{P}$ of all partitions $p = (P,\xi(\omega))$ with distinguished points on the closed interval $[q_1(\omega),q_2(\omega)]$.

Definition 3.1.8. (**Riemann Integral on a random interval**)

(i) Let $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega),q_2(\omega)]$.

The random variable $I(\omega)$ is the Riemann integral of the function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega),q_2(\omega)]$ if for every $\varepsilon > 0$

there exists $\delta > 0$ such that

$$a.s.: \left| I(\omega) - \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega) \right| < \varepsilon \tag{3.1.4}$$

for any partition $(P, \xi(\omega))$ with distinguished points on $[q_1(\omega), q_2(\omega)]$ whose mesh $\lambda(P)$ is less than δ.

(ii) Let $f: \Omega \times \Xi(\Sigma, \mathbb{R}) \to \Xi(\Sigma, \mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega), q_2(\omega)]$ such that $f: \Omega \times [q_1(\omega), q_2(\omega)] \to \Xi(\Sigma, \mathbb{R})$.

The random variable $I(\omega)$ is the Riemann integral of the function $f: \Omega \times \Xi(\Sigma, \mathbb{R}) \to \Xi(\Sigma, \mathbb{R})$ on the closed random interval $[q_1(\omega), q_2(\omega)]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: \left| I(\omega) - \sum_{i=1}^{n} f(\omega, \xi_i(\omega))\Delta x_i(\omega) \right| < \varepsilon. \tag{3.1.4$'$}$$

Since the partitions $p = (P, \xi(\omega))$ for which $\lambda(P) < \delta$ form the element B_δ of the base $\mathbf{B}$ introduced above in the set $\widetilde{P}$ of partitions with distinguished points, Definition 3.1.8 is equivalent to the statement

$$a.s.: I(\omega) = \lim_{\mathbf{B}} \Phi(p(\omega)). \tag{3.1.5}$$

that is, the integral $I(\omega)$ is the limit over $\mathbf{B}$ of the Riemann sums of the function f corresponding to partitions with distinguished points on $[q_1(\omega), q_2(\omega)]$.

It is natural to denote the base $\mathbf{B}$ by $\lambda(P) \to 0$, and then the definition of the Riemann integral on a random interval can be rewritten as

$$\begin{aligned} a.s.: I(\omega) &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega), \\ a.s.: I(\omega) &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega, \xi_i(\omega))\Delta x_i(\omega). \end{aligned} \tag{3.1.6}$$

Notation 3.1.2. The integral of $f(x(\omega))$ $(f(\omega, x(\omega)))$ over $[q_1(\omega), q_2(\omega)]$ is denoted

$$\begin{aligned} &\int_{q_1(\omega)}^{q_2(\omega)} f(x(\omega))d[x(\omega)], \\ &\int_{q_1(\omega)}^{q_2(\omega)} f(\omega, x(\omega))d[x(\omega)], \end{aligned} \tag{3.1.7}$$

in which the random variables $q_1(\omega)$ and $q_2(\omega)$ are called respectively the lower and upper limits of integration. The function f is called the integrand, $f(x(\omega))d[x(\omega)]$ $(f(\omega, x(\omega))d[x(\omega)])$ is called the differential form, and $x(\omega)$ is the random variable of integration. Thus

$$\begin{aligned} a.s.: \int_{q_1(\omega)}^{q_2(\omega)} f(x(\omega))d[x(\omega)] &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i(\omega))\Delta x_i(\omega), \\ a.s.: \int_{q_1(\omega)}^{q_2(\omega)} f(\omega, x(\omega))d[x(\omega)] &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega, \xi_i(\omega))\Delta x_i(\omega). \end{aligned} \tag{3.1.8}$$

Definition 3.1.9. A function $f: \Omega \times \Xi(\Sigma, \mathbb{R}) \to \Xi(\Sigma, \mathbb{R})$ is Riemann integrable on the closed interval $[q_1(\omega), q_2(\omega)]$ if the limit of the Riemann sums in (3.1.8) exists $a.s.$ as

$\lambda(P) \to 0$ (that is, the Riemann integral of f is defined).

Notation 3.1.3. The set of Riemann-integrable functions on a closed random interval $[q_1(\omega), q_2(\omega)]$ will be denoted $\Re[q_1(\omega), q_2(\omega)]$

By the definition of the integral (Definition 2.6.8) and its reformulation in the forms (3.1.5) and (3.1.8), an integral is the limit of a certain special function $\Phi(p(\omega)) = \sigma[f, P, \xi(\omega)]$ the random Riemann sum, defined on the set $\widetilde{P}$ of partitions $p(\omega) = (P, \xi(\omega))$ with distinguished points on $[q_1(\omega), q_2(\omega)]$. This limit is taken with respect to the base $\mathbf{B}$ in $\widetilde{P}$ that we have denoted $\lambda(P) \to 0$.

Thus the integrability or nonintegrability of a function f on $[q_1(\omega), q_2(\omega)]$ depends on the existence of this limit. By the Cauchy criterion, this limit exists a.s., if and only if for every $\varepsilon > 0$ there exists an element $B_\delta \in \mathbf{B}$ in the base such that

$$a.s.: |\Phi(p'(\omega)) - \Phi(p''(\omega))| < \varepsilon \qquad (3.1.9)$$

for any two points $p'(\omega), p''(\omega)$ in B_δ. In more detailed notation, what has just been said means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: |\sigma[f, P, \xi(\omega)] - \sigma[f, P, \xi(\omega)]| < \varepsilon \qquad (3.1.10)$$

or, what is the same,

$$a.s.: \left|\sum_{i=1}^{n'} f(\omega, \xi_i'(\omega))\Delta x_i'(\omega) - \sum_{i=1}^{n''} f(\omega, \xi_i''(\omega))\Delta x_i''(\omega)\right| < \varepsilon \qquad (3.1.11)$$

for any partitions $(P', \xi'(\omega))$ and $(P'', \xi''(\omega))$ with distinguished points on the random interval $[q_1(\omega), q_2(\omega)]$ with $\lambda(P') < \delta$ and $\lambda(P'') < \delta$.

Proposition 3.1.2. A necessary condition for a function $f(\omega, x(\omega))$ defined on $\Omega \times [q_1(\omega), q_2(\omega)]$ to be Riemann integrable on $[q_1(\omega), q_2(\omega)]$ is that f be bounded a.s. on $\Omega \times [q_1(\omega), q_2(\omega)]$.

Proof. If f is not bounded a.s. on $[q_1(\omega), q_2(\omega)]$, then for any partition (P, ξ) of $[q_1(\omega), q_2(\omega)]$ the function f is unbounded on at least one of the intervals $[x_{i-1}(\omega), x_i(\omega)]$ of (P, ξ). This means that, by choosing the point $\xi_i(\omega) \in [x_{i-1}(\omega), x_i(\omega)]$ in different ways, we can make the quantity $|f(\omega, \xi_i(\omega))\Delta x_i(\omega)|$ a.s. as large as desired. But then the Riemann sum $\sum_{i=1}^{n} f(\omega, \xi_i(\omega))\Delta x_i(\omega)$ can also be made as large as desired in absolute value by changing only the point $\xi_i(\omega)$ in this interval.

We agree that when a partition P

$$q_1(\omega) = x_0(\omega) \prec x_1(\omega) \prec \ldots x_{n-1}(\omega) \prec x_n(\omega) = q_2(\omega).$$

is given on the closed random interval $[q_1(\omega), q_2(\omega)]$, we shall use the symbol $\Delta_i(\omega)$ to denote the interval $[x_{i-1}(\omega), x_i(\omega)]$ along with $\Delta x_i(\omega)$ as a notation for the difference $x_i(\omega) - x_{i-1}(\omega)$.

If a partition $P^\star$ of the closed random interval $[q_1(\omega), q_2(\omega)]$ is obtained from the partition P by the adjunction of new points to P, we call $P^\star$ a *refinement* of P.

When a refinement $P^\star$ of a partition P is constructed, some (perhaps all) of the closed random intervals $\Delta_i(\omega) = [x_{i-1}(\omega), x_i(\omega)]$ of the partition P themselves undergo partitioning: $x_{i-1}(\omega) = x_{i_0}(\omega) \prec \ldots \prec x_{i_{n_i}}(\omega) = x_i(\omega)$. In that connection, it will be useful for us to label the points of $P^\star$ by double indices. In the notation $x_{i_j}(\omega)$ the first index means that $x_{i_j}(\omega) \in \Delta_i(\omega)$, and the second index is the ordinal number of the point on the closed random interval $\Delta_i(\omega)$. It is now natural to set $\Delta x_{i_j}(\omega) = x_{i_j} - x_{i_{j-1}}$ and $\Delta_{i_j}(\omega)$. Thus $\Delta x_i(\omega) = \Delta x_{i_1}(\omega) + \ldots + \Delta x_{i_{n_i}}(\omega)$.

As an example of a partition that is a refinement of both the partition P' and P'' one can take $P^\star = P' \cup P''$, obtained as the union of the points of the two partitions P' and P''.
We recall finally that $\Omega(f(\omega,x(\omega)),E,\omega')$ denotes the oscillation of the function $f(\omega,x)$ on the random set $E(\omega)$, that is

$$\Omega(f(\omega,x(\omega)),E(\omega)) = \sup_{a.s.:\, x_1,(\omega)x_2(\omega)\in E(\omega)} |f(\omega,x_1(\omega)) - f(\omega,x_2(\omega))|. \qquad (3.1.12)$$

In particular, $\Omega(f(\omega,x(\omega)),\Delta_i(\omega))$ is the oscillation of $f(\omega,x(\omega))$ on the closed random interval $[x_{i-1}(\omega),x_i(\omega)]$.
This oscillation is necessarily a.s. finite if $f(\omega,x)$ is a.s. bounded function of variable x.
Proposition 3.1.3. A sufficient condition for a.s. bounded function $f(\omega,x)$ to be integrable on a closed random interval $[q_1(\omega),q_2(\omega)]$ is that for every $\varepsilon > 0$ there exist a number $\delta > 0$ such that

$$\sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) < \varepsilon \qquad (3.1.13)$$

for any partition P of $[q_1(\omega),q_2(\omega)]$ with mesh $\lambda(P) < \delta$.
Proof Let P be a partition of $[q_1(\omega),q_2(\omega)]$ and $P^\star$ a refinement of P. Let us estimate the difference between the random Riemann sums $\sigma(f,P^\star,\xi^\star) - \sigma(f,P,\xi)$. Using the notation introduced above, we can write

$$\begin{gathered}
|\sigma(f,P^\star,\xi^\star(\omega)) - \sigma(f,P,\xi(\omega))| = \\
\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega))\Delta x_{i_j}(\omega) - \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega)\right| = \\
\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega))\Delta x_{i_j}(\omega) - \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_{i_j}(\omega)\right| = \\
\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} [f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))]\Delta x_{i_j}(\omega)\right| \leq \\
\sum_{i=1}^{n}\sum_{j=1}^{n_i} |[f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))]|\Delta x_{i_j}(\omega) \leq \\
\sum_{i=1}^{n}\sum_{j=1}^{n_i} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_{i_j}(\omega) = \\
\sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega).
\end{gathered} \qquad (3.1.14)$$

In this computation we have used the relation $\Delta x_i(\omega) = \sum_{j=1}^{n_i} \Delta x_{i_j}(\omega)$ and the inequality

$$a.s.:\ |f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))| \leq \Omega(f(\omega,x(\omega)),\Delta_i(\omega)), \qquad (3.1.15)$$

which holds because $a.s.:\ \xi_{i_j}(\omega) \in \Delta_{i_j}(\omega) \subset \Delta_i(\omega)$ and $a.s.:\ \xi_i(\omega) \in \Delta_i(\omega)$.
It follows from the estimate for the difference of the random Riemann sums that if the function satisfies the sufficient condition given in the statement of Proposition 3.1.3, then for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$a.s.: |\sigma(f,P^{\star},\xi^{\star}(\omega)) - \sigma(f,P,\xi(\omega))| < \varepsilon/2 \quad (3.1.16)$$

for any partition P of $[q_1(\omega),q_2(\omega)]$ with mesh $\lambda(P) < \delta$, any refinement $P^{\star}$ of P, and any choice of the sets of distinguished points $\xi(\omega)$ and $\xi^{\star}(\omega)$.
Now if (P',ξ') and (P'',ξ'') are arbitrary partitions with distinguished points on $[q_1(\omega),q_2(\omega)]$ whose meshes satisfy $\lambda(P') < \delta$ and $\lambda(P'') < \delta$, then, by what has just been proved, the partition $P^{\star} = P' \cup P''$, which is a refinement of both of them,
must satisfy

$$\begin{aligned} a.s.&: |\sigma(f,P^{\star},\xi^{\star}(\omega)) - \sigma(f,P',\xi'(\omega))| < \varepsilon/2, \\ a.s.&: |\sigma(f,P^{\star},\xi^{\star}(\omega)) - \sigma(f,P'',\xi''(\omega))| < \varepsilon/2. \end{aligned} \quad (3.1.17)$$

It follows that

$$|\sigma(f,P',\xi'(\omega)) - \sigma(f,P'',\xi''(\omega))| < \varepsilon, \quad (3.1.18)$$

provided $\lambda(P') < \delta$ and $\lambda(P'') < \delta$. Therefore, by the Cauchy criterion, the limit of the random Riemann sums exists $a.s.$:

$$a.s.: \ \exists \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega,\xi_i(\omega))\Delta x_i(\omega), \quad (3.1.19)$$

that is $f \in \Re[q_1(\omega),q_2(\omega)]$.

Definition 3.1.10. (i) A function $f(\omega,x(\omega)) : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is a.s. continuous at point $x_0(\omega) \in [q_1(\omega),q_2(\omega)]$ if
1.$f(\omega,x_0(\omega))$ is defined, so that $x_0(\omega)$ is in the domain of $f(\omega,x(\omega))$.
2. $\lim_{x(\omega)\to x_0(\omega)} f(\omega,x(\omega))$ a.s. exists for $x(\omega)$ in the domain of $f(\omega,x(\omega))$.
3. $a.s.: \lim_{x(\omega)\to x_0(\omega)} f(\omega,x(\omega)) = f(\omega,x_0(\omega))$.
(ii) A function $f(\omega,x(\omega)) : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is a.s. continuouson on the closed random interval $[q_1(\omega),q_2(\omega)]$ if $f(\omega,x(\omega))$ is a.s. continuous at any point $x(\omega) \in [q_1(\omega),q_2(\omega)]$

Notation 3.1.4.The set of a.s. continuous functions on a closed random interval $[q_1(\omega),q_2(\omega)]$ will be denoted $C_{a.s.}[q_1(\omega),q_2(\omega)]$

Definition 3.1.11.

Corollary 3.1.1. $f(\omega,x(\omega)) \in C_{a.s.}[q_1(\omega),q_2(\omega)] \Rightarrow f(\omega,x(\omega)) \in \Re[q_1(\omega),q_2(\omega)]$, that is, every a.s. continuous function $f(\omega,x(\omega))$ on a closed random interval $[q_1(\omega),q_2(\omega)]$ is integrable on that closed random interval.

Proof. If a function is continuous on a closed random interval, it is a.s. bounded there, so that the necessary condition for integrability is satisfied in this case. But a.s. continuous function on a closed random interval $[q_1(\omega),q_2(\omega)]$ is uniformly a.s. continuous on that interval. Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $a.s.: \Omega(f(\omega,x(\omega)),\Delta(\omega)) < \dfrac{\varepsilon}{q_2(\omega) - q_1(\omega)}$ on any closed interval $\Delta(\omega) \subset [q_1(\omega),q_2(\omega)]$ of length less than δ. Then for any partition P with mesh $\lambda(P) < \delta$ we have that

$$a.s.: \sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega)) < \frac{\varepsilon}{q_2(\omega)-q_1(\omega)} \sum_{i=1}^{n} \Delta_i(\omega) = \\ = \frac{\varepsilon[q_2(\omega)-q_1(\omega)]}{q_2(\omega)-q_1(\omega)} = \varepsilon. \tag{3.1.20}$$

By Proposition 3.1.3, we can now conclude that $f \in \Re[q_1(\omega),q_2(\omega)]$.

Corollary 3.1.2. If a.s. bounded function f on a closed random interval $[q_1(\omega),q_2(\omega)]$ is a.s. continuous everywhere except at a finite set of random points $x_i(\omega), i = 1,\ldots,k$ then $f \in \Re[q_1(\omega),q_2(\omega)]$.

Proof. Let $a.s.: \Omega(f(\omega,x(\omega)),[q_1(\omega),q_2(\omega)]) < C < oo$, and suppose f has k points of discontinuity on $[q_1(\omega),q_2(\omega)]$. We shall verify that the sufficient condition for integrability of the function f is satisfied.

For a given $\varepsilon > 0$ we choose the number $\delta_1 = \frac{\varepsilon}{8C \times k}$ and construct the δ_1-neighborhood of each of the k points of a.s. discontinuity of f on $[q_1(\omega),q_2(\omega)]$. The complement of the union of these neighborhoods in $[q_1(\omega),q_2(\omega)]$ consists of a finite number of closed random intervals, on each of which f is a.s. continuous and hence a.s. uniformly continuous. Since the number of these intervals is finite, given $\varepsilon > 0$ there exists $\delta_2 > 0$ such that on each interval $\Delta_i(\omega) = [x_i(\omega),x_{i-1}(\omega)], i = 2,\ldots,k$ whose length $l[\Delta_i(\omega)] = x_i(\omega) - x_{i-1}(\omega)$ a.s. is less than δ_2 and which is entirely contained in one of the closed random intervals just mentioned, on which f is a.s. continuous, we have

$$a.s.: \Omega(f(\omega,x(\omega)),\Delta_i(\omega)) < \frac{\varepsilon}{2[q_2(\omega)-q_1(\omega)]}$$

We now choose $\delta = \min\{\delta_1,\delta_2\}$. Let P be an arbitrary partition of $[q_1(\omega),q_2(\omega)]$ for which $\lambda(P) < \delta$. We break the sum $\sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))$ corresponding to the partition P into two parts:

$$\sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) = \sum{}' \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) + \\ \sum{}'' \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega). \tag{3.1.21}$$

The sum $\sum'$ contains the terms corresponding to random intervals $\Delta_i(\omega)$ of the partition having no points in common with any of the δ_1-neighborhoods of the points of discontinuity. For these random intervals $\Delta_i(\omega)$ we have

$$a.s.: \Omega(f(\omega,x(\omega)),\Delta_i(\omega)) < \frac{\varepsilon}{2(q_2(\omega)-q_1(\omega))}$$

and so

$$a.s.: \sum{}' \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) < \frac{\varepsilon}{2(q_2(\omega)-q_1(\omega))} \sum{}' \Delta x_i(\omega) \leq \\ \frac{\varepsilon}{2(q_2(\omega)-q_1(\omega))}(q_2(\omega)-q_1(\omega)) = \frac{\varepsilon}{2}. \tag{3.1.22}$$

The sum of the lengths of the remaining intervals of the partition P, as one can easily see, is at most $(\delta + 2\delta_1 + \delta) \leq \frac{4\varepsilon}{8C \times k}k = \varepsilon/2C$, and therefore

$$a.s.: \sum{}'' \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) \leq C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2}. \tag{3.1.23}$$

Thus we find that for $\lambda(P) < \delta$:

$$a.s.: \sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),\Delta_i(\omega))\Delta x_i(\omega) < \varepsilon. \quad (3.1.24)$$

that is, the sufficient condition for integrability holds, and so $f \in \Re[q_1(\omega),q_2(\omega)]$.

Corollary 3.1.3. Any a.s. monotonic function on a closed random interval $[q_1(\omega),q_2(\omega)]$ is integrable on that random interval.

Proof. It follows from the a.s. monotonicity of $f(\omega,x(\omega))$ on $[q_1(\omega),q_2(\omega)]$ that $a.s.: \sum_{i=1}^{n} \Omega(f(\omega,x(\omega)),[q_1(\omega),q_2(\omega)]) = |f(\omega,q_2(\omega)) - f(\omega,q_1(\omega))|$. Suppose $\varepsilon > 0$ is given. We set $\delta(\omega) = \dfrac{\varepsilon}{|f(\omega,q_2(\omega)) - f(\omega,q_1(\omega))|}$ We assume that $a.s.: f(\omega,q_2(\omega)) - f(\omega,q_1(\omega)) \neq 0$, since otherwise f a.s. is constant, and there is no doubt as to its integrability. Let P be an arbitrary partition of $[q_1(\omega),q_2(\omega)]$ with mesh $a.s.: \lambda(P) < \delta(\omega)$. Then, taking account of the a.s.monotonicity of f, we have

(3.1.25)

Thus / satisfies the sufficient condition for integrability, and therefore $f \in \Re[q_1(\omega),q_2(\omega)]$.

Remark 3.1.1. A monotonic function may have a (countably) infinite set of discontinu ities on a closed random $[q_1(\omega),q_2(\omega)]$interval. For example, the function defined by the relations

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3.2.Integration over generalized random interval.

Definition 3.2.1.Let for any $\omega \in \Omega$: $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are $\Omega' = \Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$,i.e., $\forall\omega[\omega \in \Omega \Rightarrow q_{1,2}(\omega,\omega') : \Omega \to \Omega' = \Xi(\Sigma,\mathbb{R})]$.

Let $\widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ be a set of the all $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$,thus $q_{1,2}(\omega,\omega') \in \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$.

Definition 3.2.2.Let $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$,i.e., $q_{1,2}(\omega,\omega') : \Omega \to \Xi(\Sigma,\mathbb{R})$.Assume that a.s.:$-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$.

Let $\widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ be a set of the all $\Xi(\Sigma,\mathbb{R})$- valued random variables defined on a generalized probability space $\Sigma = \left(\Omega,\mathcal{F},\widetilde{\mathbf{P}}\right)$.Closed generalized random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ that is a subset $[q_1(\omega,\omega'),q_2(\omega,\omega')] \subset \widetilde{\Xi}(\Omega,\Sigma,\mathbb{R})$ such that a.s.:$-\infty < q_1(\omega,\omega') < q_2(\omega,\omega') < \infty$ and

$$\begin{gathered}\forall q(\omega,\omega')\{q(\omega,\omega') \in [q_1(\omega,\omega'),q_2(\omega,\omega')] \Leftrightarrow \\ \{q(\omega,\omega') \in \Xi(\Sigma,\mathbb{R}) | a.s.(q_1(\omega,\omega') < q(\omega,\omega') < q_2(\omega,\omega'))\}\}.\end{gathered} \quad (3.2.1)$$

Notation 3.2.1.Assume that a.s.:$q_1(\omega,\omega') < q_2(\omega,\omega')$.Then we will write:

$$q_1(\omega,\omega') \prec q_2(\omega,\omega').$$

Assume that a.s.$q_1(\omega,\omega') \leq q_2(\omega,\omega')$.Then we will write:

$$q_1(\omega,\omega') \preccurlyeq q_2(\omega,\omega').$$

Definition 3.2.3.Let E be a separable complete metric space and let Σ be its Borel σ-algebra. <u>*Generalized*</u> random measure it is a function $\mu : \Sigma \to \Xi(\Sigma,\mathbb{R})$, that

satisfies:
(1) If $E_1 \subset E_2$, then a.s.:$\mu(E_1) < \mu(E_2)$.
(2) If $E_n \in \mathcal{S},\ n = 1,2,\ldots$ and $E_i \cap E_j = \varnothing\ (i \neq j)$, then a.s.:$\mu(\cup_n E_n) = \sum_n \mu(E_n)$.
If we further impose a condition $\mu(\Omega) = 1$, then the generalized random measure so defined is called the generalized random probability measure and is usually denoted by $\widetilde{\mathbf{P}}$.

Definition 3.2.4. The tuple $(\Omega,\mathcal{S},\widetilde{\mathbf{P}})$ is then called the *generalized probability space*. Suppose $(\Omega,\mathcal{S})$ and $(\Omega',\mathcal{S}')$ are two measurable spaces. The function $X : \Omega \to \Omega'$ is called a generalized random element, if for every $A \in \mathcal{S}'$, $X^{-1}(A) \in \mathcal{S}$. If the first measurable space is equipped with a generalized random probability measure $\widetilde{\mathbf{P}}$, then the generalized random element $X : \Omega \to \Omega'$ induces a generalized random probability measure on the second space $(\Omega',\mathcal{S}')$ and given by $\mathcal{P} \equiv \widetilde{\mathbf{P}} \circ X^{-1}$, called the generalized random distribution of X.

Definition 3.2.5. If the second measurable space $(\Omega',\mathcal{S}')$ is taken to be $(\Xi(\Sigma,\mathbb{R}),\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})})$, where $\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})}$ is the Borel σ-algebra on $\Xi(\Sigma,\mathbb{R})$ the function $X(\omega) : \Omega \to \Omega' = (\Xi(\Sigma,\mathbb{R}),\mathfrak{B}_{\Xi(\Sigma,\mathbb{R})})$ is called a *generalized random variable*. For this case, the generalized random distribution of X is completely determined by the *random distribution function* defined as

$$F_{X(\omega)}(x(\omega,\omega')) = \widetilde{\mathbf{P}}(X(\omega) \preccurlyeq x(\omega,\omega')) \qquad (3.2.2)$$

Definition 3.2.6. The lengths $l([q_1(\omega),q_2(\omega)])$ of the random interval $[q_1(\omega),q_2(\omega)]$ is defined by

$$l([q_1(\omega,\omega'),q_2(\omega,\omega')]) = \mathbf{ess}\sup q_3(\omega,\omega'), \qquad (3.2.2)$$

where $q_3(\omega,\omega') = q_2(\omega,\omega') - q_1(\omega,\omega')$.

Notation 3.2.2. Assume that a.s.$q_1(\omega,\omega') < q_2(\omega,\omega')$.Then we will write:

$$q_1(\omega,\omega') \prec q_2(\omega,\omega').$$

Assume that a.s.$q_1(\omega,\omega') \leq q_2(\omega,\omega')$.Then we will write:

$$q_1(\omega,\omega') \preccurlyeq q_2(\omega,\omega').$$

Definition 3.2.7. A partition P of a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is a finite system of random variables $x_0(\omega,\omega'),\ldots,x_n(\omega,\omega')$ such that

$$q_1(\omega,\omega') = x_0(\omega,\omega') \prec x_1(\omega,\omega') \prec \ldots x_{n-1}(\omega,\omega') \prec x_n(\omega,\omega') = q_2(\omega,\omega').$$

The closed random intervals $[x_{i-1}(\omega),x_i(\omega)],\ (i = 1,\ldots,n)$ are called the intervals of the partition P.

Definition 3.2.8. The largest of the lengths of the intervals of the partition P, denoted $\lambda(P)$, is called the *mesh* of the partition.

Definition 3.2.9. We speak of a partition with distinguished points (P,ξ) on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ if we have a partition P of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$and a point $\xi_1(\omega,\omega') \in [x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ has been chosen in each of the intervals of the partition $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')],\ (i = 1,\ldots,n)$.
We denote the set of points $(\xi_1(\omega,\omega'),\ldots,\xi_n(\omega,\omega'))$ by the single letter $\xi(\omega,\omega')$.

Definition 3.2.10. In the set $\widetilde{P}$ of partitions with distinguished points on a given random

interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$, we consider the following base
$\mathbf{B} = \{B_d\}$. The element $B_d, d > 0$, of the base B consists of all partitions
with distinguished points (P,ξ) on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ for which $\lambda(P) < d$.

Proposition 3.2.1. Let us verify that $\{B_d\}, d > 0$ is actually a base in $\widetilde{P}$.

Proof. First $B_d \neq \varnothing$. In fact, for any number $d > 0$, it is obvious that there exists a partition P of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with mesh $\lambda(P) < d$ (for example, a partition into n congruent closed random intervals). But then there also exists a partition $(P,\xi(\omega,\omega'))$ with distinguished points for which $\lambda(P) < d$.
Second, if $d_1 > 0, d_2 > 0$, and $d = \min\{d_1,d_2\}$, it is obvious that $B_{d_1} \cap B_{d_2} = B_d \in \mathbf{B}$. Hence $\mathbf{B} = \{B_d\}$ is indeed a base in $\widetilde{P}$.

Definition 3.2.11. (**Random Riemann Sums**) (i) If a function $f : \widetilde{\widetilde{\Xi}}(\Sigma,\mathbb{R}) \to \widetilde{\widetilde{\Xi}}(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ and $(P,\xi(\omega))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega,\omega')] = \sum_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \tag{3.2.3}$$

where $\Delta x_i(\omega,\omega') = x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$, is the random Riemann sum of the function $f : \widetilde{\widetilde{\Xi}}(\Sigma,\mathbb{R}) \to \widetilde{\widetilde{\Xi}}(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. Thus, when the function f is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega,\omega')]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\widetilde{P}$ of all partitions $p = (P,\xi(\omega,\omega'))$ with distinguished points on the closed interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.
Since there is a base $\mathbf{B}$ in $\widetilde{P}$, one can ask about the limit of the function $\Phi(p(\omega,\omega'))$ over that base.

(ii) If a function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \widetilde{\widetilde{\Xi}}(\Sigma,\mathbb{R})$ is defined on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ and $(P,\xi(\omega,\omega'))$ is a partition with distinguished points on this closed random interval, the sum

$$\sigma[f,P,\xi(\omega,\omega')] = \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \tag{3.2.3$'$}$$

where $\Delta x_i(\omega,\omega') = x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$, is the random Riemann sum of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ corresponding to the partition $(P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$. Thus, when the function

$$f : \Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')] \to \widetilde{\widetilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$$

is fixed, the random Riemann sum $\sigma[f,P,\xi(\omega,\omega')]$ is a function $\Phi(p) = \sigma[f,P]$ on the set $\widetilde{P}$ of all partitions $p = (P,\xi(\omega,\omega'))$ with distinguished points on the closed interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Definition 3.2.12. (**Riemann Integral on a random interval**)
(i) Let $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.
The random variable $I(\omega,\omega')$ is the Riemann integral of the function $f : \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: \left| I(\omega,\omega') - \sum_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega') \right| < \varepsilon \tag{3.2.4}$$

for any partition $(P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ whose mesh $\lambda(P)$ is less than δ.
(ii) Let $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ be a function restricted on a closed random interval

$[q_1(\omega), q_2(\omega)]$ such that $f : \Omega \times [q_1(\omega,\omega'), q_2(\omega,\omega')] \to \Xi(\Sigma,\mathbb{R})$.
The random variable $I(\omega,\omega')$ is the Riemann integral of the function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ on the closed random interval $[q_1(\omega,\omega'), q_2(\omega,\omega')]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: \left| I(\omega,\omega') - \sum_{i=1}^{n} f(\omega, \xi_i(\omega,\omega'))\Delta x_i(\omega,\omega') \right| < \varepsilon. \quad (3.2.4')$$

Since the partitions $p = (P, \xi(\omega,\omega'))$ for which $\lambda(P) < \delta$ form the element B_δ of the base $\mathbf{B}$ introduced above in the set $\widetilde{P}$ of partitions with distinguished points, Definition 3.2.12 is equivalent to the statement

$$a.s.: I(\omega,\omega') = \lim_{\mathbf{B}} \Phi(p(\omega,\omega')). \quad (3.2.5)$$

that is, the integral $I(\omega,\omega')$ is the limit over $\mathbf{B}$ of the Riemann sums of the function f corresponding to partitions with distinguished points on $[q_1(\omega,\omega'), q_2(\omega,\omega')]$.
It is natural to denote the base $\mathbf{B}$ by $\lambda(P) \to 0$, and then the definition of the Riemann integral on a random interval can be rewritten as

$$\begin{aligned} a.s.: I(\omega,\omega') &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \\ a.s.: I(\omega,\omega') &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega, \xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'). \end{aligned} \quad (3.2.6)$$

Notation 3.2.2. The integral of $f(x(\omega,\omega'))$ over $[q_1(\omega,\omega'), q_2(\omega,\omega')]$ is denoted

$$\begin{aligned} &\int_{q_1(\omega)}^{q_2(\omega)} f(x(\omega,\omega'))d[x(\omega,\omega')], \\ &\int_{q_1(\omega)}^{q_2(\omega)} f(\omega, x(\omega,\omega'))d[x(\omega,\omega')], \end{aligned} \quad (3.2.7)$$

in which the random variables $q_1(\omega,\omega')$ and $q_2(\omega,\omega')$ are called respectively the lower and upper limits of integration. The function f is called the integrand, $f(x(\omega,\omega'))d[x(\omega,\omega')]$ is called the differential form, and $x(\omega,\omega')$ is the random variable of integration. Thus

$$\begin{aligned} a.s.: \int_{q_1(\omega,\omega')}^{q_2(\omega,\omega')} f(x(\omega,\omega'))d[x(\omega,\omega')] &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'), \\ a.s.: \int_{q_1(\omega,\omega')}^{q_2(\omega,\omega')} f(\omega, x(\omega,\omega'))d[x(\omega,\omega')] &= \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\omega, \xi_i(\omega,\omega'))\Delta x_i(\omega,\omega'). \end{aligned} \quad (3.2.8)$$

Definition 3.2.13. A function $f : \Omega \times \Xi(\Sigma,\mathbb{R}) \to \Xi(\Sigma,\mathbb{R})$ is Riemann integrable on the closed interval $[q_1(\omega,\omega'), q_2(\omega,\omega')]$ if the limit of the Riemann sums in (3.2.8) exists $a.s.$ as $\lambda(P) \to 0$ (that is, the Riemann integral of f is defined).
Notation 3.2.3. The set of Riemann-integrable functions on a closed random interval $[q_1(\omega,\omega'), q_2(\omega,\omega')]$ will be denoted $\mathfrak{R}[q_1(\omega,\omega'), q_2(\omega,\omega')]$
By the definition of the integral (Definition 3.2.8) and its reformulation in the forms (3.2.5) and (3.2.8), an integral is the limit of a certain special function

$\Phi(p(\omega,\omega')) = \sigma[f,P,\xi(\omega,\omega')]$ the random Riemann sum, defined on the set $\widetilde{P}$ of partitions $p(\omega,\omega') = (P,\xi(\omega,\omega'))$ with distinguished points on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$.
This limit is taken with respect to the base $\mathbf{B}$ in $\widetilde{P}$ that we have denoted $\lambda(P) \to 0$.
Thus the integrability or nonintegrability of a function f on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ depends on the existence of this limit. By the Cauchy criterion, this limit exists a.s., if and only if for every $\varepsilon > 0$ there exists an element $B_\delta \in \mathbf{B}$ in the base such that

$$a.s.: |\Phi(p'(\omega,\omega')) - \Phi(p''(\omega,\omega'))| < \varepsilon \tag{3.2.9}$$

for any two points $p'(\omega,\omega'),p''(\omega,\omega')$ in B_δ. In more detailed notation, what has just been said means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a.s.: |\sigma[f,P,\xi(\omega,\omega')] - \sigma[f,P,\xi(\omega,\omega')]| < \varepsilon \tag{3.2.10}$$

or, what is the same,

$$a.s.: \left|\sum_{i=1}^{n'} f(\omega,\xi_i'(\omega,\omega'))\Delta x_i'(\omega(\omega,\omega')) - \sum_{i=1}^{n''} f(\omega,\xi_i''(\omega,\omega'))\Delta x_i''(\omega,\omega')\right| < \varepsilon \tag{3.2.11}$$

for any partitions $(P',\xi'(\omega,\omega'))$ and $(P'',\xi''(\omega,\omega'))$ with distinguished points on the random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with $\lambda(P') < \delta$ and $\lambda(P'') < \delta$.

Proposition 3.2.2. A necessary condition for a function $f(\omega,x(\omega,\omega'))$ defined on $\Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')]$ to be Riemann integrable on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is that f be bounded a.s. on $\Omega \times [q_1(\omega,\omega'),q_2(\omega,\omega')]$.

Proof. If f is not bounded a.s. on $[q_1(\omega,\omega'),q_2(\omega,\omega')]$, then for any partition (P,ξ) of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ the function f is unbounded on at least one of the intervals $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ of (P,ξ). This means that, by choosing the point
$\xi_i(\omega,\omega') \in [x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$
in different ways, we can make the quantity $|f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega')|$ a.s. as large as desired.But then the Riemann sum $\sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega')$ can also be made as large as desired in absolute value by changing only the point $\xi_i(\omega,\omega')$ in this interval.

We agree that when a partition P

$$q_1(\omega,\omega') = x_0(\omega,\omega') \prec x_1(\omega,\omega') \prec \ldots x_{n-1}(\omega,\omega') \prec x_n(\omega,\omega') = q_2(\omega,\omega').$$

is given on the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$, we shall use the symbol $\Delta_i(\omega,\omega')$ to denote the interval $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ along with $\Delta x_i(\omega,\omega')$ as a notation for the difference $x_i(\omega,\omega') - x_{i-1}(\omega,\omega')$.

If a partition $P^\star$ of the closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is obtained from the partition P by the adjunction of new points to P, we call $P^\star$ a *refinement* of P.

When a refinement $P^\star$ of a partition P is constructed, some (perhaps all) of the closed random intervals $\Delta_i(\omega,\omega') = [x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$ of the partition P themselves undergo partitioning: $x_{i-1}(\omega,\omega') = x_{i_0}(\omega,\omega') \prec \ldots \prec x_{i_{n_i}}(\omega,\omega') = x_i(\omega,\omega')$.

In that connection, it will be useful for us to label the points of $P^\star$ by double indices.

In the notation $x_{i_j}(\omega,\omega')$ the first index means that $x_{i_j}(\omega,\omega') \in \Delta_i(\omega,\omega')$, and the second index is the ordinal number of the point on the closed random interval $\Delta_i(\omega)$. It is now natural to set $\Delta x_{i_j}(\omega,\omega') = x_{i_j}(\omega,\omega') - x_{i_{j-1}}(\omega,\omega')$ and $\Delta_{i_j}(\omega,\omega')$. Thus

$$\Delta x_i(\omega,\omega') = \Delta x_{i_1}(\omega,\omega') + \ldots + \Delta x_{i_{n_i}}(\omega,\omega').$$

As an example of a partition that is a refinement of both the partition $P'$and P'' one can take $P^\star = P' \cup P''$, obtained as the union of the points of the two partitions P' and P''.

We recall finally that $\Omega(f(\omega,x(\omega,\omega')),E,\omega')$ denotes the oscillation of the function $f(\omega,x)$ on the random set $E(\omega,\omega')$, that is

$$\Omega(f(\omega,x(\omega,\omega')),E(\omega,\omega')) = \sup_{a.s.:\, x_1,(\omega,\omega'),x_2(\omega,\omega')\in E(\omega,\omega')} |f(\omega,x_1(\omega,\omega')) - f(\omega,x_2(\omega,\omega'))|. \quad (3.2.12)$$

In particular, $\Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))$ is the oscillation of $f(\omega,x(\omega,\omega'))$ on the closed random interval $[x_{i-1}(\omega,\omega'),x_i(\omega,\omega')]$.
This oscillation is necessarily a.s. finite if $f(\omega,x)$ is a.s. bounded function of variable x.
Proposition 3.1.3. A sufficient condition for a.s. bounded function $f(\omega,x)$ to be integrable on a closed random interval $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ is that for every $\varepsilon > 0$ there exist a number $\delta > 0$ such that

$$\sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega') < \varepsilon \quad (3.2.13)$$

for any partition P of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ with mesh $\lambda(P) < \delta$.
Proof Let P be a partition of $[q_1(\omega,\omega'),q_2(\omega,\omega')]$ and $P^\star$ a refinement of P. Let us estimate the difference between the random Riemann sums $\sigma(f,P^\star,\xi^\star) - \sigma(f,P,\xi)$.
Using the notation introduced above, we can write

$$\begin{gathered}
|\sigma(f,P^\star,\xi^\star(\omega,\omega')) - \sigma(f,P,\xi(\omega,\omega'))| = \\
\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega,\omega'))\Delta x_{i_j}(\omega,\omega') - \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_i(\omega,\omega')\right| = \\
\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} f(\omega,\xi_{i_j}(\omega,\omega'))\Delta x_{i_j}(\omega,\omega') - \sum_{i=1}^{n} f(\omega,\xi_i(\omega,\omega'))\Delta x_{i_j}(\omega,\omega')\right| = \\
\left|\sum_{i=1}^{n}\sum_{j=1}^{n_i} [f(\omega,\xi_{i_j}(\omega,\omega')) - f(\omega,\xi_i(\omega,\omega'))]\Delta x_{i_j}(\omega,\omega')\right| \leq \\
\sum_{i=1}^{n}\sum_{j=1}^{n_i} |[f(\omega,\xi_{i_j}(\omega)) - f(\omega,\xi_i(\omega))]|\Delta x_{i_j}(\omega,\omega') \leq \\
\sum_{i=1}^{n}\sum_{j=1}^{n_i} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_{i_j}(\omega,\omega') = \\
\sum_{i=1}^{n} \Omega(f(\omega,x(\omega,\omega')),\Delta_i(\omega,\omega'))\Delta x_i(\omega,\omega').
\end{gathered} \quad (3.2.14)$$

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4.Generalized Chapman-Kolmogorov equation.

We consider now double stochastic Markov processes. The Eq.(2.4.9) fully defines double stochastic Markov process, but it does not say anything about the random probability density function p_ω. The Generalized ChapmanKolmogorov Equation

(GCKE) states the property that the function p_ω must satisfy to describe double stochastic Markov process. To derive the GCKE, we proceed as follows. Consider two values $x_1(\omega,\omega') = X(t_1,\omega)$ and $x_3(\omega,\omega') = X(t_3,\omega)$ of the generalized random variable $X(t,\omega) : [0,T]\times\Omega \to \Omega' = \widetilde{\widetilde{\Xi}}(\Omega,\Sigma,\mathbb{R})$, measured at times t_1 and t_3 with $t_1 < t_3$, then from Eq.(2.4.8) we obtain

$$p(x_1(\omega,\omega'),t_1;x_3(\omega,\omega'),t_3) = p(x_3(\omega,\omega'),t_3|x_1(\omega,\omega'),t_1)p(x_1(\omega,\omega'),t_1). \quad (4.1.1)$$

Integrating over $x_1(\omega,\omega')$, we define the random marginal density

$$\begin{aligned} p(x_3(\omega,\omega'),t_3) &:= \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[x_1(\omega,\omega')]p(x_1(\omega,\omega'),t_1;x_3(\omega,\omega'),t_3) = \\ &= \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[x_1(\omega,\omega')]p(x_3(\omega,\omega'),t_3|x_1(\omega,\omega'),t_1)p(x_1(\omega,\omega'),t_1). \end{aligned} \quad (4.1.2)$$

Consider now an intermediate point x_2, then the joint random probability is

$$\begin{gathered} p(x_1(\omega,\omega'),t_1;x_2(\omega,\omega'),t_2;x_3(\omega,\omega'),t_3) = \\ p(x_3(\omega,\omega'),t_3|x_2(\omega,\omega'),t_2)p(x_2(\omega,\omega'),t_2|x_1(\omega,\omega'),t_1)p(x_1(\omega,\omega'),t_1). \end{gathered} \quad (4.1.3)$$

We integrate over x_2 and applying the definition in Eq.(2.1.8), we obtain

$$\begin{gathered} \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[x_2(\omega,\omega')]p(x_1(\omega,\omega'),t_1;x_2(\omega,\omega'),t_2;x_3(\omega,\omega'),t_3) = \\ \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[x_2(\omega,\omega')]p(x_3(\omega,\omega'),t_3|x_2(\omega,\omega'),t_2)p(x_2(\omega,\omega'),t_2|x_1(\omega,\omega'),t_1) \times \\ \times p(x_1(\omega,\omega'),t_1) \end{gathered} \quad (4.1.4)$$

$$\begin{gathered} p(x_1,t_1;x_3,t_3) = \\ p(x_1,t_1) \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} dx_2 p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)p(x_1,t_1;x_3,t_3)p(x_1,t_1) \end{gathered} \quad (4.1.5)$$

Thus

$$\begin{gathered} \frac{p(x_1(\omega,\omega'),t_1;x_3(\omega,\omega'),t_3)}{p(x_1(\omega,\omega'),t_1)} = \\ = \int_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} dx_2(\omega,\omega')p(x_3(\omega,\omega'),t_3|x_2(\omega,\omega'),t_2)p(x_2(\omega,\omega'),t_2|x_1(\omega,\omega'),t_1). \end{gathered} \quad (4.1.6)$$

The left-hand side of Eq.(4.1.6) is by definition a conditional probability, thus we obtain the Generalized Chapman-Kolmogorov equation

$$p(x_3(\omega,\omega'),t_3|x_1(\omega,\omega'),t_1) = \int\limits_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[x_2(\omega,\omega')]p(x_3(\omega,\omega'),t_3|x_2(\omega,\omega'),t_2)p(x_2(\omega,\omega'),t_2|x_1(\omega,\omega'),t_1). \tag{4.1.7}$$

The conditional probability $p(x_3,t_3|x_1,t_1)$ defined in Eq.(4.1.7) satisfies the normalization condition

$$\int\limits_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d\big[x_3\ (\omega,\omega')\big]p(x_3(\omega,\omega'),t_3|x_1(\omega,\omega'),t_1) = 1. \tag{4.1.8}$$

If $t_3 \to t_1$, then

$$p(x_3(\omega,\omega'),t_3|x_1(\omega,\omega'),t_1) = \delta(x_3(\omega,\omega') - x_1(\omega,\omega')). \tag{4.1.9}$$

The GCKE of the Brownian motion has the explicit form

$$p(x_3(\omega,\omega'),t_3|x_1(\omega,\omega'),t_1) = \frac{1}{\sqrt{4\pi D(t_3-t_1)}}\exp\left(-\frac{(x_3(\omega,\omega')-x_1(\omega,\omega'))^2}{4D(t_3-t_1)}\right). \tag{4.1.10}$$

5.The Path Integral for a double stochastic Markov Process.

5.1.The Path Integral for a double stochastic Wiener Process.

Here we consider one-dimensional double stochastic Markovian processes describable
through Langevin or Fokker-Planck equations

$$\dot{q} = f(q,t) + g(q,t)\eta(t,\omega,\omega'), \tag{5.1.1}$$

where $f(q,t)$ and $g(q,t)$ are known (smooth) functions, and $\eta(t,\omega,\omega')$ is a double stochastic Gaussian white noise with zero mean and δ-correlated.
As has been discussed in the section 4.1, $P(q(\omega,\omega'),t|q'(\omega,\omega'),t')$ fulfills the generalized Chapman-Kolmogorov equation $(t_1 < t_2 < t_3)$

$$P(q_3(\omega,\omega'),t_3|q_1(\omega,\omega'),t_1) = \\ = \int\limits_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[q_2(\omega,\omega')]P(q_3(\omega,\omega'),t_3|q_2(\omega,\omega'),t_2)P(q_2(\omega,\omega'),t_2|q_1(\omega,\omega'),t_1). \tag{5.1.2}$$

Such an equation allows, by making a partition of the time interval in N steps: $t_0 < t_1 < \ldots < t_f$, with $t_j = t_0 + j(t_f - t_0)/N$, to obtain a path-dependent representation of the propagator. With the given partition, we reiterate (5.1.2) N-times and get

$$P(q_f(\omega,\omega'),t_f|q_0(\omega,\omega'),t_0) = \int\limits_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')}\cdots\int\limits_{-\infty(\omega,\omega')}^{\infty(\omega,\omega')} d[q_1(\omega,\omega')]d[q_2(\omega,\omega')]\ldots d[q_{N-1}(\omega,\omega')]\times$$
$$\times P(q_f(\omega,\omega'),t_f|q_{N-1}(\omega,\omega'),t_{N-1})\times\ldots$$
$$\ldots\times P(q_2(\omega,\omega'),t_2|q_1(\omega,\omega'),t_1)P(q_1(\omega,\omega'),t_1|q_0(\omega,\omega'),t_0). \quad (5.1.3)$$

The last expression can be interpreted as an integration over all possible paths that the process could follow (corresponding to the different values of the sequence: propagator $P(q_{j+1}(\omega,\omega'),t_{j+1}|q_j(\omega,\omega'),t_j)$ in order to find the more conventional representation of the integration over paths. Note that the random probability that at a given time $t,$ the process takes a value between a and b is given by

$$P(a(\omega,\omega') < q(t) < b(\omega,\omega')) = \int\limits_{a(\omega,\omega')}^{b(\omega,\omega')} d[q(\omega,\omega')]P(q(\omega,\omega'),t|q_0(\omega,\omega'),t_0) \quad (5.1.4)$$

Likewise, the random probability that the process, starting at $q = q_0(\omega,\omega')$ at $t = t_0,$ has a value between $a_1(\omega,\omega')$ and $b_1(\omega,\omega')$ at $t_1,$ between $a_2(\omega,\omega')$ and $b_2(\omega,\omega')$ at $t_2,\ldots,$ between $a_{N-1}(\omega,\omega')$ and $b_{N-1}(\omega,\omega')$ at t_{N-1}(with $a_j(\omega,\omega') \prec b_j(\omega,\omega')$ and $t_j < t_{j+1}$),
and reaching point $q_N(\omega,\omega')$ at $t_N,$ will be given by

$$P(q_f(\omega,\omega'),t_f|q_0(\omega,\omega'),t_0) = \int\limits_{a_1(\omega,\omega')}^{b_1(\omega,\omega')}\int\limits_{a_2(\omega,\omega')}^{b_2(\omega,\omega')}\cdots\int\limits_{a_{N_1}(\omega,\omega')}^{b_{N-1}(\omega,\omega')} d[q_1(\omega,\omega')]d[q_2(\omega,\omega')]\ldots d[q_{N-1}(\omega,\omega')]\times$$
$$\times P(q_f(\omega,\omega'),t_f|q_{N-1}(\omega,\omega'),t_{N-1})\times\ldots\ldots$$
$$\times P(q_2(\omega,\omega'),t_2|q_1(\omega,\omega'),t_1)P(q_1(\omega,\omega'),t_1|q_0(\omega,\omega'),t_0). \quad (5.1.5)$$

We define now the conditional probability $P(q_f(\omega,\omega'),t_f|q_0(\omega,\omega'),t_0;\delta_1,\delta_2)$ by

$$P(q_f(\omega,\omega'),t_f|q_0(\omega,\omega'),t_0;\delta_1,\delta_2) = \int\limits_{a_1(\omega,\omega')}^{b_1(\omega,\omega')}\int\limits_{a_2(\omega,\omega')}^{b_2(\omega,\omega')}\cdots\int\limits_{a_{N_1}(\omega,\omega')}^{b_{N_1}(\omega,\omega')} d[q_1(\omega,\omega')]d[q_2(\omega,\omega')]\ldots d[q_{N-1}(\omega,\omega')]\times$$
$$\times\theta\left(\sup_{0\le n\le N-1}|q_n|;\delta_1\right)\theta\left(\sum_{j=0}^{j=N} q_j^2(t_{j+1}-t_j);\delta_2\right)\times$$
$$\times P(q_f(\omega,\omega'),t_f|q_{N-1}(\omega,\omega'),t_{N-1})\ldots P(q_2(\omega,\omega'),t_2|q_1(\omega,\omega'),t_1)\times$$
$$\times P(q_1(\omega,\omega'),t_1|q_0(\omega,\omega'),t_0), \quad (5.1.6)$$

where $\theta(x,\delta) = 1$if $|x| \le \delta$ and $\theta(x,\delta) = 0$ if $|x| > \delta.$ If we increase the number of time slices within the time partition where the intervals (a_j,b_j) are specified, and at the same time take the limit a.s.: $b_j(\omega,\omega') - a_j(\omega,\omega') \to 0,$ the trajectory is defined with higher and higher precision. Clearly, a requisite is that the trajectories be continuous. This happens in particular for the Wiener process.With all this in mind (2.3.3) can be interpreted as an integration over all the paths that the process could follow corresponding to the different values of the sequence $q_0, q_1, q_2, \ldots.., q_N = q_f$ such that the inequalities (5.1.7) hold

$$a.s.: \sup_{0 \leq n \leq N-1} |q_n(\omega, \omega')| \leq \delta_1 \text{ and } a.s.: \sum_{j=0}^{j=N} q_j^2(\omega, \omega')(t_{j+1} - t_j) \leq \delta_2. \tag{5.1.7}$$

For the Wiener process we have that

$$\begin{aligned} &P(W_2(\omega, \omega'), t_2 | W_1(\omega, \omega'), t_1; \delta_1, \delta_2) = \\ &= \frac{1}{\sqrt{2\pi D(t_2 - t_1)}} \exp\left[-\frac{(W_2(\omega, \omega') - W_1(\omega, \omega'))^2}{2D(t_2 - t_1)} \right]. \end{aligned} \tag{5.1.8}$$

By substituting (5.1.8) into (5.1.6) we get

$$\begin{aligned} &\prod_{j=1}^{N} \frac{d[W_j(\omega, \omega')]}{\sqrt{4\pi\varepsilon D}} \theta\left(\sup_{1 \leq j \leq N} |W_j(\omega, \omega')|; \delta_1 \right) \theta\left(\sum_{j=1}^{j=N} \varepsilon W_j^2(\omega, \omega'); \delta_2 \right) \times \\ &\times \exp\left[-\frac{1}{4D\varepsilon} \sum_{j=1}^{N} (W_j(\omega, \omega') - W_{j-1}(\omega, \omega'))^2 \right] \end{aligned} \tag{5.1.9}$$

which is the desired probability of following a given path ander conditions (5.1.7). When $\varepsilon \to 0$ and $N \to \infty$, we can write the exponential in (5.1.10) in the continuous limit as

$$\exp\left[-\frac{1}{4D} \int_{t_0}^{t} d\tau \left(\frac{dW(\tau, \omega, \omega')}{d\tau} \right)^2 \right]. \tag{5.1.10}$$

If we integrate the expression in (5.1.5) over all the intermediate points (which is equivalent to a sum over all the possible paths), as all the integrands are Gaussian, and the convolution of two Gaussian is again a Gaussian, we recover the result of (5.1.8) for the probability density of the Wiener process. Hence, we have expressed the probability density as a Wiener path integral

$$\begin{aligned} &P(W(\omega, \omega'), t | W_0(\omega, \omega'), t_0) = \\ &\int\limits_{\substack{W(t) = W(\omega, \omega') \\ W(0) = W_0(\omega, \omega')}} D[W(\tau, \omega, \omega')] \exp\left[-\frac{1}{4D} \int_{t_0}^{t} d\tau \left(\frac{dW(\tau, \omega, \omega')}{d\tau} \right)^2 \right]. \end{aligned} \tag{5.1.11}$$

Thus we have expressed the conditional probability density as a Wiener path integral

$$\begin{aligned} &P(W(, \omega, \omega')t | W_0(\omega, \omega'), t_0; \delta_1, \delta_2) = \\ &\int D[W(\tau, \omega, \omega')] \theta\left(\sup_{t_0 \leq \tau \leq t} W(\tau, \omega, \omega'); \delta_1 \right) \theta\left(\int_{t_0}^{t} W^2(\tau, \omega, \omega') d\tau; \delta_2 \right) \times \\ &\times \exp\left[-\frac{1}{4D} \int_{t_0}^{t} d\tau \left(\frac{dW(\tau, \omega, \omega')}{d\tau} \right)^2 \right], \end{aligned} \tag{5.1.12}$$

where the expression inside the integral represents the continuous version of the integral of (5.1.6), over all required values of the intermediate points $\{W_j(\omega,\omega')\}$.
We revrite now RHS of (5.1.12) of the form

$$P(W(\omega,\omega'),t|W_0(\omega,\omega'),t_0;\delta_1,\delta_2) = \int D[W(\tau,\omega,\omega')] \times \\ \times \exp\left[-\frac{1}{4D}\int_{t_0}^{t}\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2 d\tau + \ln\theta\left(\sup_{t_0\leq\tau\leq t} W(\tau,\omega,\omega');\delta_1\right)\right. \\ \left. + \ln\theta\left(\int_{t_0}^{t} W^2(\tau,\omega,\omega')d\tau;\delta_2\right)\right]. \tag{5.1.13}$$

We introduce now generalized conditional probability density as a Colombeau-Wiener path integral

$$(P_\epsilon(W,t|W_0,t_0;\delta_1,\delta_2))_{\epsilon\in(0,1]} = \left(\int D[W(\tau,\omega,\omega')] \times \right. \\ \exp\left[-\frac{1}{4\epsilon}\int_{t_0}^{t}\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2 d\tau + \ln\theta\left(\sup_{t_0\leq\tau\leq t} W(\tau,\omega,\omega');\delta_1\right)\right. \\ \left.\left. + \ln\theta\left(\int_{t_0}^{t} W^2(\tau,\omega,\omega')d\tau;\delta_2\right)\right]\right)_{\epsilon\in(0,1]}. \tag{5.1.14}$$

5.2.The Path Integral for a General double stochastic Markov Process

We start by writing the discrete version of the Langevin equation given by (5.1.1) (in order to simplify the notation we adopt $g(q,t) = 1$ and $f(q,t)$ to be independent of t)

$$q_{j+1}(\omega,\omega') - q_j(\omega,\omega') \simeq \{\alpha f(q_{j+1}) + (1-\alpha)f(q_j)\}\varepsilon + [W_{j+1}(\omega,\omega') - W_j(\omega,\omega')], \tag{5.2.1}$$

where $\varepsilon = t_{j+1} - t_j = (t_f - t_0)/N$, and $W_j = W(t_j)$ is the Wiener process (just formally, $dW(t,\omega,\omega') \simeq \eta(t,\omega,\omega')dt$). The parameter $\alpha(0 \leq \alpha \leq 1)$ is arbitrary,the most usual choices being $\alpha = 0, 1/2, 1$, corresponding to the prepoint, midpoint and endpoint discretization, respectively. According to the previous results, the random probability that

$$W(t_0,\omega,\omega') = 0; W_1(t_1,\omega,\omega') < W_1(\omega,\omega') + dW_1(\omega,\omega');\ldots; \\ a.s.\, W_N(t_N,\omega,\omega') < W_N(\omega,\omega') + dW_N(t_N,\omega,\omega'), \\ \sum_{j=1}^{j=N} \varepsilon W_j^2(\omega,\omega') \leq \delta_2 \tag{5.2.2}$$

is given by

$$P\left(\{W_{j-1}(\omega,\omega')\}_{j=1}^{N}\right) = \\ \prod_{j=1}^{N}\left(\frac{d[W_j(\omega,\omega')]}{\sqrt{4\pi\varepsilon D}}\right)\exp\left[-\frac{1}{4D\varepsilon}\sum_{j=1}^{N}(W_j(\omega,\omega') - W_{j-1}(\omega,\omega'))^2\right]. \tag{5.2.3}$$

As our interest is to have the corresponding conditional probability in q-space, we need to transform the probability given in the last equation (5.2.3). As is well known, to do this
we need the Jacobian of the transformation connecting both sets of stochastic variables $\{W_j(\omega,\omega')\} \to \{q_j(\omega,\omega')\}$. To find it we write Eq.(5.2.1) as

$$W_j(\omega,\omega') = q_j(\omega,\omega') - q_{j-1}(\omega,\omega') - \{\alpha f(q_j(\omega,\omega')) + +(1-\alpha)f(q_{j-1}(\omega,\omega'))\}\varepsilon + W_{j-1}(\omega,\omega'). \qquad (5.2.4)$$

The Jacobian is given by

$$\mathbf{J}(\omega,\omega') = \det\left(\frac{\partial W_j(\omega,\omega')}{\partial q_k(\omega,\omega')}\right) = \prod_{j=1}^{N}\left(1 - \varepsilon\alpha\frac{df(q_j(\omega,\omega'))}{dq_j(\omega,\omega')}\right). \qquad (5.2.5)$$

For $\varepsilon \to 0, N \to \infty$, it can be approximated as

$$\mathbf{J}(\omega,\omega') \approx \exp\left(-\varepsilon\alpha\sum_j \frac{df(q_j(\omega,\omega'))}{dq_j(\omega,\omega')}\right). \qquad (5.2.6)$$

Now, remembering that $P(\{q_j\}) = \mathbf{J}P(\{W_j\})$, and taking into account that the conditional probability $P(q,t|q_0,t_0;\delta_1,\delta_2)$ is given as a sum over all the possible paths such that the inequalities (5.2.2) hold, we get

$$\begin{gathered} P(q_f(\omega,\omega'),t|q_0(\omega,\omega'),t_0;\delta_1,\delta_2) = \\ \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{4\pi\varepsilon D}}\right)^N d[W_1(\omega,\omega')]d[W_2(\omega,\omega')]\ldots d[W_{N-1}(\omega,\omega')] \\ \times\theta(|W_N(\omega,\omega')|;\delta_1)\theta\left(\sum_{j=1}^{j=N}\varepsilon W_j^2(\omega,\omega');\delta_2\right)\times \\ \times\delta(q_f(\omega,\omega') - q_N(\omega,\omega'))\exp\left[-\frac{1}{4D\varepsilon}\sum_{j=1}^{N}(W_j(\omega,\omega') - W_{j-1}(\omega,\omega'))^2\right]. \end{gathered} \qquad (5.2.7)$$

For the double stochastic Wiener process we have that

$$\begin{gathered} P(W_2(\omega,\omega'),t_2|W_1(\omega,\omega'),t_1;\delta_1,\delta_2) = \\ = \frac{1}{\sqrt{2\pi D(t_2-t_1)}}\exp\left[-\frac{(W_2(\omega,\omega') - W_1(\omega,\omega'))^2}{2D(t_2-t_1)}\right]. \end{gathered} \qquad (5.2.8)$$

By substituting (5.2.8) into (5.2.6) we get

$$\begin{gathered} \prod_{j=1}^{N}\frac{d[W_j(\omega,\omega')]}{\sqrt{4\pi\varepsilon D}}\theta\left(\sup_{1\leq j\leq N}|W_j(\omega,\omega')|;\delta_1\right)\theta\left(\sum_{j=1}^{j=N}\varepsilon W_j^2(\omega,\omega');\delta_2\right) \\ \exp\left[-\frac{1}{4D\varepsilon}\sum_{j=1}^{N}(W_j(\omega,\omega') - W_{j-1}(\omega,\omega'))^2\right] \end{gathered} \qquad (5.2.9)$$

which is the desired probability of following a given path ander conditions (5.2.7). When $\varepsilon \to 0$ and $N \to \infty$, we can write the exponential in (5.2.9) in the continuous limit as

$$\exp\left[-\frac{1}{4D}\int_{t_0}^{t}d\tau\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2\right]. \tag{5.2.10}$$

If we integrate the expression in (5.2.5) over all the intermediate points (which is equivalent to a sum over all the possible paths), as all the integrands are Gaussian, and the convolution of two Gaussian is again a Gaussian, we recover the result of (5.2.8) for the random probability density of the double stochastic Wiener process. Hence, we have expressed the random probability density as a Wiener path integral

$$P(W(\omega,\omega'),t|W_0(\omega,\omega'),t_0) = \\ N\int_{\substack{W(t,\omega,\omega')=W(\omega,\omega')\\ W(0,\omega,\omega')=W_0(\omega,\omega')}} D[W(\tau,\omega,\omega')]\exp\left[-\frac{1}{4D}\int_{t_0}^{t}d\tau\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2\right], \tag{5.2.11}$$

where the *random Feiman measure* is defined as

$$D[W(\tau,\omega,\omega')] = \lim_{M\to\infty}\prod_{i=0}^{M}d[W(\tau_i,\omega,\omega')] \tag{5.2.11$'$}$$

where $W(\tau_i,\omega,\omega')$, are the random field configurations at the time τ_i having sliced the interval $[0,\tau]$ in M finitesimal parts ϵ with $\tau_i = i\epsilon, N$ is an normalizing constant. Thus we have expressed the conditional random probability density as a Wiener path integral

$$P(W(\omega,\omega'),t|W_0(\omega,\omega'),t_0;\delta_1,\delta_2) = \\ \int D[W(\tau,\omega,\omega')]\theta\left(\sup_{t_0\le\tau\le t}|W(\tau,\omega,\omega')|;\delta_1\right)\theta\left(\int_{t_0}^{t}W^2(\tau,\omega,\omega')d\tau;\delta_2\right)\times \\ \exp\left[-\frac{1}{4D}\int_{t_0}^{t}d\tau\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2\right], \tag{5.2.12}$$

where the expression inside the integral represents the continuous version of the integral of (5.2.6), over all required values of the intermediate points $\{W_j\}$.
We revrite now RHS of (5.2.12) of the form

$$P(W(\omega,\omega'),t|W_0(\omega,\omega'),t_0;\delta_1,\delta_2) = \int D[W(\tau,\omega,\omega')]\times \\ \times\exp\left[-\frac{1}{4D}\int_{t_0}^{t}\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2 d\tau + \ln\theta\left(\sup_{t_0\le\tau\le t}W(\tau,\omega,\omega');\delta_1\right) + \\ +\ln\theta\left(\int_{t_0}^{t}W^2(\tau,\omega,\omega')d\tau;\delta_2\right)\right]. \tag{5.2.13}$$

We introduce now generalized conditional probability density as a Colombeau-Wiener path integral

$$
\begin{aligned}
&\left(P_{\epsilon}(W(\omega,\omega'),t|W_0(\omega,\omega'),t_0;\delta_1,\delta_2)\right)_{\epsilon\in(0,1]} = \left(\int D[W(\tau,\omega,\omega')] \times \right.\\
&\exp\left[-\frac{1}{4\epsilon}\int_{t_0}^{t}\left(\frac{dW(\tau,\omega,\omega')}{d\tau}\right)^2 d\tau + \ln\theta\left(\sup_{t_0\leq\tau\leq t}|W(\tau,\omega,\omega')|;\delta_1\right) + \right.\\
&\left.\left. + \ln\theta\left(\int_{t_0}^{t} W^2(\tau,\omega,\omega')d\tau;\delta_2\right)\right]\right)_{\epsilon\in(0,1]}.
\end{aligned} \tag{5.2.14}
$$

Reerences

[1] Fréchet, M. (1948). "Les éléments aléatoires de nature quelconque dans un espace distancié". Annales de l’Institut Henri Poincaré. 10 (4): 215–310.